

Kostant Convexity for split real and complex Kac-Moody Groups

Dissertation

zur Erlangung des Doktorgrades
der Mathematisch-Naturwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
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Kiel, 2025

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Tag der mündlichen Prüfung:	8. Juli 2025

Abstract

The goal of the present thesis is a generalization of *Kostant's Convexity Theorem* to split, minimal, real or complex Kac-Moody groups. Originally, this refers to a result in Lie theory that was proved in 1973 by B. Kostant and deals with the question how the group multiplication in the Iwasawa decomposition of a semisimple Lie group $G = KAU_+$ can be described in Iwasawa coordinates. More precisely, the theorem asserts that for any arbitrarily chosen $a \in A$, the set of A -components of all elements of the form $ak \in G$ with $k \in K$ is the exponential of the convex hull of the orbit of $\log(a) \in \mathfrak{a}$ under the action of the Weyl group of G . Questions of this kind arise in problems in representation theory or analysis on semisimple Lie groups.

The notions of *Kac-Moody groups & algebras* provide natural generalizations of semisimple Lie groups & algebras to an “infinite-dimensional” scenario. The definitions of these objects are chosen in such a way that many results of Lie theory have close analogues in the Kac-Moody setting. In contrast to classical “finite-dimensional” Lie groups, however, Kac-Moody groups should primarily be viewed as abstract groups that are defined by a concrete presentation and are a priori neither endowed with a topology nor carry any sort of manifold structure. This has the consequence that standard arguments of Lie theory, which often rely on analytic methods, are not applicable in the Kac-Moody setting. Instead, Kac-Moody theory focuses on a combinatorial approach: The same role that finite root systems and finite Weyl groups play in the study of semisimple Lie groups & algebras is taken by infinite root systems and infinite Coxeter groups on the Kac-Moody level. In this way, many theorems of Lie theory can be generalized to Kac-Moody groups & algebras by combinatorial arguments.

In particular, it can be shown that every (split, minimal) real or complex Kac-Moody group G can likewise be endowed with an Iwasawa decomposition $G = KAU_+$, which coincides with the usual decomposition in the “spherical” case of a semisimple Lie group. This makes it possible to formulate the exact analogue of Kostant's Convexity Theorem also for Kac-Moody groups. However, whether this naive generalization is indeed true was hitherto an unsolved problem. The main goal of this thesis is to give a largely positive answer to this question and to prove a natural variant of Kostant's Convexity Theorem for real or complex Kac-Moody groups. Based on this, various applications of the result to study the algebraic structure of these groups will be exhibited. In particular, it will be shown that non-spherical Kac-Moody groups in general do not possess a “horospherical decomposition” $G = KU_+K$ and that the causal pre-order on the Kac-Moody symmetric space $(G/K, \mu)$ constructed in [FHHK20] is a partial order. In addition, the convexity theorem provides a useful tool to investigate sub(semi-)groups of Kac-Moody groups. This seems to be a promising approach to decide whether the compact form K of a Kac-Moody group G is a maximal (abstract) subgroup, which is closely connected to the important open question of simplicity of G .

Zusammenfassung

Das Ziel der vorliegenden Arbeit ist eine Verallgemeinerung des *Konvexitätssatzes von Kostant* auf zerfallende, minimale, reelle oder komplexe Kac-Moody Gruppen. Ursprünglich bezeichnet dies ein Resultat der Lie-Theorie, das 1973 von B. Kostant bewiesen wurde und sich um die Frage dreht, wie die Gruppenmultiplikation in der Iwasawa-Zerlegung einer halbeinfachen Lie-Gruppe $G = KAU_+$ in Iwasawa-Koordinaten beschrieben werden kann. Genauer besagt der Satz, dass für jedes beliebig gewählte $a \in A$ die Menge der A -Komponenten aller Elemente der Form $ak \in G$ mit $k \in K$ das Exponential der konvexen Hülle der Bahn von $\log(a) \in \mathfrak{a}$ unter der Wirkung der Weyl-Gruppe von G ist. Fragestellungen dieser Art treten in Problemen der Darstellungstheorie oder der Analysis auf halbeinfachen Lie-Gruppen auf.

Die Begriffe der *Kac-Moody Gruppen & Algebren* bilden natürliche Verallgemeinerungen halbeinfacher Lie-Gruppen & Algebren in ein “unendlich-dimensionales” Szenario. Die Definitionen dieser Objekte sind so gewählt, dass viele Resultate der Lie-Theorie naheliegende Entsprechungen im Kac-Moody Setting besitzen. Im Gegensatz zu klassischen “endlich-dimensionalen” Lie-Gruppen sind Kac-Moody Gruppen jedoch primär als abstrakte Gruppen zu verstehen, die durch eine konkrete Präsentation definiert sind und a priori weder eine Topologie noch irgendeine Art von Mannigfaltigkeitsstruktur tragen. Dies hat zur Folge, dass Standardargumente der Lie-Theorie, die oft auf analytischen Methoden beruhen, im Kac-Moody Setting nicht anwendbar sind. Stattdessen konzentriert sich die Kac-Moody Theorie auf einen kombinatorischen Zugang: Dieselbe Rolle, die endliche Wurzelsysteme und endliche Weyl-Gruppen im Studium halbeinfacher Lie-Gruppen & Algebren spielen, nehmen unendliche Wurzelsysteme und unendliche Coxeter-Gruppen auf der Kac-Moody Ebene ein. Auf diese Weise können viele Sätze der Lie-Theorie mithilfe kombinatorischer Argumente auf Kac-Moody Gruppen & Algebren verallgemeinert werden.

Insbesondere lässt sich zeigen, dass jede (zerfallende, minimale) reelle oder komplexe Kac-Moody Gruppe G ebenfalls mit einer Iwasawa-Zerlegung $G = KAU_+$ ausgestattet werden kann, die im “sphärischen” Fall einer halbeinfachen Lie-Gruppe mit der üblichen Zerlegung übereinstimmt. Dies ermöglicht es, das exakte Analogon des Konvexitätssatzes von Kostant auch für Kac-Moody Gruppen zu formulieren. Ob diese naive Verallgemeinerung jedoch tatsächlich korrekt ist, galt bislang als ungelöstes Problem. Das Hauptziel dieser Arbeit ist es, eine weitgehend positive Antwort auf diese Frage zu geben und eine natürliche Variante des Konvexitätssatzes von Kostant für reelle oder komplexe Kac-Moody Gruppen zu beweisen. Darauf aufbauend werden verschiedene Anwendungen des Resultats zum Studium der algebraischen Struktur dieser Gruppen dargelegt. Insbesondere wird gezeigt, dass nicht-sphärische Kac-Moody Gruppen im Allgemeinen keine “horosphärische Zerlegung” $G = KU_+K$ besitzen und dass die “kausale Prä-Ordnung” auf dem Kac-Moody symmetrischen Raum $(G/K, \mu)$, der in [FHHK20] konstruiert wurde, eine Halbordnung ist. Zusätzlich liefert der Konvexitätssatz ein praktisches Werkzeug, um Unter(halb-)gruppen von Kac-Moody Gruppen zu untersuchen. Dies scheint ein vielversprechender Ansatz zu sein, um zu entscheiden, ob die kompakte Form K einer Kac-Moody Gruppe G eine maximale (abstrakte) Untergruppe ist, was eng mit der wichtigen offenen Frage nach Einfachheit von G zusammenhängt.

Acknowledgements

*“Hátt ég lyfti huga mínum á flug,
legg við hlustir og nem vindhörpuslátt.
Brátt ég eyði öllum línun á jörð,
sýnist umhverfið allt vera eilífðarblátt.”*

The process of doing independent mathematical research was a great challenge for me, often in exciting and sometimes in despairing ways. This thesis is dedicated to everyone who accompanied me on my journey during the past three years.

First and foremost, I am endlessly indebted to my family. Your strong empowerment encouraged me to even take the risk of moving abroad for mathematical studies and without your tireless assistance, I would never have been able to establish myself in Kiel. Despite our geographical separation of almost exactly 1.000 kilometers, I constantly felt that you were present in my life and I enjoyed the many beautiful experiences we made together on our travels. In difficult times, when I was possibly not the optimal partner for pleasant conversations, your open hearts and ears always comforted me and enabled me to look forward. Without your infinite support, my thesis would not have come into existence.

Equally special thanks are due to my friends in Kiel who helped me to maintain my mental sanity on a day-to-day basis. In particular, I am inexpressibly grateful to Leonie & Christian. Your overwhelmingly charming way and your incredible ability to make me laugh in almost any situation always made me see the bright side and I am happy for every piece of your delicious pies or bowls of bread salad I was lucky to enjoy. On top of that, I am especially thankful to you, Christian, for bearing the weight of sharing the office with me for two years. Your presence regularly motivated me to brave Kiel’s unpredictable weather in order to have some nice (often, but not necessarily) mathematical discussions and share a drink after a hard day’s work.

Likewise, I sincerely want to thank Franziska & Daniel. I always felt welcome in your presence and I greatly appreciate your empathy and kindness, both in our enlightening conversations and the valuable advice on difficult decisions that you have given me. On a lighter note, I am also thankful to Leonie & Franziska for having Christian, Daniel & me on board in their knitting sessions, even though we were probably rather distracting.

Furthermore, I wish to express my deep gratitude to Anna. Ever since our first meeting, I found our conversations exceptionally simple and pleasant. I admire your impressive resilience and the seemingly uncountable number of your diverse interests. Your stories, plans and projects fascinated me over and over again. Thank you for sharing them with me, for listening to mine and for your constant support over the last three years.

Last but not least, major credit is due to my good old friend Heinrich in Austria. I deeply regret that we did not have the chance to meet very often while I was in Kiel, but I am glad that we stayed in touch and hope that we still have some fine glasses of whisky to share in the future.

During my PhD-studies, I received a scholarship from the *Studienstiftung des Deutschen Volkes*. This enormous source of support not only enabled me to focus on my mathematical research but also provided a vast number of opportunities to broaden my horizon and get in touch with many fascinating people from other scientific disciplines. Moreover, I want to thank the mathematical department at *CAU Kiel* for their hospitality and convenient assistance in attending scientific conferences. Finally, I am grateful to my advisor Ralf Köhl for suggesting the research topic of Kostant Convexity to me. I always felt that it matched my mathematical capabilities very well and I was often able to progress by my own questions and ideas. I also appreciate your efforts to connect me with the mathematical world.

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Introduction

One of the crowning achievements of 19th and 20th century mathematics is the discovery of a deep connection between two seemingly unrelated disciplines: geometry and algebra. Its origins are often attributed to Felix Klein and his *Erlangen program* who suggested to study geometric objects intrinsically via their groups of symmetries. His ideas led to profound applications of abstract group theory to geometry. Conversely, they also allowed for geometric interpretations of problems in group theory that previously seemed to be entirely algebraic. To the present day, this correspondence continues to serve as an essential tool in these fields and has influenced many other areas of mathematics as well.

A prime example of this beautiful connection is the hugely successful development of *Lie theory*. Simply put, it is the natural algebraic perspective on differential geometry. *Lie groups* frequently arise as continuous groups of symmetries of smooth geometric objects, both in the realm of pure mathematics as well as in important physical theories of the early 20th century, such as general relativity. From the algebraic viewpoint, such a group is easiest to study through its associated *Lie algebra*, which is a finite-dimensional real vector space equipped with a certain skew-symmetric bilinear operation that, in a sense, measures the “non-commutativity” of the Lie group and makes it possible to “linearize” group-theoretic problems in order to approach them by methods of linear algebra. A natural transition mechanism between a Lie group and its Lie algebra is provided by the *exponential map*. On the geometric side, Lie algebras can be interpreted as describing “infinitesimal symmetries” of geometric objects which often already encode a surprising amount of information. The algebraic correspondence between Lie groups & algebras is therefore equivalent to a strong connection between “local” and “global” geometric properties.

One of the most important results in Lie theory is the classification of finite-dimensional, complex, simple Lie algebras that was established by Élie Cartan at the end of the 19th century. Based on earlier work of Wilhelm Killing, Cartan showed that, except for five isolated cases, every such Lie algebra belongs to one of four infinite families that all have explicit realizations as matrix algebras. Moreover, the classification is entirely “combinatorial” in the sense that the “type” of such a Lie algebra is determined by an associated finite directed graph, called its *Dynkin diagram*, or equivalently, its corresponding *Cartan matrix* (cf. Table 2.1). Cartan’s result, however, did not establish the existence of the “exceptional” Lie algebras which remained an open problem for several decades and sparked many developments in modern mathematics. Concrete realizations of these Lie algebras and their associated Lie groups and “geometries” were constructed in the first half of the 20th century. Finally, a truly satisfying conclusion to Cartan’s classification was reached by Jean-Pierre Serre in 1966. Instead of giving case-by-case constructions, he proved that every finite-dimensional, complex, simple Lie algebra can be defined abstractly by an explicit presentation whose generators and relations depend only on the entries of its Cartan matrix. In particular, applying this presentation to the exceptional types provided a uniform proof of existence for the five exceptional Lie algebras.

Together with the structure theory developed along the way, Cartan’s classification also made it possible to study finite-dimensional, complex, simple Lie algebras, which are inherently algebraic objects, by combinatorial methods. Most importantly, every such Lie algebra \mathfrak{g} is associated with a finite *root system* $\Phi \subset \mathfrak{a}^*$ in a real form $\mathfrak{a}^* \subset \mathfrak{h}^*$ of the dual space of a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ and this gives rise to a finite *Weyl group* W that acts on \mathfrak{a}^* by orthogonal linear transformations preserving Φ . These groups are examples of *finite reflection groups*, which can be considered as

highly “symmetric” combinatorial objects, and their abstract theory is deeply connected to the structure of the corresponding Lie algebras. Cartan’s ideas naturally extend to finite-dimensional, real, semisimple Lie algebras, leading to the concepts of *reducible* and *relative* Weyl groups. By means of the exponential map and some basic covering theory, it also provided a classification of semisimple Lie groups for which the role of Weyl groups is equally important.

Motivated by their profound impact on Lie theory and surprising occurrences in other areas of mathematics, finite reflection groups were axiomatized and generalized in the 1930’s by Harold S.M. Coxeter and are nowadays known as *Coxeter groups*. In the finite case, they are completely classified and essentially correspond to the Weyl groups of complex simple Lie algebras, the finite dihedral groups and two exceptional examples. Infinite Coxeter groups, however, also soon began to appear in various mathematical disciplines. In particular, the theory of analogues of classical Lie groups over arbitrary fields, developed by Chevalley, Bruhat and Tits in the 1960’s, showed that *affine Coxeter groups* are naturally linked to semisimple algebraic groups over non-Archimedean local fields. While the affine groups are in a sense the “simplest” family of infinite Coxeter groups and closely related to the finite ones, other examples made significant contributions to a number of different areas, such as geometric group theory.

Further developments even established the relevance of much more refined combinatorial structures to Lie theory. In the Archimedean setting, Cartan’s work on *Riemannian symmetric spaces of non-compact type* linked semisimple Lie groups to metric spaces of non-positive curvature. The subsequent construction of their *visual boundaries* showed that every real semisimple Lie group is equipped with an action on a *spherical building*. This is a highly “symmetric” simplicial complex assembled from subcomplexes, called *apartments*, that are all isomorphic to the *Coxeter complex* of a finite reflection group and satisfy certain compatibility axioms. By imitating this construction, the theory of Bruhat & Tits established the same connection between semisimple algebraic groups over non-Archimedean local fields and *affine buildings*. These are simplicial complexes that are defined as in the spherical case but modelled on affine Coxeter groups and they can reasonably be viewed as “discrete analogues” of Riemannian symmetric spaces of non-compact type. In both situations, these structures encode a lot of algebraic information about the respective Lie groups and should be seen as their naturally associated geometric objects.

Apart from the non-Archimedean setting, a different generalization of Lie theory was developed simultaneously and independently by Victor G. Kac in the Soviet Union and Robert Moody in Canada during the 1960’s. They both introduced a family of complex Lie algebras that became to be known as *Kac-Moody algebras*. Essentially, their construction imitates Serre’s presentation of finite-dimensional, complex, semisimple Lie algebras, but allows for a broader class of *generalized Cartan matrices*. The structure of the resulting Lie algebras resembles the classical case, but some notable new phenomena arise. Every Kac-Moody algebra \mathfrak{g} , defined by a matrix A , is still associated with a suitable notion of root system $\Phi \subset \mathfrak{a}^*$ in the dual space of a finite-dimensional real Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and a corresponding Weyl group W that acts on \mathfrak{a}^* by linear maps preserving Φ . However, \mathfrak{g} is infinite-dimensional, Φ is an infinite set and W is an infinite Coxeter group, unless A was one of the “original” matrices in Cartan’s classification and the associated Kac-Moody algebra is in fact a finite-dimensional, complex, semisimple Lie algebra. The structure theory of Kac-Moody algebras is intimately related to the general theory of Coxeter groups and thus retains a strong combinatorial flavour. Soon afterwards, Kac also provided a first construction of complex *Kac-Moody groups*, which can be viewed as “infinite-dimensional” analogues of complex semisimple Lie groups. It was subsequently generalized by Tits who gave a natural definition of Kac-Moody groups over arbitrary fields. In the real and complex case, they further developed a variant of the exponential map similar to the correspondence between Lie groups & algebras. Outside the setting of classical Lie theory, however, this mechanism is much more complicated since the exponential map cannot be defined on the entire Kac-Moody algebra. Consequently, it is much more difficult to translate “non-linear” questions about Kac-Moody groups into “linear” problems about Kac-Moody algebras.

In Tits’ approach, Kac-Moody groups are defined by an explicit presentation and should therefore be primarily viewed as abstract groups that are initially not endowed with any topology, let alone any sort of manifold structure. Nevertheless, it is more or less a consequence of their construction

that real and complex Kac-Moody groups share many algebraic properties with real semisimple Lie groups. While a lot of proofs in the finite-dimensional case rely on the differentiable structure of Lie groups, it is sometimes possible to replace them by combinatorial arguments which can then be carried over more easily to the Kac-Moody setting. For example, every real or complex Kac-Moody group G has an *Iwasawa decomposition* $G = KAU_+$ with unique factorization. If G is a connected semisimple Lie group, the existence of such a decomposition is a well-known result of Lie theory. In this situation, K is a maximal compact subgroup of G (if the center of G is finite), $A = \exp(\mathfrak{a})$ is an abelian subgroup defined as the exponential image of a real form \mathfrak{a} of a Cartan subalgebra \mathfrak{h} of the Lie algebra \mathfrak{g} of G and $U_+ = \exp(\mathfrak{u}_+)$ is a contractible nilpotent subgroup that is obtained as the exponential of the direct sum \mathfrak{u}_+ of all positive *root spaces* in \mathfrak{g} . For Kac-Moody groups, these notions are not well-defined. Instead, K is defined as the fixed-point set of a certain involutive automorphism of G , $A = \exp(\mathfrak{a})$ can be constructed via the exponential map as before, and U_+ is set to be the subgroup of G generated by a (typically infinite) family of positive “root groups”. These definitions coincide with the previous ones in the Lie group case where the automorphism defining K is a *Cartan involution*.

Given a real or complex Kac-Moody group G and some $g \in G$, the unique elements $k \in K$, $a \in A$ and $u \in U_+$ such that $g = kau$ can be viewed as “coordinates” of g which naturally raises the question how these coordinates behave with respect to the group multiplication. More precisely, given two elements $g_1, g_2 \in G$, is it possible to determine the Iwasawa coordinates of the product $g_1 g_2 \in G$ from those of the individual factors? As the three subgroups do not commute with or normalize each other, this problem is very difficult in such a general form. In the case where G is a semisimple Lie group, questions in representation theory and analysis often require to determine how the A -component of an element $a \in A$ is affected by right-multiplication with another element from the maximal compact subgroup K . This problem was solved by Bertram Kostant in 1973 who proved that if one multiplies any fixed $a = \exp(h) \in A$ from the right by all elements of K and computes the A -components of all these products, the resulting subset of A is the exponential image of the convex hull of the Weyl group orbit of $h = \log(a) \in \mathfrak{a}$. This result is nowadays known as *Kostant’s Convexity Theorem*. It greatly influenced modern Lie theory, provided many applications and allowed for vast generalizations in diverse areas of mathematics.

Overview of the main results

The exact analogue of Kostant’s Convexity Theorem can be formulated for an arbitrary real or complex Kac-Moody group, but whether this naive generalization is indeed true was hitherto an open problem. The principal goal of this thesis is to give a largely positive answer to this question and to use the result to study algebraic properties of Kac-Moody groups. Throughout this section, the notation is as follows (cf. Chapters 1 to 4 for the according definitions):

- $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is a symmetrizable generalized Cartan matrix.
- $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ is a Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to \mathbb{A} and Λ^\vee is the dual \mathbb{Z} -module of Λ .
- \mathfrak{g} is the split real (or complex) Kac-Moody algebra of type \mathcal{D} and $\mathfrak{a} := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ is (a real form of) the Cartan subalgebra of \mathfrak{g} .
- G is the split, minimal, real (or complex) Kac-Moody group of type \mathcal{D} .
- $G = KAU_+$ is an Iwasawa decomposition of G , where K is the fixed-point set of the compact involution of G , $A := \exp(\mathfrak{a})$ is the image of \mathfrak{a} under the exponential map, which restricts to a bijection $\exp : \mathfrak{a} \rightarrow A$, and U_+ is the subgroup of G generated by all positive root groups.
- $\Pi_A : G \rightarrow A, g = kau \mapsto a$, is the Iwasawa projection onto the A -component.
- W is the Weyl group of \mathbb{A} which acts on \mathfrak{a} via the \mathcal{D} -representation.
- (W, S) is the associated Coxeter system, $\Sigma^*(W, S)$ its reversed Coxeter complex and $\tilde{\Sigma}^*(W, S)$ the quotient of $\Sigma^*(W, S)$ by the equivalence relation defined in (4.3.19).
- $\overline{C} \subset \mathfrak{a}$ is the closed fundamental chamber and $X_+ \subset \mathfrak{a}$ the Tits cone for the \mathcal{D} -representation of W on \mathfrak{a} .
- $\Xi_h := \text{conv}(W \cdot h) \subset \mathfrak{a}$ is the convex hull of the Weyl group orbit of a point $h \in \mathfrak{a}$.

The main result of this thesis is the following generalization of Kostant's Convexity Theorem to the class of real or complex Kac-Moody groups. For the notion of an *almost virtually singular element* of \mathfrak{a} , the reader is referred to Definition 4.1.2. If \mathbb{A} is indecomposable, it is equivalent to having finite stabilizer under the W -action on \mathfrak{a} .

Theorem A (cf. 5.1.1). *If $h \in \mathfrak{a}$ is almost virtually singular and $a := \exp(h) \in A$, then*

$$\{\Pi_A(ak) \mid k \in K\} = \exp(\Xi_h).$$

If \mathbb{A} is *spherical*, i.e. the Cartan matrix of a finite-dimensional, complex, semisimple Lie algebra, then W is finite, $X_+ = \mathfrak{a}$ and the condition on h is vacuous. In this situation, Theorem A reduces to a special case of Kostant's Convexity Theorem. In the general Kac-Moody setting, however, it is genuinely new. It is worth noting that it only gives a description of the set $\Pi_A(gK)$ for certain elements $g = a \in A$ and not for an arbitrary $g \in G$. In Kostant's original setting, this does not pose a restriction because a semisimple Lie group has a KAK -decomposition. For general Kac-Moody groups, such a decomposition is not available and the theorem cannot be generalized in this way. In addition, it is a consequence of Kostant's Convexity Theorem and the KAK -decomposition that a semisimple Lie group also has a KU_+K -decomposition. The question whether the same is true for Kac-Moody groups was formulated in [Hor17]. As a first application of Theorem A, we establish that such a decomposition only exists under very restrictive assumptions. In the next results, a Kac-Moody root datum is said to be *admissible* if the topological interior X_+° of the Tits cone $X_+ \subset \mathfrak{a}$ is not empty. This is always satisfied if no indecomposable component of \mathbb{A} is of affine type. In the affine case, however, there exist examples of non-admissible root data whose rank is equal to $n - 1$.

Theorem B (cf. 6.2.5 and 6.2.6).

- (i) *If \mathcal{D} is admissible, then $G = KU_+K$ if and only if \mathbb{A} is spherical and $d = n$.*
- (ii) *If \mathbb{A} is an indecomposable GCM of affine type and $d = n - 1$, then $G = KU_+K$.*

Another application of Theorem A yields the following algebraic property of Kac-Moody groups, where we denote by $A_+^\circ := \exp(X_+^\circ) \subset A$ the exponential image of the topological interior of the Tits cone. If \mathbb{A} is spherical, then $A_+^\circ = A$ and the first statement is simply the well-known KAK -decomposition which holds more generally for reductive Lie groups. If \mathbb{A} is not spherical, then A_+° is a proper subset of A and the second assertion is new.

Theorem C (cf. 6.2.8). *If \mathcal{D} is admissible, then exactly one of the following conditions holds:*

- (i) *If \mathbb{A} is spherical, then $KA_+^\circ K = G$.*
- (ii) *If \mathbb{A} is not spherical, then $(KA_+^\circ K)^N \subset KA_+^\circ U_+ \subsetneq G$ for all $N \in \mathbb{N}$.*

The interest in this algebraic result is rooted in a related geometric question. In [FHHK20], the authors introduce the notion of a *Kac-Moody symmetric space* which serves as a useful substitute for Riemannian symmetric spaces of non-compact type in Kac-Moody theory. Exactly as in the Lie group case, a Kac-Moody symmetric space is defined as the quotient G/K endowed with a certain “multiplication map” $\mu : G/K \times G/K \rightarrow G/K$ that mirrors properties of geodesic reflections in Riemannian symmetric spaces. In the Kac-Moody setting, however, the set G/K does not carry any differentiable structure. Alternatively, the authors use the topological interior of the Tits cone to construct a *causal structure* on G/K and show that the corresponding *causal pre-order* is either trivial (i.e. any two points precede each other) or a partial order (i.e. antisymmetric), where the first case occurs if and only if the subsemigroup of G generated by K and A_+° is equal to G . At the time of writing, they were not able to decide whether or not this condition is satisfied for Kac-Moody groups, but Theorem C provides an immediate solution to this problem.

Theorem D (cf. 6.3.18). *If \mathbb{A} is invertible and $(G/K, \mu)$ is a Kac-Moody symmetric space in the sense of [FHHK20], then exactly one of the following conditions holds:*

- (i) *If \mathbb{A} is spherical, the causal pre-order on $(G/K, \mu)$ is trivial.*
- (ii) *If \mathbb{A} is not spherical, the causal pre-order on $(G/K, \mu)$ is a partial order.*

The proof of Theorem A is based on a detailed study of the “geometry” of convex hulls of Weyl group orbits in \mathfrak{a} . If \mathbb{A} is spherical, then W is finite and Ξ_h is a *convex polytope* for every $h \in \mathfrak{a}$. Such sets can be investigated by classical techniques of convex geometry. If \mathbb{A} is not spherical, however, W is an infinite group and these tools are not available. Convex hulls of infinite point sets are rarely studied in full generality and often fail to have any desirable properties. In particular, it is not clear whether Ξ_h is a closed subset of \mathfrak{a} whose relative boundary is comprised of *faces* that determine the set itself in a satisfactory way. Nevertheless, in the case of an almost virtually singular element $h \in \mathfrak{a}$, we show that Ξ_h does indeed possess a well-behaved facial structure that admits a completely combinatorial description in terms of the Coxeter system (W, S) .

Theorem E (cf. 4.3.38). *If $h \in \mathfrak{a}$ is almost virtually singular, then Ξ_h is a closed subset of \mathfrak{a} and its face poset is isomorphic to $\tilde{\Sigma}^*(W, S)$.*

A particularly appealing special case occurs when h is a *regular* element of \mathfrak{a} , i.e. when h or $-h$ is contained in an open Weyl chamber of X_+ . In this case, the equivalence relation on $\Sigma^*(W, S)$ is trivial and the face poset of Ξ_h is isomorphic to the reversed Coxeter complex in such a way that if a face $F \subset \Xi_h$ corresponds to the standard coset $wW_J \subset W$ under this isomorphism, the dimension of F coincides with the cardinality of the finite set $J \subset I$.

The most important consequence of Theorem E is that Ξ_h can be reconstructed from its proper faces as the intersection of all its supporting half-spaces, which implies that another point in \mathfrak{a} is contained in Ξ_h if and only if it satisfies a (typically infinite) family of weak linear inequalities that can be determined explicitly (cf. Theorem 4.3.38 (v)). This fact will be crucial to prove one of the inclusions in Theorem A. To show the converse, we apply an inductive argument for which we first study the intersections of Ξ_h with the parallel translates $\{E^{(i)}(t) \mid i \in I_{\text{vis}}, t \in T_h^{(i)}\}$ of its maximal supporting hyperplanes (cf. (4.3.28) and (4.3.33)). Given any such plane, we establish that $P^{(i)}(t) := \Xi_h \cap E^{(i)}(t)$ is a union of convex hulls of orbits of the corank-1 standard subgroup $W^{(i)} := W_{I \setminus \{i\}} \subset W$, where the footpoint runs over a distinguished “small” subset of $P^{(i)}(t)$ that we call the *essential part* (cf. Definition 4.3.52).

Theorem F (cf. 4.3.55). *If $h \in \mathfrak{a}$ is almost virtually singular, $i \in I_{\text{vis}}$ and $t \in T_h^{(i)}$, then every point in $P^{(i)}(t)$ lies in the convex hull of the $W^{(i)}$ -orbit of a point in the essential part.*

One noteworthy shortcoming of our results is that both Theorem E and F crucially rely on the assumption that $h \in \mathfrak{a}$ is almost virtually singular and fail without this condition. Hence, our proof strategy does not allow us to extend Theorem A beyond the almost virtually singular case. Investigating whether it is still valid for a more general class of elements requires new ideas and is a topic for further research that is beyond the scope of this thesis.

Organization of the thesis

Following the current general introduction, the thesis begins with an optional historical overview of Kostant’s Convexity Theorem, its origins and various generalizations that is intended to motivate the subsequent discussion. Afterwards, the core content of the text is divided into two parts: Part I contains preliminary material that is largely a collection of definitions and results that are well-known in the literature, but may need to be adapted to our setting. It also serves the purpose of fixing much of the notation that will be used in Part II where we establish mostly original results and prove our main theorems. More precisely, the text is organized as follows:

- **Part I: Preliminaries**

- **Chapter 1** provides a rather brief survey of the theory of Coxeter groups. Having collected some basic definitions and properties in Section 1.1, we introduce the notion of *colourings* of Coxeter diagrams in Section 1.2, which will be very useful in order to investigate convex hulls of Weyl group orbits in Chapter 4. The material of this section is elementary but perhaps difficult to find in the literature. It also contains some more refined concepts that, to the best of the author’s knowledge, have hitherto not been introduced elsewhere in the same level of detail.

- **Chapter 2** elaborates on Kac-Moody algebras. In the first three sections, we introduce *generalized Cartan matrices* and use the concept of *Kac-Moody root data* for a flexible definition of Kac-Moody algebras that is suitable for inductive arguments. Section 2.4 contains some basic notions from the representation theory of Kac-Moody algebras that will be important at various stages in the text. In Section 2.5 and 2.6, we define the *Weyl group* of a generalized Cartan matrix and study its *\mathcal{D} -representation* provided by a Kac-Moody root datum \mathcal{D} . Finally, in Section 2.7, we discuss some assumptions and conventions that will be employed in later chapters. The vast majority of this chapter consists of standard results from Kac-Moody theory that were adapted to our definitions and will be mentioned without proof. For the reader’s convenience, however, we occasionally provide self-contained proofs of statements for which it is potentially hard to give a satisfying reference.
- **Chapter 3** introduces the class of Kac-Moody groups for which we aim to prove our main theorems. Using the notion of Kac-Moody root data, we define Kac-Moody groups via a concrete presentation, called the *constructive Tits functor*, and study their basic structure and representation theory in Section 3.1 and 3.2. Afterwards, we discuss the special cases of rank-1 groups (Section 3.3) and Kac-Moody groups over the real or complex numbers (Section 3.4), where we also establish the existence of the Iwasawa decomposition. Section 3.5 contains a lot of explicit matrix computations inside the groups $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ that will be relevant to Chapter 5 and 6 but may freely be skipped on first reading. In the final Section 3.6, we explain a certain “functoriality” in the definition of a Kac-Moody group with respect to the underlying root datum. This section is also optional and only needed for a minor aspect of the proofs of two theorems in Chapter 5. Except for the computations in Section 3.5, most results of this chapter are standard facts from Kac-Moody theory that might have been tailored to our setup. When appropriate, we therefore also give self-contained proofs of statements that appear to be known in a different context.

• **Part II: Kostant Convexity for Kac-Moody Groups**

- **Chapter 4** is devoted to studying convex hulls of Weyl group orbits obtained from the \mathcal{D} -representation. The relevant terminology as well as some important facts from convex geometry to be used throughout the chapter is introduced in Section 4.1 and 4.2. In Section 4.3, we begin to investigate the set Ξ_h in the case where $h \in \mathfrak{a}$ is *internal*, but as we shall see, it will soon be necessary to invoke the stronger assumption that $h \in \mathfrak{a}$ is almost virtually singular. In this situation, we prove Theorem E and F which are crucial ingredients for the proofs of our main theorems in Chapter 5. We also briefly discuss other orbit types in Section 4.4 where it seems difficult to establish any general results. Instead, we content ourselves with some interesting examples that highlight the problematic phenomena that arise in the structure of Ξ_h when $h \in \mathfrak{a}$ is not almost virtually singular.

Although we work in the setting of Kac-Moody theory, it is interesting to observe that most arguments in this chapter merely rely on simple properties of Coxeter groups. Convex sets related to Coxeter groups have been studied in the literature from various perspectives, but most references either focus on the case of finite groups or work in a far more abstract setting (e.g. incidence geometry) that is not suitable for our purposes because, as we shall explain, it will be important that we view Ξ_h as a “metric” object that is embedded in the real vector space \mathfrak{a} in a particular way. While some of our results can surely be deduced from the existing literature, the methods we apply are typically very simple and adapted to our purposes, so it seems appropriate to provide self-contained proofs throughout this chapter.

- **Chapter 5** is the heart of this thesis and establishes the anticipated generalization of Kostant’s Convexity Theorem to the classes of real and complex Kac-Moody groups. Section 5.1 and 5.2 contain the statement and proof of Theorem A. Section 5.3 and 5.4 then develop a “linear analogue” of Theorem A on the level of a Kac-Moody algebra.

The two results are very similar in their formulation and their proofs almost identical. Both are divided into three steps:

- * **Step 1** combines Theorem E (more precisely, the characterization of Ξ_h by weak linear inequalities in Theorem 4.3.38 (v)) with arguments about highest-weight representations of Kac-Moody groups & algebras to establish one of the inclusions in the respective theorem.
- * **Step 2** exploits Theorem F in order to prove the converse inclusion by induction on the size of the generalized Cartan matrix. The base case of this induction makes use of the matrix computations in Section 3.5.
- * **Step 3** uses the functoriality developed in Section 3.6 to extend the results of the previous two steps, which require a restriction on the underlying Kac-Moody root datum, to arbitrary root data.

Despite the resemblance between these two theorems, the author is not aware whether one of them can be deduced from the other in a straightforward way. In Section 5.5, we compare our “linear” result with a very similar convexity theorem for Kac-Moody algebras due to Kac & Peterson and discuss the relationship between Theorem A in the case of a spherical generalized Cartan matrix and Kostant’s original result for semisimple Lie groups. Finally, we sketch how our arguments can be modified to prove a close analogue of Kostant’s Convexity Theorem for groups of real or complex points of reductive algebraic \mathbb{R} -groups.

- **Chapter 6** contains various applications of Theorem A to the algebraic structure of real or complex Kac-Moody groups. In Section 6.1, we describe an algorithm that, to a limited extent, provides a recipe for computing the group multiplication in such a group $G = KAU_+$ in Iwasawa coordinates. This algorithm can then be used to investigate the structure of sub(semi-)groups of G and we illustrate one such example in Theorem 6.1.10. Inspired by [Hor17], we study decompositions of Kac-Moody groups in Section 6.2 where we prove Theorem B and C. Finally, in Section 6.3, we present the construction of Kac-Moody symmetric spaces developed in [FHHK20] and then interpret some of our theorems in this “geometric” context. Here, we closely follow that reference, but as its setting is slightly different from ours, we have to invoke some assumptions on the Kac-Moody groups in question and occasionally modify its proofs in order to be consistent with our definitions. In particular, we explain the construction of the causal pre-order on Kac-Moody symmetric spaces and establish Theorem D, which is the only original result of this section and an immediate consequence of Theorem C in that framework.
- **Chapter 7** marks the conclusion to this thesis. It summarizes the principal ideas and results as well as the difficulties that prevent further generalizations. As a consequence, we formulate some open questions that arise naturally from our discussion and might serve as interesting topics for future research.

Since the material of Part I mostly consists of standard results from the literature, the first three chapters are each preceded by an introductory paragraph containing more precise references. The final chapter is followed by a subject index and a list of symbols, where frequently used terminology and notation can be found, as well as a bibliography.

One notable omission of this thesis is that the author has chosen not to introduce the theory of buildings, although it is an essential tool in Kac-Moody theory and many algebraic properties of Kac-Moody groups mentioned in Chapter 3 are proved via this machinery. In our discussion, however, there is only one result for which such arguments are required. This is the case in Proposition 3.4.6 where precise references for the undefined terminology and the relevant results from building theory will be given. Affine buildings will also appear briefly at the end of the historical introduction but only from a motivational perspective. Thus, introducing buildings in detail would have potentially distracted the reader from the main line of reasoning. The excellent book [AB08] provides a thorough account of this subject.

Notational conventions

Throughout this thesis, we employ the following conventions on notation and terminology:

- Given two sets A and B , we write $A \subset B$ or $B \supset A$ if A is any subset of B and $A \subsetneq B$ or $B \supsetneq A$ if it is a proper subset. The symbols \subseteq and \supseteq will be avoided.
- The set of natural numbers is denoted by \mathbb{N} and does not contain zero. If we want to include zero, we write $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. If $n \in \mathbb{N}_0$ is a non-negative integer that is fixed by the context, we set $I := \{1, 2, \dots, n\}$ which is understood to be the empty set if $n = 0$.
- We write δ_{ij} for the Kronecker delta of two symbols i and j :

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- A field will always be denoted by \mathbb{K} and its multiplicative subgroup by $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$. Unless explicitly stated otherwise, all fields are assumed to have characteristic zero.
- For any field \mathbb{K} and $n \in \mathbb{N}_0$, we write $\mathbb{K}^{n \times n}$ for the \mathbb{K} -vector space of $(n \times n)$ -matrices over \mathbb{K} and \mathbb{I}_n for the corresponding $(n \times n)$ -identity matrix. Likewise, we denote the set of $(n \times n)$ -matrices with integer entries by $\mathbb{Z}^{n \times n}$. For $n = 0$, we follow the convention that $\mathbb{K}^{0 \times 0}$ (resp. $\mathbb{Z}^{0 \times 0}$) contains a unique element that may be thought of as the “empty matrix”.
- Given a vector space V over a field \mathbb{K} , we use angle brackets to denote the natural pairing between V and its dual space V^* :

$$\begin{aligned} \langle \cdot, \cdot \rangle : V^* \times V &\rightarrow \mathbb{K} \\ (f, v) &\mapsto \langle f, v \rangle := f(v) \end{aligned}$$

If Λ^\vee and Λ are dual free \mathbb{Z} -modules, the same notation will be applied to the dual pairing $\Lambda \times \Lambda^\vee \rightarrow \mathbb{Z}$.

- We usually view a basis of a \mathbb{K} -vector space V as a subset of V instead of a family of vectors. For our purposes, this will be relevant in the following situation: If V is finite-dimensional with $d := \dim(V) \in \mathbb{N}$, we typically write a basis of V as a subset $\mathcal{B} = \{v_1, \dots, v_d\} \subset V$ of cardinality d , where it is tacitly understood that the basis is ordered in the suggested way. More precisely, we identify the set \mathcal{B} with the tuple $(v_1, \dots, v_d) \in V^d$ where the indices are ordered increasingly from 1 to d . Hence, we denote the corresponding dual basis of V^* also by $\mathcal{B}^* = \{v_1^*, \dots, v_d^*\} \subset V^*$ and implicitly require that the ordering satisfies $\langle v_i^*, v_j \rangle = \delta_{ij}$ for all $i, j \in \{1, \dots, d\}$. The same convention applies to bases of free \mathbb{Z} -modules.
- If Λ^\vee is a free \mathbb{Z} -module and \mathbb{K} a field, the tensor product $V := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}$ has a natural structure of a \mathbb{K} -vector space. Given any $h \in \Lambda^\vee$ and $z \in \mathbb{K}$, we denote the elementary tensor $h \otimes z$ simply by zh . In particular, we view Λ^\vee as a subset of V by identifying it with the set $\Lambda^\vee \otimes 1 \subset V$.
- For any two real numbers $x, y \in \mathbb{R}$, the closed interval between x and y may be written both as $[x, y]$ and $[y, x]$, independent of whether $x \leq y$ or $y \leq x$. The same convention applies to open and half-open intervals.
- Whenever a symbol is indexed by \pm , it refers to two separate objects indexed by $+$ and $-$, respectively. Statements involving these indices are understood to hold for both choices of the sign simultaneously.
- The notions of an *abelian group* and a *\mathbb{Z} -module* are equivalent and used interchangeably.
- The word *poset* is used as an abbreviation for *partially-ordered set*.
- Further references for notation and terminology that is specific to certain parts of the text:
 - Assumptions: 2.7.1, 2.7.3, 2.7.5, 2.7.6, 4.1.3, 4.3.43, 5.2.2, 5.4.2, 6.3.13
 - Conventions: 1.2.3, 2.1.2, 2.7.2, 2.7.4, 2.7.7, 3.4.9, 4.1.5
 - Notation: 1.2.8, 2.1.7, 2.3.6, 2.5.4, 2.6.8, 3.3.2, 3.4.13, 4.1.1, 6.1.2

A Historical Account of Kostant Convexity

The original meaning of *Kostant Convexity* stands for an important convexity property of the Iwasawa decomposition of semisimple Lie groups & algebras. More precisely, it refers to two similar statements that are known as *Kostant's Convexity Theorems* and were proved by Bertram Kostant in 1973 (cf. [Kos73]). Historically, the origins of these results are not rooted in problems of Lie theory but rather a seemingly elementary question in linear algebra. Nowadays, the notion is more general and encompasses a wide range of similar and related theorems in various branches of mathematics, including Lie theory, symplectic geometry and the theory of buildings. In this introductory chapter, we briefly sketch some parts of the history of Kostant Convexity which is primarily intended to motivate our desire to study this concept in the Kac-Moody setting. The literature on Kostant Convexity is vast and we will only be able to scratch the surface. In our exposition, we freely use standard language from Lie theory and results from the structure theory of semisimple Lie groups & algebras, as developed in e.g. [Kna02, Chapters I, II, VI, VII].

The Schur-Horn Theorem

The concept of Kostant Convexity originates from a surprising convexity property of Hermitian matrices that is motivated by a very simple observation. Let $\mathcal{H}(n) \subset \mathbb{C}^{n \times n}$ denote the set of Hermitian $(n \times n)$ -matrices and $U(n) \subset GL_n(\mathbb{C})$ the unitary group. It is a basic result of linear algebra that every $A \in \mathcal{H}(n)$ is unitarily diagonalizable with real eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and, conversely, every Hermitian matrix with these eigenvalues is unitarily equivalent to the diagonal matrix $D_\lambda := \text{diag}(\lambda_1, \dots, \lambda_n) \in \mathcal{H}(n)$. Since conjugates of D_λ are no longer diagonal in general, it is natural to ask whether the set of diagonal vectors of the resulting matrices can be characterized in a simple way. More precisely, given any two vectors $\lambda = (\lambda_1, \dots, \lambda_n)$ and $d = (d_1, \dots, d_n)$ in \mathbb{R}^n , is it possible to find necessary and sufficient conditions for the existence of a Hermitian matrix $A = (a_{ij})_{i,j=1}^n \in \mathcal{H}(n)$ with eigenvalues $\lambda_1, \dots, \lambda_n$ and diagonal entries $a_{ii} = d_i$ for all $i \in \{1, \dots, n\}$? As A must be of the form $A = UD_\lambda U^{-1}$ for some $U = (u_{ij})_{i,j=1}^n \in U(n)$ and the trace of a matrix is invariant under conjugation, one necessary condition is the following:

$$\sum_{i=1}^n d_i = \sum_{i=1}^n \lambda_i \tag{SH1}$$

Clearly, this condition is not sufficient and further restrictions were found by Issai Schur in 1923. To formulate them, we first observe that $U(n)$ contains all permutation matrices and conjugation by them permutes the diagonal entries of any matrix. Hence, there is no loss of generality in assuming that $\lambda_1 \geq \dots \geq \lambda_n$ and $d_1 \geq \dots \geq d_n$. Moreover, the diagonal entries of $A = UD_\lambda U^{-1}$ are given by $a_{ii} = \sum_{j=1}^n |u_{ij}|^2 \lambda_j$ for $i \in \{1, \dots, n\}$, so the diagonal vector of A is the image of λ under multiplication by the matrix $P = (p_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ whose entries are defined by $p_{ij} := |u_{ij}|^2$. Since U is unitary, it follows that P is a *doubly-stochastic matrix*, i.e. the matrix entries of P are non-negative real numbers and their sum in every fixed row or column of P is equal to 1. The *Birkhoff-von Neumann Theorem* (cf. [Gru07, Theorem 5.7 or 14.4]) asserts that

the set of doubly-stochastic matrices is the convex hull in $\mathbb{R}^{n \times n}$ of the set of permutation matrices, so P can be written as a convex combination of permutation matrices. Combining this result with the assumption on the orderings of λ and d shows that the following inequalities found by Schur also have to be satisfied:

$$\sum_{i=1}^k d_i \leq \sum_{i=1}^k \lambda_i \quad \text{for all } k \in \{1, \dots, n-1\} \quad (\text{SH2})$$

Conversely, Alfred Horn proved in 1954 that (SH1) and (SH2) are in fact sufficient conditions for the existence of A (cf. [Hor54a, Theorem 5]).

Schur-Horn Theorem (Version 1). *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $d = (d_1, \dots, d_n)$ be two vectors in \mathbb{R}^n such that $\lambda_1 \geq \dots \geq \lambda_n$ and $d_1 \geq \dots \geq d_n$. There exists a Hermitian matrix $A \in \mathcal{H}(n)$ with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and diagonal vector $d \in \mathbb{R}^n$ if and only if (SH1) and (SH2) are satisfied.*

While this formulation may appear to be rather technical, the theorem attains a more pleasant form by removing the dependence on the orderings. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ now be arbitrary and denote by P_λ the *permutation polytope* of λ , i.e. the convex hull of all permutations of λ :

$$P_\lambda := \text{conv}(\{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) \mid \sigma \in S_n\}) \subset \mathbb{R}^n$$

This definition does not depend on the ordering of λ and it can be shown that another vector $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ is contained in P_λ if and only if, after putting λ and d into non-increasing order, the conditions (SH1) and (SH2) are satisfied (cf. [Hor54a, Theorem 1]). Thus, the previous result may be reformulated as follows:

Schur-Horn Theorem (Version 2). *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $d = (d_1, \dots, d_n)$ be any two vectors in \mathbb{R}^n . There exists a Hermitian matrix $A \in \mathcal{H}(n)$ with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ and diagonal vector $d \in \mathbb{R}^n$ if and only if d is contained in the permutation polytope of λ .*

At first sight, the Schur-Horn Theorem may appear to be a simple curiosity of linear algebra and, indeed, Horn's proof is completely elementary. However, it was soon noticed that a minor reformulation suggests a far-reaching generalization to the theory of Lie groups that will be discussed in the next section.

To conclude the current one, we remark that the Schur-Horn Theorem is just one instance of a variety of similar results that each characterize the set of matrices with a prescribed property by a family of inequalities. For example, the result below, which is originally due to Horn as well and was later generalized by Thompson, can to some extent be viewed as a “multiplicative analogue” of the Schur-Horn Theorem (cf. [Hor54b] and [Tho72, Theorem 1]).

Horn-Thompson Theorem. *Let $P \in \mathbb{C}^{n \times n}$ be a positive-definite matrix with real eigenvalues $\lambda_1 \geq \dots \geq \lambda_n > 0$. Given any vector $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ with $\mu_1 \geq \dots \geq \mu_n > 0$, there exists a unitary matrix $U \in \text{U}(n)$ such that PU has eigenvalues $\mu_1, \dots, \mu_n \in \mathbb{R}_{>0}$ if and only if*

$$\prod_{i=1}^k \mu_i \leq \prod_{i=1}^k \lambda_i \quad \text{for all } k \in \{1, \dots, n-1\} \quad \text{and} \quad \prod_{i=1}^n \mu_i = \prod_{i=1}^n \lambda_i. \quad (\text{HT})$$

Although this result answers a question about eigenvalues while the Schur-Horn Theorem deals with diagonal vectors, the two are undeniably similar in style. In fact, the inequalities in (HT) are equivalent to (SH1) and (SH2) under the coordinate-wise exponential map $\exp : \mathbb{R}^n \rightarrow (\mathbb{R}_{>0})^n$. Consequently, the Horn-Thompson Theorem also admits an equivalent formulation in terms of the exponential of the permutation polytope of the vector $(\log(\lambda_1), \dots, \log(\lambda_n)) \in \mathbb{R}^n$.

A Lie-theoretic reformulation

The aforementioned reinterpretation of the Schur-Horn Theorem in the language of Lie theory becomes apparent after a slight change of notation. The map $\mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$, $M \mapsto iM$, defines a bijection between the sets of Hermitian and skew-Hermitian matrices and transforms the real eigenvalues of $A \in \mathcal{H}(n)$ into the purely imaginary eigenvalues of $iA \in i\mathcal{H}(n)$. At the same time, multiplication by i on \mathbb{C}^n maps the permutation polytope $P_\lambda \subset \mathbb{R}^n$ to $P_{i\lambda} := iP_\lambda \subset i\mathbb{R}^n$. Thus, the Schur-Horn Theorem equivalently characterizes the set of diagonal vectors of skew-Hermitian matrices with prescribed imaginary eigenvalues. More formally, let $\pi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^{n \times n}$ denote the map that sets all off-diagonal entries of a matrix equal to zero and $\iota : \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$ the map which constructs from a vector in \mathbb{C}^n the corresponding diagonal matrix. Then ι defines an isomorphism between \mathbb{C}^n and the image of π and the well-defined composition $\iota^{-1} \circ \pi : \mathbb{C}^{n \times n} \rightarrow \mathbb{C}^n$ extracts the diagonal vector of a matrix $M \in \mathbb{C}^{n \times n}$, i.e. $(\iota^{-1} \circ \pi)(M) = (m_{11}, \dots, m_{nn})$. With this notation, the Schur-Horn Theorem is equivalent to the following statement, where $D_{i\lambda} := iD_\lambda \in i\mathcal{H}(n)$:

Schur-Horn Theorem (Version 3). *For every $\lambda \in \mathbb{R}^n$, it holds that*

$$\{\pi(UD_{i\lambda}U^{-1}) \mid U \in \mathrm{U}(n)\} = \iota(P_{i\lambda}). \quad (\text{SH3})$$

Both sides of (SH3) can now be viewed as objects of Lie theory. In fact, the set $i\mathcal{H}(n)$ of skew-Hermitian matrices is the Lie algebra $\mathfrak{u}(n)$ of the unitary group and the adjoint action of $\mathrm{U}(n)$ on $\mathfrak{u}(n) = i\mathcal{H}(n)$ is simply conjugation. Hence, the set $\{UD_{i\lambda}U^{-1} \mid U \in \mathrm{U}(n)\}$ can be viewed as the orbit of $D_{i\lambda} \in i\mathcal{H}(n) = \mathfrak{u}(n)$ under the adjoint action of $\mathrm{U}(n)$. Moreover, there is a $\mathrm{U}(n)$ -invariant inner product on $\mathfrak{u}(n)$ defined by:

$$\begin{aligned} \mathfrak{u}(n) \times \mathfrak{u}(n) &\rightarrow \mathbb{R} \\ (A, B) &\mapsto -\mathrm{tr}(AB) \end{aligned}$$

Let $\mathfrak{t} \subset \mathfrak{u}(n)$ denote the subspace of skew-Hermitian diagonal matrices. This is a maximal abelian subalgebra of $\mathfrak{u}(n)$ and every element of \mathfrak{t} is equal to $D_{i\lambda}$ for a unique $\lambda \in \mathbb{R}^n$. It is not difficult to verify that the restriction $\pi_{\mathfrak{t}} := \pi|_{\mathfrak{u}(n)} : \mathfrak{u}(n) \rightarrow \mathfrak{t}$ coincides with the orthogonal projection of $\mathfrak{u}(n)$ to \mathfrak{t} with respect to this inner product. Thus, we may view the set on the left-hand side of (SH3) as the image of the adjoint $\mathrm{U}(n)$ -orbit of $D_{i\lambda} \in \mathfrak{t}$ under the orthogonal projection of $\mathfrak{u}(n)$ onto a maximal abelian subalgebra.

To interpret the right-hand side, we consider the subgroup $T \subset \mathrm{U}(n)$ of unitary diagonal matrices. It is the unique connected Lie subgroup of $\mathrm{U}(n)$ with Lie algebra \mathfrak{t} and a maximal torus. The group $W := N(T)/T$, where $N(T)$ denotes the normalizer of T in $\mathrm{U}(n)$, is called the *Weyl group* of $\mathrm{U}(n)$. Since the adjoint action of $N(T) \subset \mathrm{U}(n)$ on $\mathfrak{u}(n)$ stabilizes \mathfrak{t} and $T \subset N(T)$ fixes it point-wise, it induces a well-defined action of W on \mathfrak{t} . Explicitly, the group $N(T)$ consists of the “monomial matrices” in $\mathrm{U}(n)$, i.e. those having exactly one non-zero entry in every row and column. Hence, the set of permutation matrices forms a system of representatives for $W = N(T)/T$, which is therefore isomorphic to the symmetric group S_n and under this isomorphism, the W -action on \mathfrak{t} is equivalent to the action of S_n on \mathfrak{t} by permuting the diagonal entries. Thus, the set $\iota(P_{i\lambda})$ coincides with the convex hull of the orbit of $\iota(i\lambda) = D_{i\lambda} \in \mathfrak{t}$ under the Weyl group of $\mathrm{U}(n)$. Altogether, we arrive at the fourth and final reformulation of the Schur-Horn Theorem.

Schur-Horn Theorem (Version 4). *Let $G := \mathrm{U}(n)$ be the unitary group with Lie algebra $\mathfrak{g} := \mathfrak{u}(n)$ and $T \subset G$ the maximal torus of unitary diagonal matrices with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. Denote by $W := N(T)/T \cong S_n$ the Weyl group of G and by $\pi_{\mathfrak{t}} : \mathfrak{g} \rightarrow \mathfrak{t}$ the orthogonal projection to \mathfrak{t} with respect to an $\mathrm{Ad}(G)$ -invariant inner product on \mathfrak{g} . For every $h \in \mathfrak{t}$, it holds that*

$$\{\pi_{\mathfrak{t}}(\mathrm{Ad}(g)h) \mid g \in G\} = \mathrm{conv}(W \cdot h). \quad (\text{SH4})$$

This variant of the Schur-Horn Theorem is now entirely Lie-theoretic which suggests trying to generalize it to a wider class of Lie groups. This can be achieved in two ways: On the one hand, using tools from symplectic geometry, it is possible to extend the result to connected compact

Lie groups. In this setting, it is a consequence of a more general convexity theorem by Atiyah, Guillemin & Sternberg about Hamiltonian torus actions on symplectic manifolds that we will briefly discuss in the last section of this chapter. On the other hand, B. Kostant managed in 1973 to generalize the theorem to all connected semisimple Lie groups. In his approach, it is closely related to a similar convexity property that holds “on the group level” and will be explained in the next section. It should be noted that $U(n)$ is not a semisimple Lie group since its center is one-dimensional. However, every matrix in $U(n)$ is the product of a scalar multiple of the identity, whose adjoint action on $\mathfrak{u}(n)$ is trivial, and an element in the special unitary group $SU(n)$, i.e. having determinant 1. Likewise, one can subtract a scalar multiple of the identity from $h \in \mathfrak{t}$ to make the matrix trace-free and thus contained in $\mathfrak{su}(n)$. As scalar multiples of the identity are fixed by the adjoint and W -action, (SH4) also holds with $U(n)$ replaced by the semisimple Lie group $SU(n)$ and $h \in \mathfrak{t} \cap \mathfrak{su}(n)$ and it is equivalent to this special case.

Kostant’s Convexity Theorems

A remarkable generalization of the Schur-Horn Theorem was established by B. Kostant in [Kos73]. In this paper, the author not only extends Version 4 of the Schur-Horn Theorem to the class of connected semisimple Lie groups, but also proves a much more elaborate convexity property of a certain decomposition of semisimple Lie groups. Both results are typically referred to as *Kostant’s Convexity Theorem*. Guided by [LR91], we will call the former the *linear* convexity theorem and the latter the *non-linear* one. Its origins are related to the following problem.

Let G be a connected semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of its Lie algebra \mathfrak{g} , where \mathfrak{k} is the fixed-point set of a Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ and \mathfrak{p} is the (-1) -eigenspace of θ . If $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace, the adjoint action of \mathfrak{a} on \mathfrak{g} is simultaneously diagonalizable and the eigenvalues of $\text{ad}(h) : \mathfrak{g} \rightarrow \mathfrak{g}$ depend linearly on $h \in \mathfrak{a}$. The non-zero linear functionals defined by the joint eigenvalues are the *roots* of the pair $(\mathfrak{g}, \mathfrak{a})$ and the set $\Phi \subset \mathfrak{a}^*$ of all roots is an *abstract root system* in the Euclidean vector space \mathfrak{a}^* , whose inner product is induced by the Killing form of \mathfrak{g} , with an associated *Weyl group* W , which is a finite group generated by elements of order two that acts on \mathfrak{a} and \mathfrak{a}^* by orthogonal linear maps. Let $\Phi_+ \subset \Phi$ be a set of positive roots and \mathfrak{u}_+ the subalgebra of \mathfrak{g} spanned by all positive root spaces, then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ as vector spaces which is known as the *Iwasawa decomposition* of \mathfrak{g} . A fundamental result in the theory of Lie groups asserts that the Cartan involution θ lifts to an involutive automorphism $\Theta : G \rightarrow G$ with differential θ . Its fixed-point set K is a maximal compact subgroup of G (if the center of G is finite) with Lie algebra \mathfrak{k} and contains all topological information about G in the sense that the quotient space G/K is contractible. Furthermore, G/K even has a natural structure of a *Riemannian symmetric space of non-compact type* which is an invaluable geometric tool to study the algebraic structure of the group G . The exponential map $\exp : \mathfrak{g} \rightarrow G$ is injective on \mathfrak{a} and \mathfrak{u}_+ and their exponential images A and U_+ are contractible subgroups of G . Here, A is an abelian group isomorphic to \mathfrak{a} through the bijective map $\log := (\exp|_{\mathfrak{a}})^{-1} : A \rightarrow \mathfrak{a}$, so the W -action on \mathfrak{a} lifts to A by intertwining it with the exponential map, and U_+ is nilpotent. Moreover, A normalizes U_+ and the multiplication map $K \times A \times U_+ \rightarrow G$ is a diffeomorphism which is called the *Iwasawa decomposition* of G . Given an arbitrary $g \in G$, the unique elements $k \in K$, $a \in A$ and $u \in U_+$ such that $g = kau$ can be viewed as “coordinates” of g .

However, the Iwasawa decomposition is not a homomorphism of groups which raises the question whether the multiplication map in G can be described in Iwasawa coordinates. More precisely, given two elements $g_1 = k_1 a_1 u_1$ and $g_2 = k_2 a_2 u_2$ of G in Iwasawa form, is it possible to compute the Iwasawa coordinates of the product $g_1 g_2 \in G$ from the ones of g_1 and g_2 ? The major difficulty in this problem is the fact that K does not commute with or normalize A and U_+ . A simpler question is to focus on one of the components and restrict g_1 and g_2 to certain subgroups of G . According to [Kos73], various problems in representation theory and analysis on semisimple Lie groups require control of the A -component and we denote by $\Pi_A : G \rightarrow A$, $g = kau \mapsto a$, the corresponding projection. Because A normalizes U_+ , the most difficult aspect of computing the A -component of a product in G is to determine how this coordinate changes under multiplication

by K . It is clear that $\Pi_A(kg) = \Pi_A(g)$ holds for all $g \in G$ and $k \in K$, but there is no obvious way how to compute $\Pi_A(gk)$. Using that a semisimple Lie group can be decomposed as $G = KAK$,¹ this problem can be reduced to the case $g = a \in A$. In this situation, Kostant established that if $a \in A$ and $k \in K$ are arbitrary, then $\Pi_A(ak)$ lies in the “multiplicative convex hull” of the Weyl group orbit of a and as k runs through all of K , every point of this set is attained in this way. More precisely, [Kos73, Theorem 4.1] asserts the following:

Kostant’s Non-Linear Convexity Theorem (semisimple version). *Let G be a connected semisimple Lie group with Iwasawa decomposition $G = KAU_+$ and W the associated Weyl group. For every $a \in A$, it holds that*

$$\{\Pi_A(ak) \mid k \in K\} = \exp(\text{conv}(W \cdot \log(a))). \quad (\text{K1})$$

In the KAK -decomposition of an arbitrary element $g = k_1 a k_2 \in G$, the A -component is uniquely determined up to a W -translate, so (K1) immediately implies that

$$\{\Pi_A(gk) \mid k \in K\} = \{\Pi_A(ak) \mid k \in K\} = \exp(\text{conv}(W \cdot \log(a))). \quad (\text{K2})$$

Kostant’s result in [Kos73, Corollary 4.2] is even a bit more general. As an application of his convexity theorem and the KAK -decomposition, he further shows that every connected semisimple Lie group has a *horospherical decomposition* $G = KU_+K$ (cf. [Kos73, Theorem 5.1]).

While the non-linear convexity theorem is certainly the main result of [Kos73], the author also proves a similar statement that can be viewed as a vast generalization of the Schur-Horn Theorem. To this end, consider the adjoint action $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ of G on its Lie algebra and denote by $\pi_{\mathfrak{a}} : \mathfrak{g} \rightarrow \mathfrak{a}$ the linear projection induced by the Iwasawa decomposition on the Lie algebra level, then [Kos73, Theorem 8.2] states the following:

Kostant’s Linear Convexity Theorem (semisimple version). *Let G be a connected semisimple Lie group with Lie algebra \mathfrak{g} , let $G = KAU_+$ (resp. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$) be an Iwasawa decomposition of G (resp. \mathfrak{g}) and let W denote the associated Weyl group. For every $h \in \mathfrak{a}$, it holds that*

$$\{\pi_{\mathfrak{a}}(\text{Ad}(k)h) \mid k \in K\} = \text{conv}(W \cdot h). \quad (\text{K3})$$

Although the non-linear convexity theorem may appear to be nothing but an “exponentiated version” of the linear one, the two results are not related in a straightforward way and neither is a direct consequence of the other. Instead, they bear a similarity that is reminiscent of the analogy between the Schur-Horn and the Horn-Thompson Theorem.

Remark. (i) In Kostant’s Linear Convexity Theorem, the group G does not play a major role since it mainly serves as an ambient group to define K and a more intrinsic formulation would be possible. However, if G is compact, the result is trivial as $\mathfrak{a} = \{0\}$ and $W = \{1\}$ in this case. In particular, it does not lead to the Schur-Horn Theorem when applied to the special unitary group $\text{SU}(n)$. Alternatively, if G is connected, compact and semisimple, one can replace it by its *universal complexification* $G^{\mathbb{C}}$ (cf. [HN12, Chapter 15]), which is a connected, non-compact, semisimple Lie group whose Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is the complexification of \mathfrak{g} . The direct sum $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k} := \mathfrak{g}$ and $\mathfrak{p} := i\mathfrak{g}$ is a Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ and if $\mathfrak{t} \subset \mathfrak{g} = \mathfrak{k}$ is a maximal abelian subalgebra, then $\mathfrak{a} := i\mathfrak{t} \subset i\mathfrak{g} = \mathfrak{p}$ is a maximal abelian subspace which can be used to construct Iwasawa decompositions $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ and $G^{\mathbb{C}} = GAU_+$ as above. Moreover, the Weyl group of G in the sense of compact groups coincides with that of $G^{\mathbb{C}}$ in the sense of semisimple groups, the W -actions on \mathfrak{a} and \mathfrak{t} are the same and the projection $\pi_{\mathfrak{a}}|_{\mathfrak{p}} : \mathfrak{p} \rightarrow \mathfrak{a}$ can be identified with the orthogonal projection $\mathfrak{g} \rightarrow \mathfrak{t}$ with respect to the Killing form. For example, the universal complexification of $G = \text{SU}(n)$ is $G^{\mathbb{C}} = \text{SL}_n(\mathbb{C})$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition of $\mathfrak{sl}_n(\mathbb{C})$ into the subspaces

¹The KAK -decomposition is occasionally also called “Cartan decomposition” in the literature. We avoid this terminology on purpose in order not to confuse it with the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra of G which also has an analogue on the group level (cf. [Kna02, Theorem 6.31]).

of trace-free skew-Hermitian and Hermitian matrices. If $\mathfrak{t} \subset \mathfrak{g}$ is taken to be the maximal abelian subalgebra of trace-free, purely imaginary, diagonal matrices, then A is the group of diagonal matrices with positive real entries and determinant 1 and W is isomorphic to the symmetric group S_n . Thus, Kostant's Linear Convexity Theorem for the group $\mathrm{SL}_n(\mathbb{C})$ reduces to the Schur-Horn Theorem. A slightly different variant of this result which applies more easily to compact groups and directly yields the Schur-Horn Theorem in the case of the unitary group will be discussed in the next section.

- (ii) The main reason why Kostant's two convexity theorems are not related in a straightforward way is that the exponential map $\exp : \mathfrak{g} \rightarrow G$ does not intertwine the Iwasawa projections in the sense that $\Pi_A \circ \exp \neq \exp \circ \pi_{\mathfrak{a}}$. This can already be observed in the simplest example of $G = \mathrm{SL}_2(\mathbb{R})$ with its standard Iwasawa decomposition (cf. also [LR91, Example 2.3]). However, Duistermaat established that for every $X \in \mathfrak{p}$, there exists an element $\psi(X) \in K$, which depends analytically on X , such that

$$(\Pi_A \circ \exp)(\mathrm{Ad}(\psi(X)^{-1})X) = (\exp \circ \pi_{\mathfrak{a}})(X)$$

and the map $k \mapsto k\psi(\mathrm{Ad}(k^{-1})X)$ defines a diffeomorphism of K (cf. [Dui84, Theorem 1.1]).

To conclude this section, it is worth emphasizing that the decomposition $G = KAK$, which was known before the publication of [Kos73], and the horospherical decomposition $G = KU_+K$, which follows from Kostant's Non-Linear Convexity Theorem, have interesting reformulations in the language of differential geometry. As mentioned above, there exists a natural Riemannian metric on the quotient space G/K that turns it into a Riemannian symmetric space of non-compact type on which G acts by isometries. This allows for a geometric interpretation of the groups K , A and U_+ from the Iwasawa decomposition. By definition, the group K is the stabilizer of the “base point” $o := 1 \cdot K \in G/K$. The orbit $F_0 := A \cdot o$ is a flat, totally geodesic submanifold of G/K that is isometric to $\mathbb{R}^{\dim(\mathfrak{a})}$ via the exponential map. Such a submanifold is called a *flat* in G/K and it can be shown that every flat $F \subset G/K$ is contained in $g(F_0)$ for some $g \in G$. Hence, the maximal flats of G/K are all isometric and simply given by the G -translates of F_0 which are sometimes called the *standard flats*. Moreover, every maximal flat containing the base point o is of the form $k(F_0)$ for some $k \in K$, so K acts transitively on maximal flats through o . From this discussion, it follows that the decomposition $G = KAK$ is equivalent to the assertion that every point of G/K lies in a maximal flat containing o . Using that the G -action on G/K is isometric and transitive, it also implies that any two points of G/K are contained in a common maximal flat.

Likewise, a G -translate of the orbit $U_+ \cdot o$ is sometimes called a *horosphere* in G/K . The motivation for this terminology comes from the fact that hyperbolic n -space can be identified with the symmetric space G/K , where G is the identity component of the semisimple Lie group $\mathrm{SO}(n, 1)$ and the maximal compact subgroup $K \subset G$ is $\mathrm{SO}(n)$. For this space, the U_+ -orbits coincide with the usual horospheres in the sense of hyperbolic geometry. By the same reasoning as before, the horospherical decomposition $G = KU_+K$ is equivalent to the statement that every point of G/K is contained in a horosphere through the base point o or, equivalently, that any two points of G/K lie on a common horosphere.

Kostant's Non-Linear Convexity Theorem now admits a geometric formulation in the framework of symmetric spaces. If we write the Iwasawa decomposition equivalently as $G = U_+AK$ and denote by $\Pi_A : G \rightarrow A$ the projection onto the A -component,² then Π_A factors to a well-defined map $\bar{\Pi}_A : G/K \rightarrow F_0$ which can be viewed as the “projection” of a point in G/K onto the standard flat F_0 . Elements of A are in bijective correspondence with points in F_0 , so the Weyl group acts on F_0 . Straight lines in \mathfrak{a} correspond to geodesics in $F_0 = \exp(\mathfrak{a}) \cdot o$ and multiplying an element of A on the left by all of K is equivalent to taking the K -orbit of the corresponding point in F_0 . Thus, we arrive at the following geometric statement:

Kostant's Non-Linear Convexity Theorem (geometric version). *For every point $p \in F_0$, the projection $\bar{\Pi}_A(K \cdot p)$ of the K -orbit of p onto the standard flat F_0 is the geodesic convex hull of the Weyl group orbit of p .*

²For the usual Iwasawa decomposition $G = KAU_+$, analogous statements are true for the right quotient $K \backslash G$ which can be turned into a Riemannian symmetric space of non-compact type with the same properties as G/K .

Related results

Since the publication of [Kos73], Kostant's Convexity Theorems have both been reformulated and generalized in many ways. To conclude our historical introduction, we describe two interesting variations, which are of a completely different nature, and briefly mention some other results.

On the one hand, a close analogue of Kostant's Linear Convexity Theorem, which applies more easily to compact groups, can be obtained by methods of symplectic geometry. In this setting, it is derived from the following far more general result that was established independently by Atiyah and Guillemin-Sternberg (cf. [Ati82], [GS82] and the generalizations [GS84] and [Kir84]). In its formulation, a *compact torus* is defined to be a connected, compact, abelian Lie group. The necessary terminology from symplectic geometry is explained in the cited references.

Atiyah-Guillemin-Sternberg Convexity Theorem. *If (M, ω) denotes a connected, compact, symplectic manifold and T a compact torus with a Hamiltonian action on M with moment map $\mu : M \rightarrow \mathfrak{t}^*$, then $\mu(M) \subset \mathfrak{t}^*$ is a convex polytope. More precisely, let M^T be the set of global fixed-points of the T -action on M , then $\mu(M^T) \subset \mathfrak{t}^*$ is finite and $\mu(M) = \text{conv}(\mu(M^T))$.*

We now explain how this result can be applied to Lie theory. Let G be an arbitrary Lie group with Lie algebra \mathfrak{g} . The dual of the adjoint representation $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ defines a G -action on \mathfrak{g}^* that is called the *coadjoint representation* and denoted by $\text{Ad}^* : G \rightarrow \text{GL}(\mathfrak{g}^*)$. For every $\xi \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\xi := \text{Ad}^*(G)\xi \subset \mathfrak{g}^*$ can be endowed with a canonical closed, non-degenerate, differential 2-form $\omega_{\text{KKS}} \in \Omega^2(\mathcal{O}_\xi)$, called the *Kirillov-Kostant-Souriau form*, which turns \mathcal{O}_ξ into a symplectic manifold. The G -action on $(\mathcal{O}_\xi, \omega_{\text{KKS}})$ is Hamiltonian with moment map $\mathcal{O}_\xi \rightarrow \mathfrak{g}^*$ being the inclusion of the orbit.

If G is connected and compact, so is \mathcal{O}_ξ . In this case, given a maximal torus $T \subset G$ and an element $\xi \in \mathfrak{t}^* \subset \mathfrak{g}^*$, the restriction of the G -action to T turns $(\mathcal{O}_\xi, \omega_{\text{KKS}})$ into a Hamiltonian T -space whose moment map $\mathcal{O}_\xi \rightarrow \mathfrak{t}^*$ is the composition of the inclusion $\mathcal{O}_\xi \hookrightarrow \mathfrak{g}^*$ with the natural projection $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$ induced by restriction. As the normalizer of T in G stabilizes $\mathfrak{t}^* \subset \mathfrak{g}^*$ under the coadjoint action and T fixes it point-wise, the Weyl group $W := N(T)/T$ acts on \mathfrak{t}^* and the global fixed-points for the T -action on \mathcal{O}_ξ are precisely the W -translates of ξ . Moreover, by compactness of G , there exists an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} which provides an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ that is W -equivariant, intertwines the adjoint and coadjoint actions and transforms the restriction $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$ to the orthogonal projection $\pi_{\mathfrak{t}} : \mathfrak{g} \rightarrow \mathfrak{t}$ with respect to this inner product. Therefore, the Atiyah-Guillemin-Sternberg Convexity Theorem implies the following:

Kostant's Linear Convexity Theorem (compact version). *Let G be a connected compact Lie group, $T \subset G$ a maximal torus, $W := N(T)/T$ the Weyl group of G and $\pi_{\mathfrak{t}} : \mathfrak{g} \rightarrow \mathfrak{t}$ the orthogonal projection with respect to an $\text{Ad}(G)$ -invariant inner product on \mathfrak{g} . For every $h \in \mathfrak{t}$, it holds that*

$$\{\pi_{\mathfrak{t}}(\text{Ad}(g)h) \mid g \in G\} = \text{conv}(W \cdot h). \quad (\text{K4})$$

If $G = \text{U}(n)$, its Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$ is the space of skew-Hermitian $(n \times n)$ -matrices. Its dual space can be identified with the set $\mathcal{H}(n)$ of Hermitian matrices via the following isomorphism:

$$\begin{aligned} \phi : \mathcal{H}(n) &\rightarrow \mathfrak{g}^* \\ A &\mapsto \phi(A) := [B \mapsto \phi(A)(B) := \text{Im}(\text{tr}(AB))] \end{aligned}$$

Under this isomorphism, the coadjoint action of G on \mathfrak{g}^* is equivalent to its action on $\mathcal{H}(n)$ by conjugation and if $T \subset G$ is the maximal torus of unitary diagonal matrices, the projection $\mathfrak{g}^* \rightarrow \mathfrak{t}^*$ corresponds to the map that sets all off-diagonal entries of a Hermitian matrix equal to zero. Let now $A \in \mathcal{H}(n)$ be arbitrary and $\xi := \phi(A) \in \mathfrak{g}^*$. The set of all Hermitian matrices with the same eigenvalues as A can be identified with the coadjoint orbit \mathcal{O}_ξ and the diagonal vectors of these matrices with the image of \mathcal{O}_ξ in $\mathfrak{t}^* \cong \mathbb{R}^n$ under the moment map. In addition, it is not difficult to compute that the global fixed-points for the T -action on this orbit are precisely the diagonal matrices. Thus, the set of diagonal vectors of Hermitian matrices with the same eigenvalues as A

is a convex polytope in \mathbb{R}^n whose vertices are the vectors obtained from all possible permutations of the eigenvalues of A , as claimed by the Schur-Horn Theorem. For other choices of G , the same argument yields natural analogues of Kostant's Linear Convexity Theorem. Several explicit statements for the classical compact groups were worked out by Tam in [Tam97]. A reformulation of the non-linear convexity theorem in the language of symplectic geometry was developed by Lu and Ratiu in [LR91].

On the other hand, Kostant's Non-Linear Convexity Theorem has been generalized to the setting of affine buildings by Schwer³ in [Hit10]. Parallel to the way in which a real semisimple Lie group is naturally associated with a Riemannian symmetric space of non-compact type, affine buildings can be viewed as “discrete analogues” of symmetric spaces and are the geometric objects naturally linked to semisimple algebraic groups over non-Archimedean local fields, for example p -adic Lie groups such as $\mathrm{SL}_n(\mathbb{Q}_p)$. The next result (cf. [Hit10, Theorem 3.1]) is similar to the geometric formulation of Kostant's Non-Linear Convexity Theorem in terms of symmetric spaces. We point to the reference for the relevant terminology. It should be remarked that the notion of “convexity” is different from the standard definition in the theory of buildings (cf. [Hit10, Definition 2.12]).

Schwer's Convexity Theorem for affine buildings. *Let \mathcal{X} be a thick affine building of type (W, S) and $\partial\mathcal{X}$ its spherical building at infinity of type (\check{W}, \check{S}) . Let \mathcal{A} be an apartment of \mathcal{X} and $\partial\mathcal{A}$ the corresponding apartment of $\partial\mathcal{X}$. Choose a fundamental alcove $\mathfrak{c} \in \mathcal{A}$, denote by \mathcal{C} the corresponding fundamental Weyl chamber in \mathcal{A} and by $\partial\mathcal{C}$ the associated chamber of $\partial\mathcal{A}$. Let $r : \mathcal{X} \rightarrow \mathcal{A}$ be the canonical retraction centered at \mathfrak{c} and $\rho : \mathcal{X} \rightarrow \mathcal{A}$ the retraction centered at the chamber $\partial\mathcal{C}^{\mathrm{op}} \in \partial\mathcal{A}$ opposite to $\partial\mathcal{C}$. If $x \in \mathcal{A}$ is a special vertex, then $\rho(r^{-1}(\check{W} \cdot x))$ is the set of all points in $\mathrm{conv}(\check{W} \cdot x)$ having the same type as x .*

Compared to the geometric version of Kostant's Non-Linear Convexity Theorem, in the discrete setting of a thick affine building \mathcal{X} , maximal flats are replaced by apartments, the projection onto a maximal flat is substituted by the retraction onto a chosen apartment $\mathcal{A} \subset \mathcal{X}$ centered at a chamber in the spherical building at infinity and the role of the K -orbit is played by the pre-image of a Weyl group orbit of a special vertex $x \in \mathcal{A}$ under the retraction centered at an alcove of \mathcal{A} . It should be observed that the sets in the assertion of the theorem only involve the underlying finite Weyl group \check{W} of the affine Weyl group W . Interestingly, Schwer's result is entirely geometric and does not require any group action on the building to begin with. However, if G is a group acting on \mathcal{X} by type-preserving automorphisms and K is the stabilizer of a chosen “origin” $o \in \mathcal{A}$, then under suitable assumptions on G (cf. [Hit10, Section 4]), the set $r^{-1}(\check{W} \cdot x)$ coincides with the K -orbit of x . In this case, it resembles Kostant's original theorem more closely. Moreover, if G is a semisimple algebraic group over a non-Archimedean local field, then K is also a maximal compact subgroup of G with respect to its natural non-Archimedean topology, for example $\mathrm{SL}_n(\mathbb{Z}_p) \subset \mathrm{SL}_n(\mathbb{Q}_p)$. Schwer even extended her result to *affine Λ -buildings* in [Hit11].

Some other noteworthy generalizations and variations of Kostant's Convexity Theorems that we will not explain here any further are the following:

- Based on the interpretation of Kostant's Non-Linear Convexity Theorem as a statement about Riemannian symmetric spaces, it has been generalized by van den Ban and Neeb to the more general framework of *semisimple symmetric spaces* (cf. [vdB86] and [Nee94]). The two results have similar formulations, but the authors consider distinct forms of the Iwasawa decomposition and their proofs apply different methods. More recently, van den Ban's result was further improved by Bălibanu and himself in [BvdB16].
- The connection between Kostant's two original convexity theorems was studied by Heckman in his PhD-thesis. He succeeded in deriving the non-linear theorem from the linear one by a homotopy argument. His findings precede the work of Duistermaat that was mentioned in the previous section (cf. [Hec80, Chapter 1] and [Dui84]).
- An analogue of Kostant's Linear Convexity Theorem in the Kac-Moody setting has been established by Kac & Peterson in [KP84]. It will be discussed in more detail in Section 5.5. For the non-linear one, no such generalization appears to be known in this context.

³née Hitzelberger

Part I

Preliminaries

Chapter 1

Coxeter Groups

The notion of a *Coxeter group* provides a natural way of generalizing the well-known concept of a *finite reflection group* to an infinite setting. While finite reflection groups are, among other things, famous for their deep connection to Lie groups & algebras, the general theory of Coxeter groups is an essential tool to study the structure of Kac-Moody groups & algebras. This chapter summarizes their most important properties and fixes a lot of notation and terminology that will be used in later chapters. Unless stated otherwise, all assertions in Section 1.1 are basic facts that are proved in standard textbooks on Coxeter groups such as [AB08, Chapters 1-3] or [BB05, Chapters 1-4]. The former also contains a thorough treatment of finite reflection groups and motivates the general definition of a Coxeter group. In Section 1.2, we introduce *coloured Coxeter diagrams* and some useful related notions. This material is elementary, but has apparently not been studied in the literature in the same level of detail, so we give self-contained proofs of all results there.

1.1 Basic notions

The classical theory of finite reflection groups naturally suggests to define the following notion of a “generalized” reflection group:

Definition 1.1.1. A group W is called a *Coxeter group* if there exists a finite generating set $S = \{s_1, \dots, s_n\} \subset W$ such that W admits a presentation of the form

$$W \cong \langle S \mid (s_i s_j)^{m_{ij}} \text{ for all } i, j \in I \rangle, \quad (1.1.1)$$

where $m_{ii} = 1$ for all $i \in I$ and $m_{ij} = m_{ji} \in \{2, 3, \dots\} \cup \{\infty\}$ whenever $i \neq j \in I$. The pair (W, S) is called a *Coxeter system*, the symmetric matrix $M = (m_{ij})_{i, j \in I}$ is the *Coxeter matrix* of (W, S) and $n := |S| \in \mathbb{N}_0$ is the *rank* of (W, S) .

Here, we employ the convention that if $m_{ij} = m_{ji} = \infty$, there is no relation between s_i and s_j . Further, we allow the case $n = 0$ in which $W = \{1\}$ is the trivial group. By definition, the set S consists of involutions, i.e. elements of order two. It is called the set of *simple reflections* and their conjugates $R := \{ws_i w^{-1} \mid w \in W, i \in I\}$ form the set of all *reflections* of (W, S) which also have order two. More generally, given any $i, j \in I$, it can be shown that m_{ij} is equal to the order of $s_i s_j$. In particular, $m_{ij} = 2$ if and only if $i \neq j$ and the simple reflections s_i and s_j commute. If $m_{ij} = \infty$, the subgroup generated by s_i and s_j is isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$.

The *Coxeter diagram* of (W, S) is the graph $\Delta(W, S)$ whose vertices correspond to the set I and two distinct vertices $i \neq j \in I$ are connected by an edge if and only if $m_{ij} \geq 3$. In addition, if $m_{ij} > 3$, the edge is labelled by m_{ij} . This is well-defined as M is symmetric. Hence, two vertices $i \neq j \in I$ are not joined by an edge if and only if the simple reflections s_i and s_j commute. We say that (W, S) is *irreducible* if $\Delta(W, S)$ is connected and *reducible* otherwise. In the latter case, let $I_1, \dots, I_m \subset I$ denote the vertex sets of the connected components $\Delta_1, \dots, \Delta_m$ of $\Delta(W, S)$,

then $I = I_1 \sqcup \dots \sqcup I_m$ is the disjoint union of at least two non-empty, proper subsets such that $m_{ij} = 2$ whenever $i, j \in I$ are not both contained in I_k for some $k \in \{1, \dots, m\}$. Hence, the Coxeter group W splits as a direct product $W \cong W_1 \times \dots \times W_m$, where for each $k \in \{1, \dots, m\}$, W_k is the subgroup of W generated by $S_k := \{s_i \mid i \in I_k\}$. The pairs $(W_1, S_1), \dots, (W_m, S_m)$ are irreducible Coxeter systems which are called the *irreducible factors* of (W, S) . If the Coxeter system is clear from the context, we will often write $\Delta := \Delta(W, S)$ and simply call I_1, \dots, I_m the connected components of I and W_1, \dots, W_m the irreducible factors of W .

Example 1.1.2. Some simple examples of Coxeter groups are the following:

- (i) Given an arbitrary $n \in \mathbb{N}_0$, let W be the symmetric group S_{n+1} . It is generated by the set $S = \{s_1, \dots, s_n\}$ of adjacent transpositions which all have order two. If $|i - j| > 1$, then s_i and s_j commute and if $|i - j| = 1$, then $s_i s_j$ has order three. In fact, it can be shown that (W, S) is a Coxeter system of rank n whose Coxeter diagram is $\bullet \text{---} \bullet \text{---} \dots \text{---} \bullet \text{---} \bullet$. The symmetric group acts on \mathbb{R}^{n+1} by permuting the coordinates and for every $i \in I$, the action of s_i coincides with the orthogonal reflection at the hyperplane $\{x \in \mathbb{R}^{n+1} : x_i = x_{i+1}\}$. Hence, it can also be viewed as a finite reflection group.
- (ii) For every integer $m \geq 3$, the finite dihedral group D_{2m} is the symmetry group of a regular m -gon in the Euclidean plane, which is generated by the orthogonal reflections at two lines in the plane intersecting at an angle of $\frac{\pi}{m}$. It is a Coxeter group of rank two with respect to this generating system whose Coxeter diagram is $\bullet \text{---}^m \text{---} \bullet$.
- (iii) The infinite dihedral group D_∞ is the Coxeter group with Coxeter diagram $\bullet \text{---}^\infty \text{---} \bullet$. It is the only infinite Coxeter group of rank two and isomorphic to the free product $\mathbb{Z}_2 * \mathbb{Z}_2$. It can also be identified with the group acting on the real line that is generated by the reflections at 0 and at $\frac{1}{2}$, i.e. the maps $x \mapsto -x$ and $x \mapsto 1 - x$ for $x \in \mathbb{R}$. Thus, it is the simplest example of an “infinite reflection group”.

In a Coxeter system (W, S) , we denote the length function of W with respect to the generating set S by $l : W \rightarrow \mathbb{N}_0$. Trivially, it satisfies $l(w) = 0$ if and only if $w = 1$ and $l(w) = 1$ if and only if $w = s_i$ for some $i \in I$. Given any $w \in W$, every expression of w as a product of simple reflections

$$w = s_{i_1} \dots s_{i_k} \quad \text{with} \quad i_1, \dots, i_k \in I \quad (1.1.2)$$

is called a *decomposition* of w and it is said to be *reduced* if $k = l(w)$. Due to the existence of relations between the simple reflections, w typically has several distinct reduced decompositions. However, Jacques Tits proved that the “word problem” in Coxeter groups has a simple solution. If $w = s_{i_1} \dots s_{i_k}$ is any decomposition, then every reduced decomposition can be obtained from the given one by successively applying one of the following two operations:

- (I) Remove a subword of the form s_i^2 for some $i \in I$.
- (II) For $i \neq j \in I$ with $m_{ij} < \infty$, replace an alternating subword of the form $s_i s_j s_i s_j \dots$ with m_{ij} factors by the word $s_j s_i s_j s_i \dots$ with m_{ij} factors.

In particular, any two reduced decompositions of w only differ by moves of type (II), which implies that there exists a unique set $S(w) \subset S$ such that every reduced decomposition of w involves precisely the simple reflections in $S(w)$. We call $S(w)$ the *simple factors* of w and its index set $I(w) := \{i \in I : s_i \in S(w)\}$ the *support* of w . For $i \in I$, we either have $l(ws_i) = l(w) + 1$ or $l(ws_i) = l(w) - 1$ and the same is true for multiplication from the left. The former necessarily holds if $i \notin I(w)$, but the converse is not true.

Given any subset $J \subset I$, we denote by W_J the subgroup of W generated by $S_J := \{s_j \mid j \in J\}$. The pair (W_J, S_J) is again a Coxeter system of rank $|J|$ whose Coxeter matrix (resp. diagram) is the submatrix M_J of M (resp. the full subdiagram Δ_J of Δ) whose rows and columns (resp. vertices) are indexed by J . It is called the *standard subgroup* of type J of (W, S) . Trivially, we have $W_\emptyset = \{1\}$ and $W_I = W$. More generally, for $w \in W$ and $J, K \subset I$, it holds that $w \in W_J$ if and only if $I(w) \subset J$ and hence $W_J \cap W_K = W_{J \cap K}$. Every left (resp. right) coset of the form wW_J (resp. $W_J w$) for some $w \in W$ and $J \subset I$ is referred to as a *standard coset* and the conjugate $wW_J w^{-1}$ as a *parabolic subgroup* of type J of (W, S) .

For every $w \in W$ and $J \subset I$, the standard left (resp. right) coset wW_J (resp. W_Jw) contains a unique element of minimal length that will be denoted by $\pi_J^r(w)$ (resp. $\pi_J^l(w)$). If $w = \pi_J^r(w)$ (resp. $w = \pi_J^l(w)$), we say that w is *right* (resp. *left*) J -reduced. Note that w is right J -reduced if and only if w^{-1} is left J -reduced. Equivalently, this property can be characterized as follows:

$$\begin{aligned} w = \pi_J^r(w) &\iff l(w) < l(ws_j) \text{ for all } j \in J \\ w = \pi_J^l(w) &\iff l(w) < l(s_jw) \text{ for all } j \in J \end{aligned} \quad (1.1.3)$$

More generally, if $w \in W$ is right (resp. left) J -reduced, then $l(ww') = l(w) + l(w')$ (resp. $l(w'w) = l(w') + l(w)$) for all $w' \in W_J$. Furthermore, given any $i \in I$, the *folding condition* asserts that if w is right J -reduced, then exactly one of the following three conditions holds for s_iw :

- a) $l(s_iw) < l(w)$ and s_iw is right J -reduced.
- b) $l(s_iw) > l(w)$ and s_iw is right J -reduced.
- c) $l(s_iw) > l(w)$ and s_iw is not right J -reduced. In this case, there exists some $j \in J$ such that $s_iw = ws_j$.

The obvious analogues of these statements hold for left J -reduced elements. Finally, given any $w \in W$ and any two subsets $J, K \subset I$, the double coset W_JwW_K also contains a unique element of minimal length that we denote by $\pi_{J,K}(w)$. If $w = \pi_{J,K}(w)$, we say that w is (J, K) -reduced which is the case if and only if w is both left J -reduced and right K -reduced.

We define a partial order on the set of standard left cosets by “reverse inclusion”:

$$w_1W_J \leq w_2W_K \iff w_2W_K \subset w_1W_J \quad \text{for all } w_1, w_2 \in W \text{ and } J, K \subset I \quad (1.1.4)$$

Definition 1.1.3. The poset of all standard left cosets of (W, S) ordered by reverse inclusion is called the *Coxeter complex* of (W, S) and denoted by $\Sigma(W, S)$. Likewise, we denote by $\Sigma^*(W, S)$ the poset of all standard left cosets of (W, S) ordered by (actual) inclusion and call it the *reversed Coxeter complex*.

The Coxeter complex is (the underlying poset of) an $(n-1)$ -dimensional simplicial complex whose barycentric subdivision is isomorphic to the order complex of $\Sigma(W, S)$. Under this identification, the dimension of a simplex wW_J is equal to $n-1-|J|$. The $(n-1)$ -dimensional simplices, called *chambers*, are singleton sets $\{w\} = wW_\emptyset$ with $w \in W$ and the $(n-2)$ -dimensional simplices, called *panels*, are two-element sets of the form $\{w, ws_i\} = wW_{\{i\}}$ with $w \in W$ and $i \in I$. For $w_1 \neq w_2 \in W$ and $i \in I$, the chambers $\{w_1\}$ and $\{w_2\}$ are said to be s_i -adjacent if $w_2 = w_1s_i$ or, equivalently, they both contain the panel $w_1W_{\{i\}} = w_2W_{\{i\}}$. Two chambers are called *adjacent* if they are s_i -adjacent for some $i \in I$. There is a unique minimal simplex, namely the entire group $W = W_I$, which has dimension -1 and can be viewed as the “empty simplex”. By definition, every simplex of $\Sigma(W, S)$ has a well-defined type and the natural left-action of W induces an isomorphism between W and the group of type-preserving simplicial automorphisms of $\Sigma(W, S)$.

Remark 1.1.4. On the one hand, the simplicial Coxeter complex and its geometric realization are very important tools to investigate a Coxeter group since they provide a way of transitioning between combinatorial and geometric arguments. The reversed Coxeter complex, on the other hand, is not simplicial which is why it appears much less often in the literature, even though its definition might appear more natural at first glance. However, we will see in Chapter 4 that it arises in Kac-Moody theory from the study of convex hulls of Weyl group orbits and is therefore much more relevant to our main results than the “usual” Coxeter complex.

A special role in the theory of Coxeter groups is played by the finite ones which are precisely the classical finite reflection groups. If (W, S) is a Coxeter system with $|W| < \infty$, there exists a unique element $w_+ \in W$ of maximal length.¹ It is called the *longest word* of W and satisfies

$$l(w_+w) = l(w_+) - l(w) = l(ww_+) \quad \text{for all } w \in W. \quad (1.1.5)$$

¹In the literature on Coxeter groups, it is customary to denote this element by w_0 rather than w_+ . We do not use that notation on purpose since it will sometimes be convenient to be able to list elements of W as w_0, w_1, w_2, \dots where the first element should not be confused with w_+ if W is finite.

In particular, this shows that w_+ is of order two and normalizes the set S , so for every $i \in I$, there exists a unique $\sigma(i) \in I$ such that $w_+ s_i w_+ = s_{\sigma(i)}$. The map $i \mapsto \sigma(i)$ defines an involutive automorphism σ of the Coxeter diagram Δ of (W, S) . It is the identity if and only if w_+ lies in the center of W . Finite Coxeter groups are completely classified in terms of their Coxeter diagrams and in each case, the automorphism induced by conjugation with the longest word is known (cf. [AB08, Sections 1.3, 1.5.6 and 5.7.4]). A “generic” Coxeter group should be thought of as infinite. The following notion is useful to measure the extent to which a Coxeter group fails to be finite. It is motivated by the fact that the geometric realization of the Coxeter complex of a finite Coxeter group triangulates a sphere.

Definition 1.1.5. Let (W, S) be a Coxeter system of rank $n \in \mathbb{N}_0$. A subset $J \subset I$ is said to be *spherical* if the standard subgroup W_J is finite and *non-spherical* otherwise. Given an integer $k \in \{1, \dots, n\}$, we say that (W, S) is *k-spherical* if every subset $J \subset I$ with $|J| \leq k$ is spherical. If $k = n$, we simply call (W, S) *spherical*.

Since the simple reflections have order two, every (non-trivial) Coxeter system is 1-spherical. It is 2-spherical if and only if the subgroup generated by s_i and s_j is a finite dihedral group or, equivalently, $m_{ij} < \infty$ for all $i \neq j \in I$ and spherical if and only if W is a finite group.

In Chapter 4, we will occasionally have to deal with the question when a standard subgroup of a Coxeter group has finite index. This is obviously the case if the group itself is finite. If it is infinite, however, finite-index standard subgroups only exist in a rather trivial situation.

Theorem 1.1.6. *Let (W, S) be a Coxeter system, let $(W_1, S_1), \dots, (W_m, S_m)$ be the irreducible factors of (W, S) and define the following set:*

$$I_\infty := \bigcup_{k \in \{1, \dots, m\}: |W_k| = \infty} I_k \quad (1.1.6)$$

Then W_{I_∞} is the smallest finite-index standard subgroup of W . If W_J is a finite-index standard subgroup of W , then $I_\infty \subset J$ and $W_J \cong W_{I_\infty} \times W_{J \setminus I_\infty}$. In particular, if (W, S) is irreducible and not spherical, then W does not contain a non-trivial standard subgroup of finite index.

Proof. [Hos02, Corollary 3.3 and 3.4] □

1.2 Colourings of Coxeter diagrams

Throughout this section, we fix a Coxeter system (W, S) with Coxeter diagram $\Delta := \Delta(W, S)$. We denote by I_1, \dots, I_m the connected components of I , i.e. the vertex sets of the connected components $\Delta_1, \dots, \Delta_m$ of Δ , and by $(W_1, S_1), \dots, (W_m, S_m)$ the irreducible factors of (W, S) . Every subset $I^\bullet \subset I$ induces a *colouring* of Δ by declaring the vertices in I^\bullet to be black and those in $I \setminus I^\bullet$ to be white. Conversely, every colouring of Δ by black and white vertices arises in this way. Having chosen a particular colouring, the vertices can be divided into two disjoint sets, depending on whether or not they “see” a white vertex.

Definition 1.2.1. Suppose that Δ is coloured by a set $I^\bullet \subset I$. We say that a vertex $i \in I$ is *visible* (with respect to this colouring) if the unique connected component $I_k \subset I$ with $i \in I_k$ contains a white vertex, i.e. if $I_k \not\subset I^\bullet$. Otherwise, we say that i is *invisible*. We denote by I_{vis} (resp. I_{inv}) the set of visible (resp. invisible) vertices. We further say that I is *visible* if $I = I_{\text{vis}}$.

By definition, the sets I_{vis} and I_{inv} are unions of connected components of I and partition this set, so that $W \cong W_{I_{\text{vis}}} \times W_{I_{\text{inv}}}$. Moreover, I is visible if and only if every connected component of Δ contains a white vertex.

Remark 1.2.2. The notions introduced in this section will be important in order to study the geometry of convex hulls of Weyl group orbits in Chapter 4. However, it was pointed out to the author that some of the results established there have previously been proved in [Max89] in a

different context. In particular, upon denoting the set I^\bullet instead by T , our notion of a “visible” subset of I coincides with that of a “ T -minimal” subset in the sense of [Max89] or, equivalently, with that of an “ $(I \setminus T)$ -reduced” subset in the terminology of [Tit74]. In addition, Maxwell also acknowledges that his definitions and results can be interpreted in the framework of “shadows” introduced by Tits (cf. [Tit74, Appendix 1]).

On the one hand, it appears that most of the other definitions in this section refine the concept of a “visible subset” in a useful way that has not been made explicit in these references. Thus, the author believes that the following discussion goes beyond the work of Maxwell and Tits, although it surely does not lead to any significant new insights. On the other hand, some results of Section 4.3 were certainly already established by Maxwell and in every such situation, we will provide self-contained proofs of the statements in question that follow our terminology as well as explicit references to [Max89]. It is also likely that some of them can be deduced from Tits’ theory of shadows, which is formulated in the much more abstract language of incidence geometry, and Scharlau’s work on geometric realizations of shadow geometries (cf. [Sch90]).

Having coloured Δ by a set $I^\bullet \subset I$, we will frequently be interested in subgraphs of Δ which are formed by unions of paths connecting black and white vertices in a particularly well-behaved way. To prepare for the definition, we fix the following convention:

Convention 1.2.3. Since a Coxeter diagram does not have multiple edges, every path in Δ is uniquely determined by its sequence of vertices. Thus, if γ is a path in Δ of length $\delta \in \mathbb{N}_0$ starting at a vertex $i_0 \in I$, running through $i_1, \dots, i_{\delta-1} \in I$ and ending at $i_\delta \in I$, we may write $\gamma = (i_0, \dots, i_\delta)$ where the dots represent the vertices $i_1, \dots, i_{\delta-1}$ determined by γ . We will always assume that a path is not stammering, i.e. $i_k \neq i_{k-1}$ for all $k \in \{1, \dots, \delta\}$, and that it does not have backtracking, i.e. $i_{k+1} \neq i_{k-1}$ for all $k \in \{1, \dots, \delta-1\}$. However, we explicitly allow paths to have length zero, i.e. to consist of a single vertex.

Hence, a vertex $i \in I$ is visible (with respect to the colouring defined by I^\bullet) if and only if there exists a path $\gamma = (i_0, \dots, i_{\delta-1}, i_\delta = i)$ in Δ that starts at a white vertex $i_0 \in I \setminus I^\bullet$ and ends at i . In this case, we denote the minimal length of such a path by $\delta(i) \in \mathbb{N}_0$. Note that $\delta(i) = 0$ if and only if i is white, i.e. $i \in I \setminus I^\bullet$. If i is invisible, we set $\delta(i) := \infty$.

Definition 1.2.4. Suppose that Δ is coloured by a set $I^\bullet \subset I$. A subgraph Δ° of Δ is called an *exposing forest* for Δ if there exist paths $\gamma_1, \dots, \gamma_s$ in Δ with the following properties:

- (EF1) $\Delta^\circ = \bigcup_{r=1}^s \gamma_r$ and Δ° contains every vertex of Δ .
- (EF2) For every $r \in \{1, \dots, s\}$, the path γ_r has the property that, if its sequence of vertices is $\gamma_r = (i_{r,0}, \dots, i_{r,\delta_r})$ for some $\delta_r \in \mathbb{N}_0$, then $\delta(i_{r,k}) = k$ for all $k \in \{0, \dots, \delta_r\}$.
- (EF3) For all $r_1 \neq r_2 \in \{1, \dots, s\}$, the intersection $\gamma_{r_1} \cap \gamma_{r_2}$ is either empty or a proper initial segment of both γ_{r_1} and γ_{r_2} .

These axioms imply that if Δ° is an exposing forest for Δ , then for every $i \in I$, there exists a unique path in Δ° which starts at a white vertex and ends at i . Moreover, this path has length $\delta(i)$. In particular, the white vertices of Δ are precisely the starting points of the paths $\gamma_1, \dots, \gamma_s$, so $s \geq |I \setminus I^\bullet|$, but it is possible for different paths to start at the same point. By definition, an exposing forest for Δ is a union of paths that each contain precisely one white vertex and can only meet in initial segments. It is connected if and only if Δ contains exactly one white vertex since in this case, the paths $\gamma_1, \dots, \gamma_s$ must all start at that vertex and otherwise, two distinct white vertices cannot lie on the same path. More generally, this argument shows that $|I \setminus I^\bullet|$ is equal to the number of connected components of Δ° . Clearly, if there exists an exposing forest for Δ , then every connected component of Δ has to contain a white vertex or, equivalently, I must be visible. The converse is also true.

Lemma 1.2.5. *If I is visible, there exists an exposing forest for Δ .*

Proof. In general, an exposing forest for Δ is a disjoint union of m exposing forests, one for each of the connected components of Δ , so we may w.l.o.g. assume that (W, S) is irreducible or, equivalently, that Δ is connected. Since Δ is finite and I is visible, we have $\delta := \max_{j \in I} \delta(j) < \infty$.

If $\delta = 0$, then all vertices of Δ are white, i.e. $I^\bullet = \emptyset$, and we may take Δ° to be the subgraph of Δ consisting of all vertices but no edges. If $\delta > 0$, we choose an arbitrary $i \in I$ such that $\delta = \delta(i)$ and a minimal path

$$\gamma_1 = (i_0, \dots, i_\delta = i)$$

in Δ which connects i to a white vertex $i_0 \in I \setminus I^\bullet$. If γ_1 runs through every vertex of Δ , we are done by taking $\Delta^\circ := \gamma_1$. Otherwise, there exists a vertex of Δ that is not contained in γ_1 and we repeat the construction as follows: Set $I' := I \setminus \{i_0, \dots, i_\delta\}$ and $\delta' := \max_{j \in I'} \delta(j)$. If $\delta' = 0$, then all vertices in I' are white and we may set Δ° to be the union of γ_1 and all vertices in I' as isolated nodes. If $\delta' > 0$, we choose any $i' \in I'$ with $\delta' = \delta(i')$ and a minimal path

$$\gamma_2 = (i'_0, \dots, i'_{\delta'} = i')$$

in Δ which connects i' to a white vertex $i'_0 \in I \setminus I^\bullet$. If $\gamma_1 \cap \gamma_2 = \emptyset$, then either $\gamma_1 \cup \gamma_2$ contains every vertex of Δ and we are done by taking $\Delta^\circ := \gamma_1 \cup \gamma_2$, or we repeat the previous step.

If $\gamma_1 \cap \gamma_2 \neq \emptyset$, we claim that we may choose γ_2 (without changing γ_1) in such a way that $\gamma_1 \cap \gamma_2$ is a proper initial segment of γ_1 and γ_2 . To see this, let $k \in \{0, \dots, \delta\}$ and $k' \in \{0, \dots, \delta'\}$ be the largest numbers with $i_k = i'_{k'}$. Then $k \neq \delta$ since γ_2 does not contain $i = i_\delta$ and $k' \neq \delta'$ since γ_1 does not contain $i' = i'_{\delta'}$. If we had $k' < k$, then $(i'_0, \dots, i'_{k'} = i_k, \dots, i_\delta = i)$ would be a path of length less than δ connecting i with a white vertex, contradicting the assumption that $\delta = \delta(i)$. The same argument with k and k' interchanged shows that $k < k'$ is also impossible, so that we must have $k = k'$. Hence, we may replace γ_2 by the path $(i_0, \dots, i_k = i'_{k'}, \dots, i'_{\delta'} = i')$ which is a minimal path of length δ' that connects i' to a white vertex (see Figure 1.1).

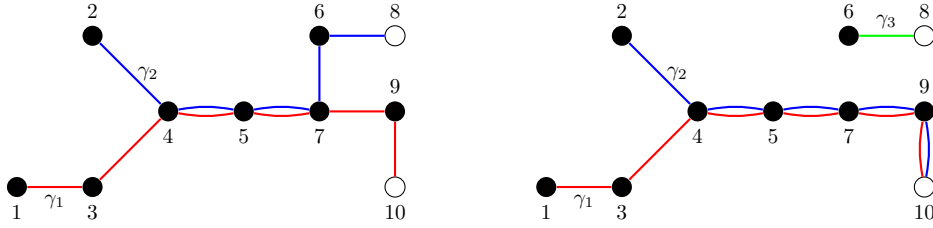


Figure 1.1: Replacing an initial segment

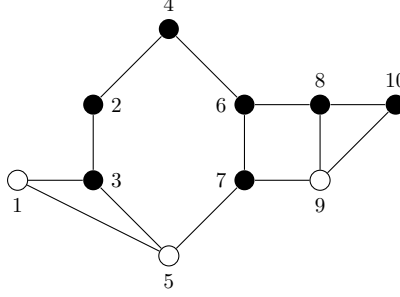
Once $\gamma_1 \cap \gamma_2$ is a proper initial segment of both paths, the subgraph $\gamma_1 \cup \gamma_2$ of Δ satisfies **(EF2)** and **(EF3)**. If it also contains every vertex of Δ , we are done by taking $\Delta^\circ := \gamma_1 \cup \gamma_2$. Otherwise, we repeat the process in the obvious way. As Δ is finite, it terminates after finitely many steps and the resulting subgraph $\Delta^\circ := \bigcup_{r=1}^s \gamma_r$ is an exposing forest for Δ . \square

Having chosen an exposing forest Δ° for Δ , there is a natural way of associating elements of W to the vertices of Δ whose simple factors can be read off from Δ° .

Definition 1.2.6. Let Δ° be an exposing forest for Δ . For $i \in I$, let $\gamma = (i_0, \dots, i_\delta = i)$, where $\delta := \delta(i) \in \mathbb{N}_0$, be the unique path in Δ° that connects i with a white vertex $i_0 \in I \setminus I^\bullet$. We associate to $i \in I$ the element $w_i := s_i s_{i_{\delta-1}} \dots s_{i_0} \in W$ and call the resulting set $\{w_i \mid i \in I\} \subset W$ the *exposing set* for W defined by Δ° .

In particular, if $i \in I \setminus I^\bullet$ is a white vertex, the associated element $w_i \in W$ is the simple reflection $s_i \in S$. More generally, it is immediate from the solution to the word problem in Coxeter groups that for every $i \in I$, the decomposition $w_i = s_i s_{i_{\delta-1}} \dots s_{i_0}$ is reduced, so that $l(w_i) = \delta(i) + 1$ and $S(w_i) = \{s_{i_k} \mid 0 \leq k \leq \delta(i)\}$. This also shows that an exposing forest is uniquely determined by its exposing set as the colouring on Δ can be read off from the simple reflections in that set and the paths in Δ° can be reconstructed iteratively. For example, the edges emanating from the white vertices are determined by the simple factors of elements of length two in the exposing set. In turn, the edges emanating from the black vertices obtained in the previous step are determined by words of length three and so on.

Example 1.2.7. Exposing forests are usually not unique. Suppose that the Coxeter diagram Δ of (W, S) has the following shape and colouring:



Since I is visible, there exists an exposing forest for Δ and we construct it following the proof of Lemma 1.2.5. The unique vertex of this graph which is at maximal distance from a white vertex is 4 and we start by choosing a minimal path from 4 to a white vertex. As every white vertex of Δ has distance three to the node 4, already the first choice in the construction is not unique. Two possible configurations are depicted in Figure 1.2. The exposing set corresponding to the exposing forest on the left is given by $\{s_1, s_2s_3s_5, s_3s_5, s_4s_2s_3s_5, s_5, s_6s_7s_5, s_7s_5, s_8s_9, s_9, s_{10}s_9\}$, whereas the one for the graph on the right is $\{s_1, s_2s_3s_1, s_3s_1, s_4s_6s_8s_9, s_5, s_6s_8s_9, s_7s_5, s_8s_9, s_9, s_{10}s_9\}$.

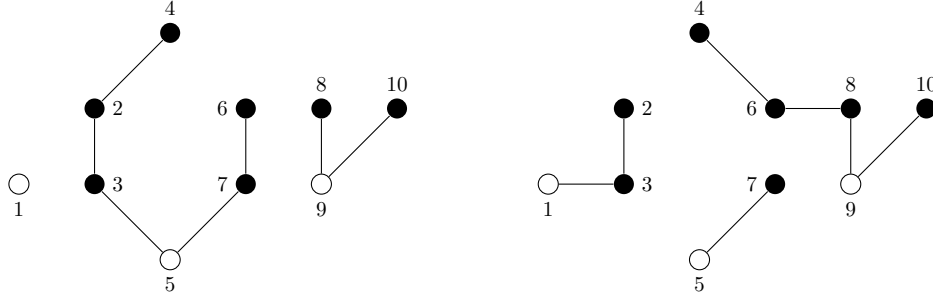


Figure 1.2: Two distinct exposing forests for the same coloured Coxeter diagram

We slightly generalize the notions of visible and invisible vertices to subsets of I . Given any $J \subset I$, the colouring on Δ defined by I^\bullet induces the colouring on the Coxeter diagram Δ_J of (W_J, S_J) defined by $J^\bullet := J \cap I^\bullet$. We say that a vertex $j \in J$ is *visible in J* if it is visible with respect to this colouring on Δ_J and *invisible in J* otherwise. We write J_{vis} (resp. J_{inv}) for the set of vertices that are visible (resp. invisible) in J . We further say that J is *visible* if $J = J_{\text{vis}}$ and denote the set of all visible subsets of I by $\mathcal{P}_{\text{vis}}(I)$. As before, J_{vis} and J_{inv} are unions of connected components of J and form a partition of this set, so that $W_J \cong W_{J_{\text{vis}}} \times W_{J_{\text{inv}}}$. For $j \in J_{\text{vis}}$, we denote by $\delta_J(j) \in \mathbb{N}_0$ the minimal distance in Δ_J from j to a white vertex in $J \setminus J^\bullet$ and for $j \in J_{\text{inv}}$, we set $\delta_J(j) := \infty$. If $J = I$, these notions reduce to the previous ones and we simplify the notation by writing $W_{\text{vis}} := W_{I_{\text{vis}}}$ and $W_{\text{inv}} := W_{I_{\text{inv}}}$. An important special case occurs when $J = I \setminus \{i\}$ for some $i \in I$ for which it will be convenient to introduce the following convention:

Notation 1.2.8. If $J = I \setminus \{i\}$ for some $i \in I$, we replace the subscript $I \setminus \{i\}$ by the superscript (i) in all expression where it occurs. For example, we write $W^{(i)} := W_{I \setminus \{i\}}$ for the corresponding standard subgroup of W , $S^{(i)} := S_{I \setminus \{i\}}$ for its canonical generating set and $\Delta^{(i)} := \Delta_{I \setminus \{i\}}$ for the Coxeter diagram of $(W^{(i)}, S^{(i)})$. Also, we denote the set itself by $I^{(i)} := I \setminus \{i\}$.

Given any $i \in I_k \subset I$, the connected components of the Coxeter diagram $\Delta^{(i)}$ of $(W^{(i)}, S^{(i)})$ are the same as those of Δ , except for the component Δ_k containing i which decomposes into a finite number of connected components after removing i . If i is visible, then Δ_k contains a white vertex

and every other vertex in I_k is visible too. However, once the vertex i is removed from Δ_k , it may happen that some vertices of $I_k \setminus \{i\}$ are no longer visible in $I^{(i)}$. We call $i \in I_{\text{vis}}$ *degenerate* if there exists a vertex $j \in I^{(i)}$ that is visible in I but invisible in $I^{(i)}$, and *non-degenerate* otherwise. We write I_{deg} (resp. I_{nd}) for the set of degenerate (resp. non-degenerate) vertices of I_{vis} . This gives rise to the following partition:

$$I = I_{\text{nd}} \sqcup I_{\text{deg}} \sqcup I_{\text{inv}} \quad (1.2.1)$$

More generally, if $J \subset I$ is any subset, we call a vertex $j \in J_{\text{vis}}$ *degenerate in J* if there exists a vertex in $J^{(j)} := J \setminus \{j\}$ that is visible in J but invisible in $J^{(j)}$. Equivalently, j is degenerate in J if and only if removing j from Δ_J results in a Coxeter diagram that has at least one more connected component consisting entirely of black vertices. We write J_{deg} (resp. J_{nd}) for the set of vertices of J_{vis} that are degenerate (resp. non-degenerate) in J and we have the analogous partition $J = J_{\text{nd}} \sqcup J_{\text{deg}} \sqcup J_{\text{inv}}$.

Example 1.2.9. Figure 1.3 depicts a coloured Coxeter diagram where all three types of vertices occur. In this example, we have $I_{\text{inv}} = \{1, 2, 3\}$, $I_{\text{nd}} = \{4, 6, 7, 10\}$ and $I_{\text{deg}} = \{5, 8, 9\}$. The invisible vertices are surrounded by the dashed box and the degenerate vertices are marked by a small dotted circle around them. Observe that an invisible vertex is necessarily black, but the diagram contains degenerate and non-degenerate vertices of both colours.

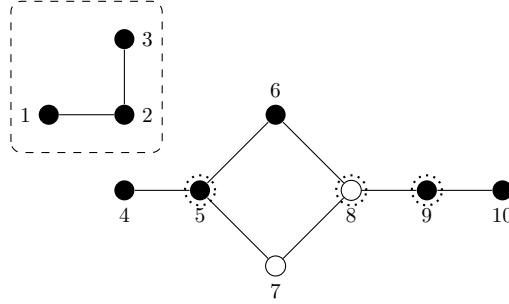


Figure 1.3: The partition $I = I_{\text{nd}} \sqcup I_{\text{deg}} \sqcup I_{\text{inv}}$

In the cases of the two trivial colourings $I^\bullet = \emptyset$ and $I^\bullet = I$, we have $I_{\text{nd}} = I$ if all vertices are white and $I_{\text{nd}} = \emptyset$ if all vertices are black. For a non-trivial colouring, it may not be immediately obvious whether non-degenerate vertices always exist, but this is indeed the case.

Proposition 1.2.10. $|I_{\text{nd}}| \geq |I \setminus I^\bullet|$

Proof. If $I^\bullet = I$, then $I_{\text{nd}} = \emptyset = I \setminus I^\bullet$ and the inequality is trivially satisfied, so we may assume that $I^\bullet \neq I$. We prove the assertion by induction on $n = |I|$. If $n = 1$, then $I^\bullet = \emptyset$ under this assumption, so that $|I_{\text{nd}}| = |I| = 1$. If $n \geq 2$, we establish the statement using the following simple graph-theoretic property:

Claim. If Γ is a finite connected graph with at least two vertices, then there exist at least two distinct vertices $v_1 \neq v_2$ of Γ such that the full subgraphs $\Gamma \setminus \{v_1\}$ and $\Gamma \setminus \{v_2\}$ are connected.

Proof of Claim. Let $d(\cdot, \cdot)$ denote the path metric of Γ . Since Γ is finite and connected, we may choose vertices $v_1 \neq v_2$ such that $d(v_1, v_2)$ is finite and maximal among all pairs of vertices of Γ . If v is any other vertex of Γ , then v can be connected to v_2 by a minimal path which does not pass through v_1 because $d(v, v_2) \leq d(v_1, v_2)$. Hence, $\Gamma \setminus \{v_1\}$ is connected and the same argument shows that $\Gamma \setminus \{v_2\}$ is connected as well.

Conclusion of the proof. Suppose that $n \geq 2$ and set $k := |I \setminus I^\bullet|$, so that $k > 0$ by assumption. To begin with, we further assume that (W, S) is irreducible or, equivalently, that Δ is connected. In this case, the claim above shows that there exist distinct vertices $i \neq j \in I$ such that removing

any one of them from Δ leads to a connected graph. Suppose first that one of them, say i , is black. Then $\Delta^{(i)}$ is connected and contains k white vertices. By the induction hypothesis, we find $i_1, \dots, i_k \in I^{(i)}$ that are non-degenerate in $I^{(i)}$. For each $l \in \{1, \dots, k\}$, every connected component of $\Delta^{(i_l)}$ is either equal to a connected component of $\Delta_{I \setminus \{i, i_l\}}$, the union of a finite number of such components that are all joined at the vertex i , or equal to the isolated node i . The last case occurs if and only if the only vertex of Δ that is connected to i is i_l . Since i_l is non-degenerate in $I^{(i)}$, no connected component of the former two types consists only of black vertices, so i_l is degenerate in I if and only if i is an isolated node of $\Delta^{(i_l)}$. This is only possible for one value of l , so at least $k - 1$ of these vertices are also non-degenerate in I . But the vertex i itself is also non-degenerate in I because $\Delta^{(i)}$ is connected and contains a white vertex, so there exist at least k non-degenerate vertices.

If i and j are both white, then $k \geq 2$ and we arbitrarily choose one of them, say i . As $\Delta^{(i)}$ is connected and contains $k - 1 > 0$ white vertices, the induction hypothesis yields $k - 1$ vertices $i_1, \dots, i_{k-1} \in I^{(i)}$ that are non-degenerate in $I^{(i)}$. For each $l \in \{1, \dots, k\}$, the description of the connected components of $\Delta^{(i_l)}$ is exactly the same as in the previous case. This time, however, each of the vertices i_1, \dots, i_{k-1} is non-degenerate in I since if i happens to be an isolated node of $\Delta^{(i_l)}$, this connected component consists of a white vertex. As before, the vertex i itself is also non-degenerate in I since $\Delta^{(i)}$ is connected and still contains a white vertex, so we again find at least k non-degenerate vertices.

Finally, if (W, S) is not necessarily irreducible, we set $I_k^\bullet := I_k \cap I^\bullet$ for all $k \in \{1, \dots, m\}$. By the irreducible case, there exist at least $|I_k \setminus I_k^\bullet|$ vertices that are non-degenerate in I_k . For $i \in I_k$, the connected components of $\Delta^{(i)}$ are the same as those of Δ , except for the component Δ_k which decomposes into a finite number of connected components after removing i . Therefore, i is non-degenerate in I if and only if it is non-degenerate in I_k , which implies the following:

$$|I_{\text{nd}}| = \sum_{k=1}^m |(I_k)_{\text{nd}}| \geq \sum_{k=1}^m |I_k \setminus I_k^\bullet| = |I \setminus I^\bullet| \quad \square$$

Remark 1.2.11. (i) The proof of Proposition 1.2.10 is admittedly technical and a much simpler argument was suggested to the author in a personal communication:² As above, we may assume that Δ is connected and $I^\bullet \neq I$, so that I is visible. Let Δ° be an exposing forest for Δ and $\gamma_1, \dots, \gamma_s$ the paths in Δ° appearing in Definition 1.2.4 that join every vertex of Δ to a white vertex in a unique way. Then $s \geq |I \setminus I^\bullet|$ and every such path γ_r with $r \in \{1, \dots, s\}$ contains at least one non-degenerate vertex. To see this, write $\gamma_r = (i_{r,0}, \dots, i_{r,\delta_r})$ for some $\delta_r \in \mathbb{N}_0$ and $i_{r,0} \in I \setminus I^\bullet$ and set $i := i_{r,\delta_r}$. Any other vertex $j \neq i$ of Δ either lies on γ_r between $i_{r,0}$ and i or on $\gamma_{r'}$ for some $r' \neq r$ which does not contain i . In both cases, j can still be connected to a white vertex of Δ after removing i , which shows that i is non-degenerate. Because the endpoints of $\gamma_1, \dots, \gamma_s$ are all distinct, the assertion follows.

- (ii) For the validity of the claim in our proof of Proposition 1.2.10, the assumption that Γ is a finite graph is essential. In fact, consider an infinite graph where each vertex has degree two (i.e. an infinite line), then removing any vertex makes the resulting graph disconnected. If the line is finite (i.e. if there exist precisely two vertices of degree one), the two nodes at the ends can be removed without destroying connectedness. If every vertex has degree two but their number is finite, the graph is a circle and removing any vertex leaves it connected.
- (iii) If $W = S_{n+1}$ is the symmetric group with its canonical generating set S consisting of the adjacent transpositions, then the Coxeter diagram of (W, S) is the finite line with n vertices. Suppose that $n \geq 2$ and that $I^\bullet = I^{(n)} = \{1, \dots, n-1\}$, so that n is the only white vertex. Then 1 is the only non-degenerate vertex, so the inequality in Proposition 1.2.10 is optimal, even if I^\bullet is neither empty nor equal to I .

At this point, we conclude our discussion about coloured Coxeter diagrams, even though we have only established a lot of definitions and relatively few results. However, the terminology developed in this section, in particular the notion of an exposing forest and its existence result in Lemma 1.2.5, will be very useful in studying convex hulls of Weyl group orbits in Chapter 4.

²The author is thankful to Richard Weidmann for proposing this simplified proof.

Chapter 2

Kac-Moody Algebras

The definition of *Kac-Moody algebras* is motivated by a famous theorem of Jean-Pierre Serre which states that a finite-dimensional, complex, semisimple Lie algebra is uniquely determined by its Cartan matrix (or, equivalently, its Dynkin diagram) and gives an explicit presentation of the Lie algebra whose relations depend only on this matrix (cf. [Kna02, Theorem 2.108 and 2.111]). In fact, Kac-Moody algebras are defined via presentations analogous to Serre's but allowing for a larger class of "generalized" Cartan matrices in the construction. Historically, they were first introduced by Victor G. Kac and Robert Moody in the 1960's. In their work, the authors associated to every generalized Cartan matrix a unique Kac-Moody algebra defined over the field of complex numbers. A refined approach to Kac-Moody theory, which is also better suited for the construction of Kac-Moody groups to be discussed in Chapter 3, was developed by Jacques Tits in the 1980's. It allows for the definition of Kac-Moody algebras over arbitrary rings and provides another useful form of flexibility. In Tits' construction, every generalized Cartan matrix is associated not only to one but a family of Kac-Moody algebras with similar structural properties that only depend on the underlying matrix.

Standard references for Kac-Moody algebras in the sense of Kac & Moody are [Kac90] and [MP95]. Our definition is based on Tits' approach which is contained in [Tit87, Section 4] and also explained and compared to the classical case in [Rém02, Chapter 7]. Except for some results in Section 2.6, most of the material discussed in this chapter and many more details can be found in Chapters 1-5, 9 and 11 of [Kac90], Chapters 4-6 of [MP95], Chapters 3-7 of the excellent textbook [Mar18] or Chapter 7 of [Rém02]. In some circumstances, however, our setup will be slightly more general, so for the reader's convenience, we occasionally give self-contained proofs of some results for which it is difficult to point to a satisfying reference. As in the previous chapter, we write $I := \{1, \dots, n\}$ if $n \in \mathbb{N}_0$ is a non-negative integer fixed by the context, unless Convention 2.1.2 applies.

2.1 Generalized Cartan matrices

Serre's presentation of a finite-dimensional, complex, semisimple Lie algebra in terms of its Cartan matrix suggests to construct a more general class of Lie algebras by allowing for a larger family of matrices. The following definition is the fundamental notion of all of Kac-Moody theory:

Definition 2.1.1. An integer matrix $\mathbb{A} = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$, where $n \in \mathbb{N}_0$, is called a *generalized Cartan matrix* ("GCM") if it satisfies the following three conditions for all $i, j \in I$:

- (GCM1) $a_{ii} = 2$.
- (GCM2) $a_{ij} \leq 0$ if $i \neq j$.
- (GCM3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

It is said to be *symmetrizable* if there exists an invertible diagonal matrix D and a symmetric matrix B of size $n \times n$ such that $\mathbb{A} = DB$. It is called *decomposable* if there exist non-empty, disjoint subsets $I_1, I_2 \subset I$ such that $I = I_1 \sqcup I_2$ and $a_{ij} = 0$ for all $i \in I_1$ and $j \in I_2$. Otherwise, \mathbb{A} is said to be *indecomposable*.

In general, a GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is not necessarily invertible and we set $\ell := \text{rank}(\mathbb{A}) \leq n$. Upon applying a suitable permutation to the index set I , we may assume that the first ℓ columns of \mathbb{A} are linearly independent, so that it has the form

$$\mathbb{A} = \begin{pmatrix} A_1 & A_2 \end{pmatrix}, \quad (2.1.1)$$

where A_1 is an $(n \times \ell)$ -matrix of rank ℓ and A_2 has size $n \times (n - \ell)$. Given any subset $J \subset I$, we denote by $\mathbb{A}_J := (a_{ij})_{i,j \in J}$ the submatrix of \mathbb{A} whose rows and columns are indexed by J , which is a GCM as well, and we set $\ell_J := \text{rank}(\mathbb{A}_J) \leq |J|$. Observe that we allow the vacuous case $n = 0$ in Definition 2.1.1, so that \mathbb{A}_\emptyset is also defined and can be thought of as the “empty matrix”.

If \mathbb{A} is decomposable and $I_1, I_2 \subset I$ are as in Definition 2.1.1, the matrix has a “block structure” with respect to these subsets that we write as $\mathbb{A} = \mathbb{A}_{I_1} \oplus \mathbb{A}_{I_2}$. By refining this decomposition, it follows that there exists a unique partition $I = I_1 \sqcup \dots \sqcup I_m$ with the property that the submatrix $\mathbb{A}_k := \mathbb{A}_{I_k}$ is indecomposable for every $k \in \{1, \dots, m\}$ and $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_m$. We call $\mathbb{A}_1, \dots, \mathbb{A}_m$ the *indecomposable components* of \mathbb{A} . Upon reordering the set I , we may assume that $I_k = \{i_{k-1}, \dots, i_k\}$ for $k \in \{1, \dots, m\}$, where $1 = i_0 < i_1 < \dots < i_{m-1} < i_m = n$. With such an ordering, the matrix \mathbb{A} has the following “block-diagonal” structure, where the zeroes denote rectangular zero-matrices of appropriate sizes:

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_1 & 0 & \dots & 0 \\ 0 & \mathbb{A}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbb{A}_m \end{pmatrix} \quad (2.1.2)$$

The *Dynkin diagram* of \mathbb{A} is the graph $\Gamma(\mathbb{A})$ with vertex set I whose edges are defined as follows: If $i \neq j \in I$ satisfy $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$, then i and j are connected by $|a_{ij}|$ edges. If $|a_{ij}| > 1$, this edge is decorated with an arrow pointing towards i . If $a_{ij}a_{ji} > 4$, then i and j are joined by a single edge labelled by the pair $(|a_{ij}|, |a_{ji}|)$. The matrix \mathbb{A} is indecomposable if and only if $\Gamma(\mathbb{A})$ is connected and, more generally, the sets $I_1, \dots, I_m \subset I$ indexing the indecomposable components of \mathbb{A} coincide with the vertex sets of the connected components of $\Gamma(\mathbb{A})$.

There is a classification of indecomposable GCM's that is formulated in the following way (cf. [Kac90, Theorem 4.3]): For a vector $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, we write $x > 0$ if $x_i > 0$ for all $i \in I$ and analogously, we define $x < 0$, $x \geq 0$ and $x \leq 0$. If \mathbb{A} is indecomposable, then exactly one of the three conditions below holds for both \mathbb{A} and \mathbb{A}^T . According to these cases, we say that \mathbb{A} (and hence \mathbb{A}^T) is of *finite*, *affine* or *indefinite type*.

- (Fin)** $\det(\mathbb{A}) \neq 0$; there exists $x > 0$ such that $\mathbb{A}x > 0$; $\mathbb{A}x \geq 0$ implies $x > 0$ or $x = 0$ for all $x \in \mathbb{R}^n$.
- (Aff)** $\text{rank}(\mathbb{A}) = n - 1$; there exists $x > 0$ such that $\mathbb{A}x = 0$; $\mathbb{A}x \geq 0$ implies $\mathbb{A}x = 0$ for all $x \in \mathbb{R}^n$.
- (Ind)** There exists $x > 0$ such that $\mathbb{A}x < 0$; $\mathbb{A}x \geq 0$ and $x \geq 0$ imply $x = 0$ for all $x \in \mathbb{R}^n$.

Furthermore, the GCM's of finite and affine type can be completely classified in terms of their Dynkin diagrams. In fact, a GCM \mathbb{A} is of finite type if and only if \mathbb{A} (resp. $\Gamma(\mathbb{A})$) is the Cartan matrix (resp. Dynkin diagram) of a finite-dimensional, complex, simple Lie algebra. It follows from Cartan's classification that every such matrix is invertible and the corresponding diagram belongs either to one of the four infinite families $(A_n)_{n \geq 1}, (B_n)_{n \geq 3}, (C_n)_{n \geq 2}, (D_n)_{n \geq 4}$ or is one of the five exceptional types E_6, E_7, E_8, F_4, G_2 (cf. Table 2.1).

The GCM's of affine type (resp. their Dynkin diagrams) are classified as well and arise as certain “extensions” of those of finite type by a single row and column (resp. vertex). They also fall into a small number of infinite families and a few exceptional cases and are usually separated into three classes to distinguish the so-called “untwisted type” from two kinds of “twisted types” (cf. Table 2.2 and 2.3). In order to explain the labels in these diagrams and align our notation with the existing literature, the following typical convention is useful:

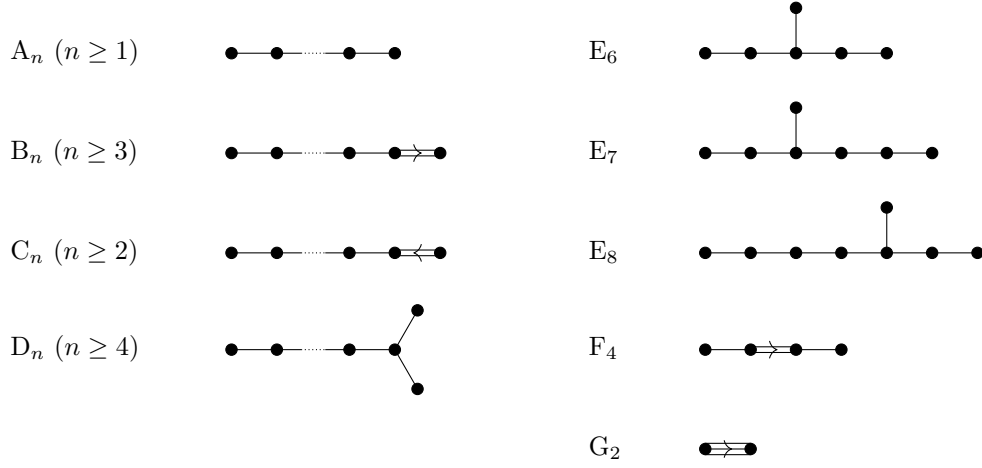


Table 2.1: The Dynkin diagrams of GCM's of finite type

Convention 2.1.2. If \mathbb{A} is a GCM of affine type and $\Gamma(\mathbb{A})$ its Dynkin diagram, then removing the white vertex from $\Gamma(\mathbb{A})$ results in a Dynkin diagram of finite type. Hence, every Dynkin diagram of affine type is obtained from a finite one by adjoining a single node and some edges in a certain way. To encompass this “extension process”, we will usually consider \mathbb{A} to be a matrix of size $(n+1) \times (n+1)$ and write $I := \{0, \dots, n\}$ in this case. We order I in such a way that the white vertex of $\Gamma(\mathbb{A})$ corresponds to 0. In addition, we set $\dot{I} := \{1, \dots, n\}$ and always replace the subscript i by a superscript \circ above the respective symbol. For example, we denote by $\mathring{\mathbb{A}} := \mathbb{A}_{\dot{I}}$ the $(n \times n)$ -submatrix of \mathbb{A} obtained by deleting the “first” (i.e. indexed by 0) row and column. Hence, $\mathring{\mathbb{A}}$ is of finite type and we think of \mathbb{A} as an extension of $\mathring{\mathbb{A}}$. Moreover, $\mathring{\mathbb{A}}$ is invertible, while $\ker(\mathbb{A})$ is spanned by a vector $x > 0$. In fact, one can choose $x = (a_0, a_1, \dots, a_n)^T \in \mathbb{Z}^{n+1}$, where for $i \in I$, the positive integer $a_i \in \mathbb{N}$ is the numerical label of the vertex i in $\Gamma(\mathbb{A})$. Note that $a_0 = 1$ in all but two cases and $a_0 = 2$ in the remaining two. We call $\mathring{\mathbb{A}}$ the *reduced matrix* of \mathbb{A} . By inspecting the list of affine Dynkin diagrams, it is apparent that \mathbb{A} is not uniquely determined by $\mathring{\mathbb{A}}$ in general.

The Dynkin diagram $\Gamma(\mathbb{A}^T)$ is obtained from $\Gamma(\mathbb{A})$ by reversing all arrows but keeping the ordering of the set I . At the same time, the numerical labels a_0, \dots, a_n of $\Gamma(\mathbb{A})$ must be replaced by the corresponding labels $a_0^\vee, \dots, a_n^\vee$ of $\Gamma(\mathbb{A}^T)$. As the white vertex remains fixed, it follows by inspecting the tables that $a_0^\vee = 1$ in all cases. For example, if $\Gamma(\mathbb{A})$ is the diagram of twisted type $A_{2n}^{(2)}$ for some $n \geq 1$ where $a_0 = 2$, then $\Gamma(\mathbb{A}^T)$ is the same diagram but the arrows now point away from the white vertex. Hence, the white vertex of $\Gamma(\mathbb{A}^T)$ is the rightmost vertex of $\Gamma(\mathbb{A})$ which is labelled by 1. The same argument applies in all other cases. Note that if $\Gamma(\mathbb{A})$ is of untwisted affine type, it may happen that $\Gamma(\mathbb{A}^T)$ is of twisted type and vice versa. For instance, if $n \geq 2$, the diagrams of type $C_n^{(1)}$ and $D_{n+1}^{(2)}$ are related in this way.

A GCM of affine type is not invertible, but has the property that deleting any node from its Dynkin diagram results in a union of diagrams of finite type. For GCM's of finite or indefinite type, this is impossible.

Lemma 2.1.3. *If $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is an indecomposable GCM that is not invertible and has the property that there exists some $i \in I$ such that every indecomposable component of the submatrix $\mathbb{A}^{(i)}$ is of finite type, then \mathbb{A} is of affine type.*

Proof. Upon reordering the index set I , we may assume that $i = 1$. Thus, every indecomposable component of the submatrix $\mathbb{A}' := \mathbb{A}^{(1)}$ is of finite type and we may write \mathbb{A} as a block matrix

$$\mathbb{A} = \begin{pmatrix} 2 & x^T \\ y & \mathbb{A}' \end{pmatrix}$$

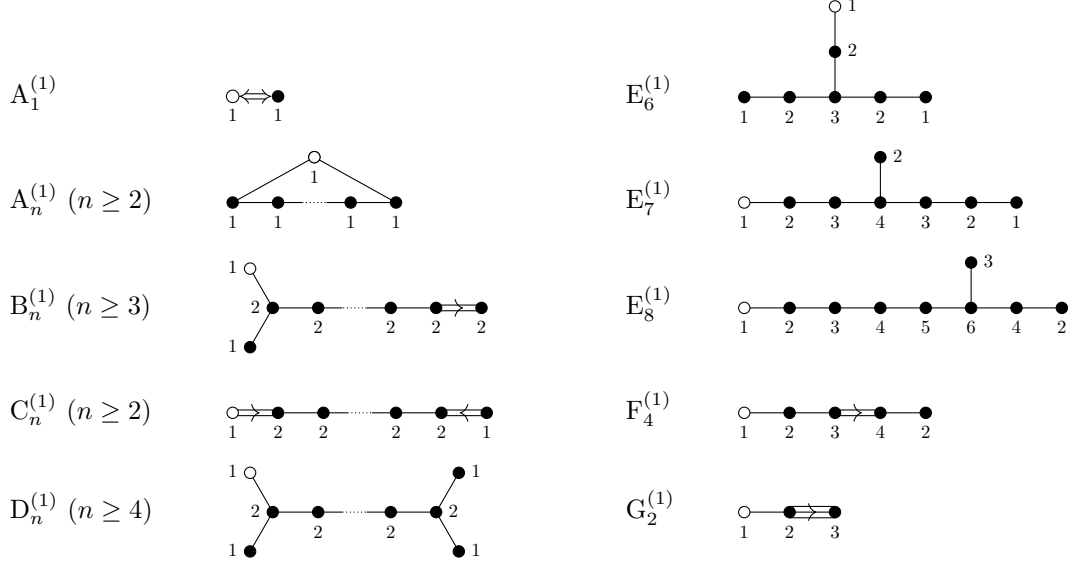


Table 2.2: The Dynkin diagrams of GCM's of untwisted affine type

for some $x, y \in \mathbb{Z}^{n-1} \setminus \{0\}$ with $x, y \leq 0$. Because \mathbb{A} is not invertible, there exists a non-zero vector $v = (v_1, \dots, v_n)^T \in \mathbb{R}^n$ such that $\mathbb{A}v = 0$ and we may w.l.o.g. suppose that $v_1 \geq 0$. We set $v' := (v_2, \dots, v_n)^T$, so that $v = (v_1, v')^T$. If $v_1 = 0$, then $v' \neq 0$ and $\mathbb{A}'v' = 0$. This is impossible since a GCM of finite type is invertible, so we must have $v_1 > 0$.

Assume that \mathbb{A} is not of affine and hence of indefinite type. As $v_1 > 0$ and $\mathbb{A}v = 0 \geq 0$, it follows from **(Ind)** that some coordinate of v' must be negative. For every $j \in \{2, \dots, n\}$, the j -th entry of $\mathbb{A}v$ is equal to zero. If $y_j = 0$, this implies that the j -th entry of $\mathbb{A}'v'$ must already be zero. If $y_j < 0$, then $v_1 > 0$ forces the j -th entry of $\mathbb{A}'v'$ to be positive. Since $y \neq 0$, the latter occurs for at least one $j \in \{2, \dots, n\}$, so it follows that $\mathbb{A}'v' \geq 0$ and $\mathbb{A}'v' \neq 0$. As every indecomposable component of \mathbb{A}' is of finite type, **(Fin)** requires that $v' \geq 0$, a contradiction. \square

Remark 2.1.4. Given an arbitrary GCM \mathbb{A} , the Dynkin diagram $\Gamma(\mathbb{A})$ not only encodes information about decomposability of \mathbb{A} , but also provides a graphic criterion for symmetrizability. In fact, it is not difficult to show that \mathbb{A} is symmetrizable if and only if it satisfies

$$a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} \dots a_{i_k i_{k-1}} a_{i_1 i_k} \quad \text{for all } k \geq 2 \text{ and } i_1, \dots, i_k \in I. \quad (2.1.3)$$

By **(GCM3)**, both sides of the equation are zero if the vertices i_1 and i_k , or i_{l-1} and i_l for some $l \in \{2, \dots, k\}$, are not connected in $\Gamma(\mathbb{A})$, so it suffices to verify (2.1.3) under the assumption that (i_1, \dots, i_k, i_1) is a closed path in $\Gamma(\mathbb{A})$ which may even be assumed to be simple. In particular, the condition is always satisfied if $\Gamma(\mathbb{A})$ does not contain any non-trivial cycles.

More generally, a precise interpretation of (2.1.3) is the following: Suppose that we replace each edge of $\Gamma(\mathbb{A})$ connecting two vertices $i \neq j \in I$ by a double-edge that is oriented in both directions and labelled by a_{ij} when going from i to j and by a_{ji} in the other direction. Given a closed path $\gamma = (i_1, \dots, i_k, i_1)$ in $\Gamma(\mathbb{A})$, the product $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1} \neq 0$ can then be interpreted as a “mass” of γ . From this perspective, (2.1.3) can be expressed by saying that for every simple closed path γ in $\Gamma(\mathbb{A})$, the mass of γ is the same as the mass of the path obtained by following γ in the opposite direction.

Based on this criterion, it follows by inspecting their Dynkin diagrams that GCM's of finite or affine type are always symmetrizable. In fact, the only such diagrams that contain a non-trivial cycle are those of type $A_n^{(1)}$ with $n \geq 2$, but they do not have any arrows, so the corresponding GCM's are already symmetric. More precisely, it can be shown that \mathbb{A} is of finite (resp. affine) type if and only if it can be written as $\mathbb{A} = DB$, where D is an invertible diagonal matrix and

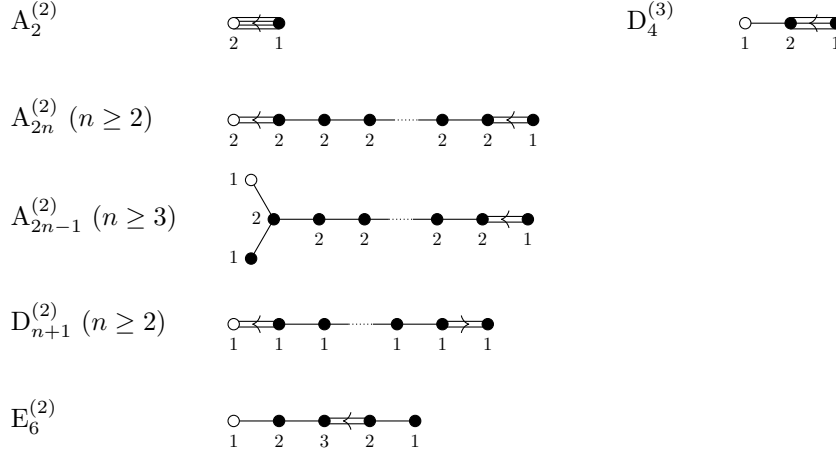


Table 2.3: The Dynkin diagrams of GCM's of twisted affine type

B is symmetric and positive-definite (resp. positive-semidefinite with corank 1). However, these matrices make up a very small subclass of all GCM's and a “generic” indecomposable GCM is of indefinite type and not necessarily symmetrizable. The simplest example of a non-symmetrizable GCM is the following:

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \iff \Gamma(\mathbb{A}) = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}$$

Definition 2.1.5. Let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a GCM. A subset $J \subset I$ is said to be *spherical* if every indecomposable component of the submatrix \mathbb{A}_J is of finite type and *non-spherical* otherwise. Given an integer $k \in \{1, \dots, n\}$, we say that \mathbb{A} is *k-spherical* if every subset $J \subset I$ with $|J| \leq k$ is spherical. If $k = n$, we simply call \mathbb{A} *spherical*.

Hence, a GCM is spherical if and only if every indecomposable component is of finite type or, equivalently, it is the Cartan matrix of a finite-dimensional, complex, semisimple Lie algebra. The only GCM of size 1×1 is (2) which is obviously of finite type. It is the Cartan matrix of the simple Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its Dynkin diagram is A_1 . Every GCM of size 2×2 has the form $\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ for some $a, b \in \mathbb{N}_0$. It is indecomposable if and only if $a, b > 0$ and spherical if and only if $ab \leq 3$. It is of affine type if $ab = 4$ and of indefinite type if $ab > 4$. Thus, an arbitrary GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is 2-spherical if and only if $a_{ij}a_{ji} \leq 3$ for all $i \neq j \in I$. If $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is of affine type, then it follows by inspecting Table 2.1, 2.2 and 2.3 that \mathbb{A} is n -spherical but not spherical. The converse is not true, however.

Remark 2.1.6. Definition 2.1.5 introduces the same concepts for GCM's as Definition 1.1.5 did for Coxeter systems. In particular, both define the notion of a “spherical subset” of I . The reason for this abuse of terminology will be justified in Section 2.5 and 2.6, where we introduce the Weyl group of a GCM and its associated Coxeter system, which is spherical if and only if the GCM is spherical (cf. Proposition 2.6.5 (vi)).

Notation 2.1.7. As we shall see repeatedly throughout our discussion, the only GCM of size 1×1 plays a special role in Kac-Moody theory. Therefore, it will be convenient to denote it by $\mathbb{A}^0 := (2)$ in order to distinguish it from an arbitrary GCM.

Example 2.1.8. Although a “randomly chosen” GCM is usually invertible, the affine type is of particular interest in many applications of Kac-Moody theory. This is the main motivation for developing the theory for arbitrary GCM's, even though the possibility that the matrix may not

have full rank adds several technicalities to many arguments. While GCM's of finite type are always invertible and those of affine type never, there also exist non-invertible GCM's that are of indefinite type, e.g. the following:

$$\mathbb{A} = \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix} \iff \Gamma(\mathbb{A}) = \bullet \longleftrightarrow \bullet \text{---} \bullet \longleftrightarrow \bullet$$

In fact, this is just the simplest instance of a more general method that combines two affine GCM's into a bigger one which is also indecomposable and has corank 1, but is no longer of affine type. This is achieved by connecting two affine Dynkin diagrams by a single node and two edges (cf. [Kac90, Exercise 4.12]). Here, we have applied this procedure to the affine diagram $A_1^{(1)}$.

It is perhaps less obvious that there also exist indecomposable GCM's of higher corank which are necessarily of indefinite type. The construction below shows that for every $n \in \mathbb{N}$, there exists an indecomposable GCM $\mathbb{A}_n = (a_{ij})_{i,j=1}^{2n}$ with $\text{corank}(\mathbb{A}_n) = n - 1$ or, equivalently, $\text{rank}(\mathbb{A}_n) = n + 1$. It is defined as follows:¹

- $a_{ii} = 2$ for all $i \in \{1, \dots, 2n\}$
- $a_{ij} = a_{ji} = -6$ for all $i \neq j \in \{1, \dots, n\}$
- $a_{i,n+i} = a_{n+i,i} = -4$ for all $i \in \{1, \dots, n\}$
- $a_{ij} = a_{ji} = 0$ for all other $i \neq j \in \{1, \dots, 2n\}$

In other words, if $\mathbb{B}_n = (b_{ij})_{i,j=1}^n$ is the GCM with $b_{ij} = -6$ whenever $i \neq j \in \{1, \dots, n\}$, then \mathbb{A}_n has the following block structure:

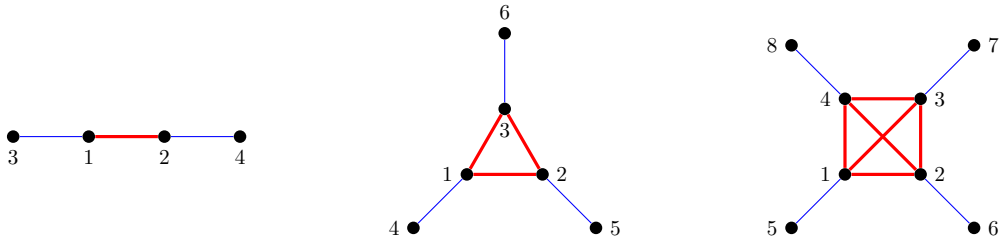
$$\mathbb{A}_n = \begin{pmatrix} \mathbb{B}_n & -4\mathbb{I}_n \\ -4\mathbb{I}_n & 2\mathbb{I}_n \end{pmatrix}$$

It is not difficult to verify that the vectors $v_2, \dots, v_n \in \mathbb{R}^{2n}$ defined by

$$v_j := e_1 - e_j + 2e_{n+1} - 2e_{n+j} \quad \text{for } j \in \{2, \dots, n\},$$

where $\{e_1, \dots, e_{2n}\}$ denotes the standard basis of \mathbb{R}^{2n} , are linearly independent and lie in the kernel of \mathbb{A}_n . Thus, $\text{corank}(\mathbb{A}_n) \geq n - 1$ and, conversely, the corank cannot be larger since the last $n + 1$ column vectors of \mathbb{A}_n are linearly independent.

The Dynkin diagram $\Gamma(\mathbb{A}_n)$ is obtained by first forming a complete graph on n vertices and labelling each edge by $(6, 6)$. Afterwards, one adds n nodes to this graph and connects each of them to a unique vertex of the complete subgraph by an edge with label $(4, 4)$. For $n \in \{2, 3, 4\}$, the resulting graphs look as follows, where we have drawn the edges labelled by $(6, 6)$ as thick red lines and the ones with label $(4, 4)$ as thin blue ones. The numbers next to the vertices simply denote the ordering of the set $I = \{1, \dots, 2n\}$ and should not be confused with the labels of the nodes in affine Dynkin diagrams.



This construction can be slightly generalized. In fact, it is possible to replace the number -4 in the definition of \mathbb{A}_n by an arbitrary negative even integer $-2k$ for some $k \in \mathbb{N}$ and then substitute -6 by $2 - 2k^2$. For $k = 1$, the resulting matrix is decomposable. For $k \geq 2$, it is indecomposable and the case $k = 2$ is the example above.

¹The author is grateful to Timothée Marquis who suggested this example to him in a personal communication. It is presented here with his permission.

2.2 Kac-Moody root data

Given a generalized Cartan matrix \mathbb{A} , there are many ways of constructing a Kac-Moody algebra \mathfrak{g} from it. First, it is certainly desirable to be able to define \mathfrak{g} over arbitrary fields.² Second, even once the field is fixed, there typically exist several non-isomorphic Kac-Moody algebras that can reasonably be “associated” to \mathbb{A} . For example, in classical Lie theory, the Cartan matrix $\mathbb{A}^0 = (2)$ is usually associated to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, but one may just as well view $\mathfrak{gl}_2(\mathbb{C})$ as a Lie algebra belonging to \mathbb{A}^0 as it is simply a trivial central extension of $\mathfrak{sl}_2(\mathbb{C})$ and the structure of these algebras is almost identical. In general, we want to be able to encompass such “degree-zero modifications” which can be viewed as generalizing the distinction between semisimple and reductive Lie algebras. The following definition provides enough flexibility for our purposes:

Definition 2.2.1. A *Kac-Moody root datum* is a quintuple

$$\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I}), \quad (2.2.1)$$

where $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}_0$ is a set indexing a GCM $\mathbb{A} = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$, Λ is a free \mathbb{Z} -module of rank $d \in \mathbb{N}$, $(c_i)_{i \in I}$ is a family of elements in Λ and $(h_i)_{i \in I}$ is a family in the dual \mathbb{Z} -module Λ^\vee such that the dual pairing between Λ and Λ^\vee satisfies $\langle c_j, h_i \rangle = a_{ij}$ for all $i, j \in I$. The *rank* of \mathcal{D} is defined as the rank $d \in \mathbb{N}$ of the free \mathbb{Z} -module Λ . We further say that \mathcal{D} is

- (i) *free* (resp. *cofree*) if $(c_i)_{i \in I}$ (resp. $(h_i)_{i \in I}$) is linearly independent.
- (ii) *adjoint* (resp. *coadjoint*) if $(c_i)_{i \in I}$ spans Λ (resp. $(h_i)_{i \in I}$ spans Λ^\vee).
- (iii) *cotorsion-free* if $\Lambda^\vee / \sum_{i \in I} \mathbb{Z}h_i$ is torsion-free.

If we want to specify the GCM \mathbb{A} appearing in (2.2.1) without writing down the entire quintuple, we say that \mathcal{D} is a Kac-Moody root datum *associated to* \mathbb{A} . Note that the previous definition allows the case $n = 0$ in which I is the empty set and \mathbb{A} is the “empty matrix”. By convention, the Kac-Moody root datum \mathcal{D} then simply consists of a (non-trivial) free \mathbb{Z} -module Λ without any additional structure. We call such a root datum *trivial*.

Remark 2.2.2. (i) The condition $\langle c_j, h_i \rangle = a_{ij}$ for all $i, j \in I$ in Definition 2.2.1 implies that if \mathcal{D} is not cofree (resp. free), then the rows (resp. columns) of the GCM \mathbb{A} are linearly dependent over \mathbb{Z} . Conversely, every \mathbb{Z} -linear dependence between the rows (resp. columns) of \mathbb{A} can be used to construct a root datum of rank $n = |I|$ that is not cofree (resp. free). Hence, a GCM is invertible (over \mathbb{Q}) if and only if all Kac-Moody root data associated to it are free and cofree.

(ii) A Kac-Moody root datum is cofree and cotorsion-free if and only if the family $(h_i)_{i \in I}$ can be extended to a \mathbb{Z} -basis of Λ^\vee . Since Λ^\vee is not a vector space over a field, this property is stronger than linear independence and the difference between these notions is measured by the torsion subgroup of $\Lambda^\vee / \sum_{i \in I} \mathbb{Z}h_i$. Whereas the first condition is always satisfied in the invertible case, the same is not true for the second. For example, the minimal adjoint root datum (as defined in Example 2.2.3 (iii)) associated to $\mathbb{A}^0 = (2)$ is not cotorsion-free.

Example 2.2.3. Let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a GCM of rank $\ell \leq n$ and suppose that $\mathbb{A} = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$ as in (2.1.1). The three most important Kac-Moody root data associated to \mathbb{A} are the following:

- (i) The *Kac root datum* $\mathcal{D}_{\text{Kac}}^\mathbb{A}$: There exists a free, cofree and cotorsion-free Kac-Moody root datum of rank $2n - \ell$ associated to \mathbb{A} which is constructed as follows: Let $\Lambda_{\text{Kac}}^\vee$ and Λ_{Kac} be dual free \mathbb{Z} -modules of rank $2n - \ell$ and choose dual bases $\{v_1, \dots, v_{2n-\ell}\} \subset \Lambda_{\text{Kac}}^\vee$ and $\{v_1^*, \dots, v_{2n-\ell}^*\} \subset \Lambda_{\text{Kac}}$. For $i \in I$, set $h_i := v_i$ and $c_i := \sum_{r=1}^{2n-\ell} B_{ri} v_r^*$, where $B_{ri} \in \mathbb{Z}$ is the (r, i) -th entry of the matrix

$$B := \begin{pmatrix} A_1 & A_2 \\ 0 & \mathbb{I}_{n-\ell} \end{pmatrix}$$

²Kac-Moody algebras can even be defined over arbitrary rings, but in Section 2.3, we restrict our attention to fields of characteristic zero on purpose because the general definition is more complicated to state than our Definition 2.3.1 and later, we also want to be able to work with vector spaces instead of modules over a ring. However, in order to develop the theory of affine Kac-Moody algebras in detail, it is actually necessary to allow rings in the definition (cf. Example 2.3.7 (ii)).

which has size $(2n - \ell) \times n$ and rank n . It is minimal with these properties in the sense that any free and cofree root datum associated to \mathbb{A} must have rank $d \geq 2n - \ell$ and, if it is also cotorsion-free, contain a copy of $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ (cf. [Mar18, Exercise 7.20]). Although $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ is free, the family $(c_i)_{i \in I}$ cannot necessarily be extended to a \mathbb{Z} -basis of Λ_{Kac} since $\Lambda_{\text{Kac}} / \sum_{i \in I} \mathbb{Z}c_i$ need not be torsion-free.

- (ii) The *simply connected root datum* $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$: There is a unique Kac-Moody root datum of rank n associated to \mathbb{A} that is cofree and coadjoint. It can be constructed from $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ by defining $\Lambda_{\text{sc}}^{\vee}$ as the submodule of $\Lambda_{\text{Kac}}^{\vee}$ spanned by $(h_i)_{i \in I}$ and restricting the functionals $(c_i)_{i \in I}$ to this submodule. If \mathbb{A} is invertible, it coincides with $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$.
- (iii) The *minimal adjoint root datum* $\mathcal{D}_{\text{min}}^{\mathbb{A}}$: Similar to (ii), there exists a unique Kac-Moody root datum of rank n associated to \mathbb{A} that is free and adjoint. It can be constructed from $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ by defining Λ_{min} as the submodule of Λ_{Kac} spanned by $(c_i)_{i \in I}$ and restricting the family $(h_i)_{i \in I}$, which we may view as functionals on Λ_{Kac} , to this submodule. Whereas $\mathcal{D}_{\text{Kac}}^{\mathbb{A}} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ if \mathbb{A} is invertible, these root data need not coincide with $\mathcal{D}_{\text{min}}^{\mathbb{A}}$. In fact, the family $(c_i)_{i \in I}$ is linearly independent, but does not necessarily span $\Lambda_{\text{Kac}} = \Lambda_{\text{sc}}$, so that Λ_{min} may be a proper finite-index subgroup of Λ_{Kac} . As a consequence, when restricted from Λ_{Kac} to Λ_{min} , the family $(h_i)_{i \in I}$ remains linearly independent, but does not have to span $\Lambda_{\text{min}}^{\vee}$, so that $\mathcal{D}_{\text{min}}^{\mathbb{A}}$ is not cotorsion-free in general. This phenomenon is easy to observe for the GCM $\mathbb{A}^0 = (2)$ and will be made explicit in Section 3.3.

Example 2.2.4. There are four ways of constructing new Kac-Moody root data from given ones that will be important to our discussion:

- (i) *Restriction*: Given a Kac-Moody root datum $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ and any subset $J \subset I$, the *J-restriction* of \mathcal{D} is defined as

$$\mathcal{D}_J := (J, \mathbb{A}_J, \Lambda, (c_j)_{j \in J}, (h_j)_{j \in J}). \quad (2.2.2)$$

Note that the free \mathbb{Z} -module Λ is the same in \mathcal{D} and \mathcal{D}_J . In particular, \mathcal{D}_{\emptyset} is the trivial root datum on Λ . Moreover, if $\mathcal{D} = \mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ and $J \subsetneq I$, then \mathcal{D}_J does not coincide with $\mathcal{D}_{\text{Kac}}^{\mathbb{A}_J}$ and the analogous statements are true for $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ and $\mathcal{D}_{\text{min}}^{\mathbb{A}}$.

- (ii) *Duality*: The *dual* of \mathcal{D} is the Kac-Moody root datum

$$\mathcal{D}^* := (I, \mathbb{A}^T, \Lambda^{\vee}, (c_i^*)_{i \in I}, (h_i^*)_{i \in I}) \quad (2.2.3)$$

defined by $c_i^* := h_i$ and $h_i^* := c_i$ for all $i \in I$, where we have used the canonical identification $\Lambda \cong (\Lambda^{\vee})^{\vee}$.

- (iii) *Union*: Given two Kac-Moody root data $\mathcal{D}_k = (I_k, \mathbb{A}_k, \Lambda, (c_i)_{i \in I_k}, (h_i)_{i \in I_k})$, $k \in \{1, 2\}$, that are defined on the same free \mathbb{Z} -module Λ and have the property that $\langle c_j, h_i \rangle = 0$ whenever $i \in I_1$ and $j \in I_2$ or $i \in I_2$ and $j \in I_1$, we define their *union* by

$$\mathcal{D}_1 \cup \mathcal{D}_2 := (I_1 \sqcup I_2, \mathbb{A}_1 \oplus \mathbb{A}_2, \Lambda, (c_i)_{i \in I_1 \sqcup I_2}, (h_i)_{i \in I_1 \sqcup I_2}). \quad (2.2.4)$$

In particular, if \mathcal{D} is an arbitrary Kac-Moody root datum associated to a GCM \mathbb{A} with indecomposable components $\mathbb{A}_1, \dots, \mathbb{A}_m$ indexed by $I_1, \dots, I_m \subset I$, then $\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_m$, where for every $k \in \{1, \dots, m\}$, we denote by $\mathcal{D}_k := \mathcal{D}_{I_k}$ the I_k -restriction of \mathcal{D} . We call $\mathcal{D}_1, \dots, \mathcal{D}_m$ the *indecomposable components* of \mathcal{D} .

- (iv) *Direct sum*: For two arbitrary root data $\mathcal{D}_k = (I_k, \mathbb{A}_k, \Lambda_k, (c_i)_{i \in I_k}, (h_i)_{i \in I_k})$, $k \in \{1, 2\}$, their *direct sum* is defined as

$$\mathcal{D}_1 \oplus \mathcal{D}_2 := (I_1 \sqcup I_2, \mathbb{A}_1 \oplus \mathbb{A}_2, \Lambda_1 \times \Lambda_2, (c_i)_{i \in I_1 \sqcup I_2}, (h_i)_{i \in I_1 \sqcup I_2}). \quad (2.2.5)$$

Here, we identify Λ_1 (resp. Λ_2) with the submodule $\Lambda_1 \times \{0\}$ (resp. $\{0\} \times \Lambda_2$) of $\Lambda_1 \times \Lambda_2$ and likewise for $(\Lambda_1 \times \Lambda_2)^{\vee} \cong \Lambda_1^{\vee} \times \Lambda_2^{\vee}$. We call a Kac-Moody root datum *reducible* if it is the direct sum of two root data and *irreducible* otherwise. We allow the factors to be trivial, so a reducible root datum need not be associated to a decomposable GCM.

For GCM's of affine type, there exist Kac-Moody root data that exhibit a certain unpleasant phenomenon that we want to avoid later. We introduce these root data now, their relevance will be discussed in Section 2.6. Recall that Convention 2.1.2 is employed in the affine case.

Example 2.2.5. Let $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ be a GCM of affine type and $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ a Kac-Moody root datum associated to \mathbb{A} . By omitting $c_0 \in \Lambda$ and $h_0 \in \Lambda^\vee$, i.e. by forming the \check{I} -restriction of \mathcal{D} , we obtain a Kac-Moody root datum $\mathcal{D}_{\check{I}} = (\check{I}, \check{\mathbb{A}}, \Lambda, (c_i)_{i \in \check{I}}, (h_i)_{i \in \check{I}})$ associated to the reduced matrix $\check{\mathbb{A}}$, which is free and cofree as $\check{\mathbb{A}}$ is invertible. If \mathcal{D} is not free as well, there exists a relation $\sum_{j \in I} k_j c_j = 0$, where $k_0, \dots, k_n \in \mathbb{Z}$ are not all equal to zero, which implies

$$0 = \sum_{j \in I} k_j \langle c_j, h_i \rangle = \sum_{j \in I} k_j a_{ij} \quad \text{for all } i \in I.$$

Hence, the non-zero vector $(k_0, \dots, k_n)^T \in \mathbb{Z}^{n+1}$ lies in the kernel of \mathbb{A} , so that, up to a scalar multiple, these integers coincide with the numerical labels $a_0, \dots, a_n \in \mathbb{N}$ of the vertices in the Dynkin diagram $\Gamma(\mathbb{A})$. Similarly, if \mathcal{D} is not cofree, the same argument shows that $\sum_{i \in I} a_i^\vee h_i = 0$, where $a_0^\vee, \dots, a_n^\vee \in \mathbb{N}$ are the labels of the vertices in $\Gamma(\mathbb{A}^T)$.

Conversely, let $\check{\mathcal{D}} = (\check{I}, \check{\mathbb{A}}, \Lambda, (c_i)_{i \in \check{I}}, (h_i)_{i \in \check{I}})$ be a Kac-Moody root datum associated to $\check{\mathbb{A}}$. Given any $c' \in \Lambda$ and $h' \in \Lambda^\vee$ that satisfy $\langle c', h_j \rangle = 0 = \langle c_j, h' \rangle$ for all $j \in \check{I}$ and $\langle c', h' \rangle = 0$, we can extend $\check{\mathcal{D}}$ to a Kac-Moody root datum \mathcal{D} associated to \mathbb{A} by defining³

$$c_0 := - \sum_{i \in \check{I}} \frac{a_i}{a_0} c_i + c' \in \Lambda \quad \text{and} \quad h_0 := - \sum_{i \in \check{I}} \frac{a_i^\vee}{a_0^\vee} h_i + h' \in \Lambda^\vee.$$

Since $\check{\mathcal{D}}$ is cofree, the functional c' can only be non-zero if the quotient $\Lambda^\vee / \sum_{i \in \check{I}} \mathbb{Z} h_i$ has positive rank or, equivalently, $(h_i)_{i \in \check{I}}$ is not a \mathbb{Q} -basis of $\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, which is the case if and only if Λ has rank $d > n$. The analogous statement is true for h' . Thus, if $d = n$, then \mathcal{D} is uniquely determined by $\check{\mathcal{D}}$ and neither free nor cofree. In particular, this is the case if $\check{\mathcal{D}} = \mathcal{D}_{\text{Kac}}^{\check{\mathbb{A}}} = \mathcal{D}_{\text{sc}}^{\check{\mathbb{A}}}$ or $\check{\mathcal{D}} = \mathcal{D}_{\text{min}}^{\check{\mathbb{A}}}$. If $d \geq n+1$, then $\check{\mathcal{D}}$ can be extended either to a free or to a cofree root datum associated to \mathbb{A} . However, due to the additional constraint that $\langle c', h' \rangle = 0$, an extension that is free and cofree requires $d \geq n+2$.

The simplest instance of this construction will serve as a running example throughout this thesis, so we make it more explicit. Consider the GCM of untwisted affine type $A_1^{(1)}$:

$$\mathbb{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \quad \Longleftrightarrow \quad \Gamma(\mathbb{A}) = \begin{array}{c} \circ \\ \text{---} \text{---} \bullet \\ 1 \quad 1 \end{array}$$

Its reduced matrix is $\check{\mathbb{A}} = \mathbb{A}^0 = (2)$ and the simply connected root datum $\mathcal{D}_{\text{sc}}^{\check{\mathbb{A}}} = (\{1\}, \mathbb{A}^0, \Lambda, c_1, h_1)$ is constructed as follows: Let Λ and Λ^\vee be dual free \mathbb{Z} -modules of rank 1, i.e. $\Lambda^\vee = \mathbb{Z}v$ and $\Lambda = \mathbb{Z}v^*$ with $\langle v^*, v \rangle = 1$, and set $h_1 := v$ and $c_1 := 2v^*$. We extend $\mathcal{D}_{\text{sc}}^{\check{\mathbb{A}}}$ to a root datum

$$\mathcal{D}^+ := (\{0, 1\}, \mathbb{A}, \Lambda, (c_0, c_1), (h_0, h_1))$$

associated to \mathbb{A} by setting $h_0 := -v$ and $c_0 := -2v^*$. As $h_0 + h_1 = 0$ and $c_0 + c_1 = 0$, we see that \mathcal{D}^+ is neither free nor cofree.

The existence of Kac-Moody root data associated to GCM's of affine type that are neither free nor cofree is a problematic obstacle that we occasionally have to avoid in Part II. The exact undesirable property of these root data is the content of Lemma 2.6.2.

³As discussed in Convention 2.1.2, our choice of ordering on I ensures that $a_0^\vee = 1$ in all cases, so the expression for h_0 is always well-defined. However, if $\Gamma(\mathbb{A})$ is of twisted type $A_{2n}^{(2)}$ for some $n \in \mathbb{N}$ (which has $n+1$ vertices for $n \geq 2$) where $a_0 = 2$, then $a_i = 2$ for all $i \in \{1, \dots, n-1\}$ but $a_n = 1$. Hence, c_0 cannot be defined in this way if $c_n \in \Lambda$ is not divisible by 2, so for this type, we need to impose the assumption that $c_n = 2c'_n$ for some $c'_n \in \Lambda$. For example, this restriction excludes the case $\check{\mathcal{D}} = \mathcal{D}_{\text{min}}^{\check{\mathbb{A}}}$ where c_n is contained in a \mathbb{Z} -basis of Λ .

To conclude this section and prepare for the definition of Kac-Moody algebras in the next one, we introduce several notions that a priori only depend on the index set I of a GCM \mathbb{A} and not on the matrix itself. However, they are naturally related to any Kac-Moody root datum associated to \mathbb{A} . For the moment, our interest lies in the last part of the following definition, but the relevance of the other notions and their connections to the GCM \mathbb{A} will soon be apparent.

Definition 2.2.6. Let $I = \{1, \dots, n\}$ for some $n \in \mathbb{N}_0$ be a finite set and define two abstract sets $\Phi_0 := \{\alpha_i \mid i \in I\}$ and $\Phi_0^\vee := \{\alpha_i^\vee \mid i \in I\}$.

- (i) The elements of Φ_0 (resp. Φ_0^\vee) are called *simple roots* (resp. *simple coroots*).
- (ii) The free abelian groups

$$Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i \quad \text{and} \quad Q^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee \quad (2.2.6)$$

are called the *root lattice* and the *coroot lattice*. We also define

$$Q_\pm := \pm \bigoplus_{i \in I} \mathbb{N}_0 \alpha_i \quad \text{and} \quad Q_\pm^\vee := \pm \bigoplus_{i \in I} \mathbb{N}_0 \alpha_i^\vee \quad (2.2.7)$$

and introduce a partial order on Q (resp. Q^\vee) by declaring $\alpha \leq \beta$ if and only if $\beta - \alpha \in Q_+$ (resp. Q_+^\vee).

- (iii) For an element $\alpha = \sum_{i \in I} k_i \alpha_i \in Q$ (resp. $\alpha = \sum_{i \in I} k_i \alpha_i^\vee \in Q^\vee$) with $k_1, \dots, k_n \in \mathbb{Z}$, the *support* and *height* of α are defined by

$$I(\alpha) := \{i \in I : k_i \neq 0\} \quad \text{and} \quad \text{ht}(\alpha) := \sum_{i \in I} k_i. \quad (2.2.8)$$

- (iv) Given a Kac-Moody root datum $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$, the \mathbb{Z} -linear maps

$$\begin{aligned} c : Q &\rightarrow \Lambda & h : Q^\vee &\rightarrow \Lambda^\vee \\ \sum_{i \in I} k_i \alpha_i &\mapsto \sum_{i \in I} k_i c_i & \text{and} & \sum_{i \in I} k_i \alpha_i^\vee \mapsto \sum_{i \in I} k_i h_i \end{aligned} \quad (2.2.9)$$

are called the *character map* and *cocharacter map* of \mathcal{D} . For $\alpha \in Q$ (resp. $\alpha^\vee \in Q^\vee$), we will often write $c_\alpha := c(\alpha) \in \Lambda$ (resp. $h_\alpha := h(\alpha^\vee) \in \Lambda^\vee$).

If \mathcal{D} is free (resp. cofree), then c (resp. h) is injective and if \mathcal{D} is adjoint (resp. coadjoint), then c (resp. h) is surjective. Thus, for the simply connected root datum $\mathcal{D}_{\text{sc}}^\mathbb{A}$, the cocharacter map is an isomorphism and the coroot lattice Q^\vee can be identified with the free \mathbb{Z} -module Λ_{sc}^\vee by mapping α_i^\vee to h_i for all $i \in I$. Similarly, for the minimal adjoint root datum $\mathcal{D}_{\text{min}}^\mathbb{A}$, the character map is an isomorphism and the root lattice Q can be identified with Λ_{min} . Note, however, that Q and Q^\vee are considered to be abstract abelian groups and not dual \mathbb{Z} -modules. For any root datum \mathcal{D} , the cocharacter map $h : Q^\vee \rightarrow \Lambda^\vee$ can be viewed as a “morphism” of Kac-Moody root data in the sense of Definition 3.6.1.

2.3 Kac-Moody algebras

With the notion of Kac-Moody root data at hand, we now proceed to define Kac-Moody algebras. Historically, these Lie algebras were first introduced by V.G. Kac and R. Moody in the 1960’s and the definition in [Kac90] essentially coincides with ours in the case of the field $\mathbb{K} = \mathbb{C}$ and the root datum $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A}$, which explains the notation. Our notion of a Kac-Moody algebra is slightly more general, but the analogy to Serre’s presentation of a finite-dimensional, complex, semisimple Lie algebra in terms of its Cartan matrix is still clearly visible. In [MP95], Kac-Moody algebras are treated from the perspective of “contragredient Lie algebras” that we will not follow. As stated in the notational conventions of this thesis, we assume all fields to have characteristic zero.

Definition 2.3.1. Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum and let \mathbb{K} be a field. The (split) Kac-Moody algebra of type \mathcal{D} over \mathbb{K} is the free Lie algebra $\mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ over \mathbb{K} generated by the \mathbb{K} -vector space $\mathfrak{h}_{\mathcal{D}}(\mathbb{K}) := \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{K}$ and $2n$ symbols $(e_i)_{i \in I}, (f_i)_{i \in I}$ subject to the following relations:

$$[h, h'] = 0 \quad \text{for } h, h' \in \mathfrak{h}_{\mathcal{D}}(\mathbb{K}) \quad (\text{KMA1})$$

$$[h, e_i] = \langle c_i, h \rangle e_i \text{ and } [h, f_i] = -\langle c_i, h \rangle f_i \quad \text{for } h \in \mathfrak{h}_{\mathcal{D}}(\mathbb{K}) \text{ and } i \in I \quad (\text{KMA2})$$

$$[e_i, f_j] = -\delta_{ij} h_i \quad \text{for } i, j \in I \quad (\text{KMA3})$$

$$\text{ad}(e_i)^{1-a_{ij}} e_j = \text{ad}(f_i)^{1-a_{ij}} f_j = 0 \quad \text{for } i, j \in I \text{ with } i \neq j \quad (\text{KMA4})$$

We call $\mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ the *Cartan subalgebra* and $(e_i)_{i \in I}, (f_i)_{i \in I}$ the *Chevalley generators* of $\mathfrak{g}_{\mathcal{D}}(\mathbb{K})$.

Remark 2.3.2. Let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a GCM of rank $\ell \leq n$. Our definition of Kac-Moody algebras differs from those in the standard references [Mar18] and [Kac90] in the following ways:

- (i) In [Mar18, Definition 3.17], the author defines a complex Kac-Moody algebra, denoted by $\mathfrak{g}(\mathbb{A})$ there, by first constructing a “realization” of \mathbb{A} , which is a certain pairing between the simple roots and coroots that is modelled on a complex vector space \mathfrak{h} of dimension $2n - \ell$, and then forming the same presentation as in Definition 2.3.1 over the field \mathbb{C} . He proceeds to develop the basic theory of $\mathfrak{g}(\mathbb{A})$, but the language of Kac-Moody root data is only incorporated much later in the book. In [Mar18, Definition 7.13], he introduces a complex Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}$ with Cartan subalgebra $\mathfrak{h}_{\mathcal{D}} := \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}$ for any Kac-Moody root datum \mathcal{D} associated to \mathbb{A} and asserts that, by inspecting the proofs, all results of earlier chapters that were proved for $\mathfrak{g}(\mathbb{A})$ also hold for $\mathfrak{g}_{\mathcal{D}}$ in a very similar way. The only exception arises when the statement involves the dimension of the Cartan subalgebra $\mathfrak{h}_{\mathcal{D}}$, which need not be $2n - \ell$ in general, in which case it can easily be adapted. Intuitively, one may view $\mathfrak{g}_{\mathcal{D}}$ as a “degree-zero modification” of $\mathfrak{g}(\mathbb{A})$ in the sense that the two Kac-Moody algebras only differ in the dimensions of their Cartan subalgebras, but otherwise “have the same structure” (cf. Remark 2.3.3 for a more precise statement). This can be interpreted as the natural analogue of the similarity between semisimple and reductive Lie algebras in the Kac-Moody setting. In the important special cases $\mathcal{D} = \mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ and $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$, Marquis’ definitions imply that $\mathfrak{g}_{\mathcal{D}_{\text{Kac}}^{\mathbb{A}}} = \mathfrak{g}(\mathbb{A})$ and $\mathfrak{g}_{\mathcal{D}_{\text{sc}}^{\mathbb{A}}} = [\mathfrak{g}(\mathbb{A}), \mathfrak{g}(\mathbb{A})]$ is the derived algebra of $\mathfrak{g}(\mathbb{A})$. Guided by the approach developed by J. Tits in [Tit87, Section 4], he further explains how to construct appropriate analogues of the complex Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}$ over arbitrary rings (cf. [Mar18, Section 7.2 and 7.3]).

Compared to our terminology, Marquis’ notion of a “realization” of \mathbb{A} coincides with that of a “ \mathbb{C} -realization” of \mathbb{A} in the sense of Definition 2.3.4. More generally, our Definition 2.3.1, which is taken from [R  m02, Section 7.3.1], is obtained by replacing the complex numbers in [Mar18, Definition 7.13] by an arbitrary field \mathbb{K} . As we only consider fields of characteristic zero, this is equivalent to the more abstract construction in [Mar18, Section 7.2 and 7.3]. Hence, given a Kac-Moody root datum \mathcal{D} and a field \mathbb{K} , our definition of the Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ of type \mathcal{D} over \mathbb{K} coincides with the \mathbb{K} -form of Marquis’ complex Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}$. Furthermore, it also follows by inspecting the arguments that the assertions of [Mar18, Chapters 3-6] naturally carry over to arbitrary fields of characteristic zero and are thus applicable in our setting. For the most part, however, we will only be interested in the case $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and transferring the proofs from the complex to the real numbers is straightforward. For fields of positive characteristic, on the other hand, our definition is not suitable because, as we shall see in Section 2.4, arguments in the representation theory of Kac-Moody algebras often involve the use of certain “exponential series” which are a priori not well-defined in that situation.

- (ii) Apart from Tits’ approach, the exposition in [Mar18] further builds upon the work of Kac in [Kac90] who proceeds slightly differently. He begins by defining the same notion of a “realization” of \mathbb{A} on a $(2n - \ell)$ -dimensional complex vector space \mathfrak{h} , but then introduces an auxiliary complex Lie algebra $\tilde{\mathfrak{g}}(\mathbb{A})$ which is constructed by forming the same presentation as in Definition 2.3.1 over the field \mathbb{C} but only imposing the relations (KMA1)-(KMA3).

He shows that there exists a unique maximal ideal $\mathfrak{i} \subset \tilde{\mathfrak{g}}(\mathbb{A})$ that intersects \mathfrak{h} trivially and defines the complex Lie algebra $\tilde{\mathfrak{g}}(\mathbb{A})/\mathfrak{i}$ which is denoted by $\mathfrak{g}(\mathbb{A})$ in [Kac90] and by $\mathfrak{g}(\mathbb{A})_{\text{Kac}}$ in [Mar18]. To be precise, the two authors use slightly different conventions in their definitions of a “realization” of \mathbb{A} and (KMA3) does not contain the minus sign in [Kac90, §1.2], but these choices are insignificant (cf. [Mar18, Remark 3.19]). In contrast to Tits and Marquis, however, Kac neither introduces any notion of root data nor constructs analogues of his Kac-Moody algebras over arbitrary rings.

- (iii) In general, the two definitions of $\mathfrak{g}(\mathbb{A})$ in [Kac90] and [Mar18] are not necessarily equivalent. By [Mar18, Lemma 3.16], the ideal $\mathfrak{i} \subset \tilde{\mathfrak{g}}(\mathbb{A})$ defined by Kac contains the elements

$$\text{ad}(e_i)^{1-a_{ij}}e_j \quad \text{and} \quad \text{ad}(f_i)^{1-a_{ij}}f_j \quad \text{for all } i, j \in I \text{ with } i \neq j. \quad (2.3.1)$$

Therefore, the equations (KMA4), which are often called the *Serre relations*, hold in $\tilde{\mathfrak{g}}(\mathbb{A})/\mathfrak{i}$, so the Kac-Moody algebra $\mathfrak{g}(\mathbb{A})$ defined by Kac is a quotient of Marquis’, but it is not known whether the two always coincide. However, if \mathbb{A} is symmetrizable, the *Gabber-Kac Theorem* (cf. [Kac90, Theorem 9.11]) asserts that the ideal $\mathfrak{i} \subset \tilde{\mathfrak{g}}(\mathbb{A})$ is generated by the elements in (2.3.1). This implies that, apart from the previously mentioned irrelevant conventions, the two definitions of $\mathfrak{g}(\mathbb{A})$ are equivalent in the symmetrizable case.

As a consequence of this and other more elaborate assertions that can exclusively be found in [Kac90], symmetrizability has become a standard assumption in Kac-Moody theory and will also be necessary for the proofs of our main theorems. Many basic results in the field, however, do not need this assumption and can be established for both definitions by the same arguments. Hence, we will not require symmetrizability from the start and explicitly point out when it is necessary.

Returning to Definition 2.3.1, if the Kac-Moody root datum \mathcal{D} and the field \mathbb{K} are clear from the context, we will often simply write $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ and $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$. Using this notation, the Kac-Moody algebra \mathfrak{g} of type \mathcal{D} over \mathbb{K} has a natural gradation by the root lattice

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{u}_{\alpha} \quad (2.3.2)$$

induced by the assignment $\deg(e_i) = -\deg(f_i) := \alpha_i$ for $i \in I$ and $\deg(h) := 0$ for $h \in \mathfrak{h}$. Here, we have $\mathfrak{u}_0 = \mathfrak{h}$. The elements of $\Phi^{\infty} := \{\alpha \in Q \setminus \{0\} : \mathfrak{u}_{\alpha} \neq \{0\}\}$ are called the *roots* of \mathfrak{g} ,⁴ and for $\alpha \in \Phi^{\infty}$, the non-trivial subspace $\mathfrak{u}_{\alpha} \subset \mathfrak{g}$ is the *root space* belonging to α . Hence, \mathfrak{g} has the following *root space decomposition*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^{\infty}} \mathfrak{u}_{\alpha} \quad (2.3.3)$$

The defining relations (KMA1)-(KMA4) imply that every root is either contained in Q_+ or Q_- . More precisely, for every $\alpha \in Q_+$ (resp. $\alpha \in Q_-$), the root space $\mathfrak{u}_{\alpha} \subset \mathfrak{g}$ is spanned by all “iterated brackets” of the form

$$[e_{i_1}, \dots, e_{i_k}] := \text{ad}(e_{i_1}) \dots \text{ad}(e_{i_{k-1}})e_{i_k} \quad (\text{resp. } [f_{i_1}, \dots, f_{i_k}]),$$

where $k = \text{ht}(\alpha)$ (resp. $\text{ht}(-\alpha)$) and (i_1, \dots, i_k) runs over all k -tuples in I^k such that $\sum_{j=1}^k \alpha_{i_j} = \alpha$ (resp. $-\alpha$). In particular, it follows that

$$[\mathfrak{u}_{\alpha}, \mathfrak{u}_{\beta}] \subset \mathfrak{u}_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \Phi^{\infty}, \quad (2.3.4)$$

which is $\{0\}$ if $\alpha + \beta \notin \Phi^{\infty}$. Moreover, the set Φ_0 of simple roots is contained in Φ^{∞} since for every $i \in I$, the root space belonging to α_i is $\mathfrak{u}_{\alpha_i} = \mathbb{K}e_i$. Likewise, we have $-\Phi_0 \subset \Phi^{\infty}$ with $\mathfrak{u}_{-\alpha_i} := \mathfrak{u}_{-\alpha_i} = \mathbb{K}f_i$ for all $i \in I$. Every root is a \mathbb{Z} -linear combination of simple roots with

⁴In the literature on Kac-Moody algebras, it is customary to denote the set of roots by Φ rather than Φ^{∞} . We do not follow this convention on purpose because we want to reserve the letter Φ for the subset of real roots (cf. Definition 2.5.2) which will occur much more often than the entire set Φ^{∞} . Our notation is motivated by the fact that Φ^{∞} is the “root string closure” of the real roots (cf. [Mar18, Exercise 6.17]).

coefficients of the same sign and due to the symmetry in (KMA4), the negative of any root is also a root. Hence, the two sets $\Phi_{\pm}^{\infty} := \Phi^{\infty} \cap Q_{\pm}$ satisfy $\Phi_{+}^{\infty} = -\Phi_{-}^{\infty}$ and $\Phi^{\infty} = \Phi_{+}^{\infty} \sqcup \Phi_{-}^{\infty}$. Elements of Φ_{+}^{∞} (resp. Φ_{-}^{∞}) are called *positive* (resp. *negative*) *roots*. The linear subspaces

$$\mathfrak{u}_{\pm} := \bigoplus_{\alpha \in \Phi_{\pm}^{\infty}} \mathfrak{u}_{\alpha} \quad (2.3.5)$$

are subalgebras of \mathfrak{g} and they give rise to the *triangular decomposition*

$$\mathfrak{g} = \mathfrak{u}_{+} \oplus \mathfrak{h} \oplus \mathfrak{u}_{-}. \quad (2.3.6)$$

Furthermore, the relations (KMA1)-(KMA4) imply that the map $e_i \mapsto f_i$, $f_i \mapsto e_i$ ($i \in I$) and $h \mapsto -h$ ($h \in \mathfrak{h}$) uniquely extends to an involutive \mathbb{K} -linear automorphism $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ that is called the *Chevalley involution* of \mathfrak{g} . It stabilizes $\mathfrak{h} = \mathfrak{u}_0$ and maps \mathfrak{u}_{α} bijectively to $\mathfrak{u}_{-\alpha}$ for every $\alpha \in \Phi^{\infty}$. In particular, it exchanges \mathfrak{u}_{+} and \mathfrak{u}_{-} .

As a consequence of (KMA2), the Cartan subalgebra $\mathfrak{h} = \mathfrak{u}_0 \subset \mathfrak{g}$ acts diagonally on \mathfrak{g} by the adjoint representation and we have $\mathfrak{u}_{\alpha} \subset \{x \in \mathfrak{g} : [h, x] = \langle c_{\alpha}, h \rangle x \ \forall h \in \mathfrak{h}\}$ for all $\alpha \in \Phi^{\infty} \cup \{0\}$, but this inclusion is strict if the character map $c : Q \rightarrow \Lambda$ is not injective or, equivalently, \mathcal{D} is not free. Hence, \mathfrak{g} also has an eigenspace decomposition for the adjoint action of \mathfrak{h} , namely

$$\mathfrak{g} = \bigoplus_{\mu \in \mathfrak{h}^*} \bar{\mathfrak{u}}_{\mu} \quad \text{where} \quad \bar{\mathfrak{u}}_{\mu} := \bigoplus_{\alpha \in \Phi^{\infty} \cup \{0\} : c_{\alpha} = \mu} \mathfrak{u}_{\alpha}, \quad (2.3.7)$$

which is coarser than the root space decomposition if \mathcal{D} is not free. Finally, it also follows from (KMA1) and (KMA2) that if $h \in \mathfrak{h}$ satisfies $\langle c_i, h \rangle = 0$ for all $i \in I$, then h lies in the center \mathfrak{z} of \mathfrak{g} . By the proof of [Kac90, Proposition 1.6] and Remark 2.3.2, the converse is true under the assumption of symmetrizability where the Gabber-Kac Theorem holds, i.e.

$$\mathfrak{z} = \{h \in \mathfrak{h} : \langle c_i, h \rangle = 0 \ \forall i \in I\} \quad \text{if } \mathbb{A} \text{ is symmetrizable.} \quad (2.3.8)$$

Remark 2.3.3. Given a Kac-Moody root datum \mathcal{D} and a field \mathbb{K} , the structure of the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ “mostly” only depends on the GCM \mathbb{A} appearing in \mathcal{D} and not on the specific root datum in the following sense:

- (i) If \mathcal{D}' is another Kac-Moody root datum associated to \mathbb{A} , the corresponding Kac-Moody algebra $\mathfrak{g}' := \mathfrak{g}_{\mathcal{D}'}(\mathbb{K})$ over \mathbb{K} only “differs from \mathfrak{g} in degree zero” in the sense that \mathfrak{g} and \mathfrak{g}' have the same roots and their root spaces are naturally isomorphic. Indeed, the root spaces are spanned by iterated brackets of the Chevalley generators $(e_i)_{i \in I}$ or $(f_i)_{i \in I}$ and by (KMA4), the relations among them only depend on \mathbb{A} . Upon identifying the Chevalley generators of \mathfrak{g} and \mathfrak{g}' , it follows that $\mathfrak{u}_{\alpha} \cong \mathfrak{u}'_{\alpha}$ for all $\alpha \in Q \setminus \{0\}$, so \mathfrak{g} and \mathfrak{g}' only differ in the Cartan subalgebras $\mathfrak{h} = \mathfrak{u}_0$ and $\mathfrak{h}' = \mathfrak{u}'_0$. In addition, Φ^{∞} only depends on \mathbb{A} and may simply be called the set of *roots of \mathbb{A}* . Occasionally, we will also denote it by $\Phi^{\infty}(\mathbb{A})$ when the GCM is not clear from the context.
- (ii) If $i, j \in I$ satisfy $a_{ij} = a_{ji} = 0$, the corresponding Chevalley generators of \mathfrak{g} commute by (KMA4). Hence, if $\mathbb{A}_1, \dots, \mathbb{A}_m$ denote the indecomposable components of \mathbb{A} indexed by $I_1, \dots, I_m \subset I$, then $[e_i, e_j] = [f_i, f_j] = 0$ whenever $i, j \in I$ are not both contained in I_k for some $k \in \{1, \dots, m\}$. From the description of the root spaces in terms of iterated brackets, it therefore follows that $\mathfrak{u}_{\alpha} \subset \mathfrak{g}$ can only be non-trivial for a given $\alpha \in Q$ if $I(\alpha) \subset I$ is entirely contained in I_k for some $k \in \{1, \dots, m\}$. In this case, it is a root of \mathbb{A}_k and, conversely, every root of these submatrices defines a root of \mathbb{A} , so that

$$\Phi^{\infty}(\mathbb{A}) = \coprod_{k=1}^m \Phi^{\infty}(\mathbb{A}_k). \quad (2.3.9)$$

Given any $\alpha \in \Phi^{\infty}(\mathbb{A})$, the same argument even shows that the full subdiagram of $\Gamma(\mathbb{A})$ with vertices in $I(\alpha) \subset I$ is connected. Furthermore, if $\mathcal{D}_1, \dots, \mathcal{D}_m$ denote the indecomposable

components of \mathcal{D} and $\mathfrak{g}^k := \mathfrak{g}_{\mathcal{D}^k}(\mathbb{K})$ for every $k \in \{1, \dots, m\}$, then \mathfrak{g}^k can be identified with the subalgebra of \mathfrak{g} spanned by \mathfrak{h} and all u_α with $\alpha \in \Phi^\infty(\mathbb{A}_k)$, so we conclude that

$$\mathfrak{g} = \sum_{k=1}^m \mathfrak{g}^k \quad \text{with} \quad \mathfrak{g}^k \cap \mathfrak{g}^l = \mathfrak{h} \text{ for all } k \neq l \in \{1, \dots, m\}. \quad (2.3.10)$$

- (iii) As we only work over fields, the notion of cotorsion-freeness is largely irrelevant on the level of Kac-Moody algebras. More precisely, the \mathbb{K} -linear subspace of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ spanned by $(h_i)_{i \in I}$ does not depend on the torsion part of $\Lambda^\vee / \sum_{i \in I} \mathbb{Z}h_i$. This has the effect that distinct root data can define the same Kac-Moody algebra. For example, if \mathbb{A} is invertible, the minimal adjoint Kac-Moody root datum does not necessarily coincide with the simply connected one since the former need not be cotorsion-free. Nevertheless, the associated Kac-Moody algebras are identical because $\mathcal{D}_{\min}^\mathbb{A}$ is obtained from $\mathcal{D}_{\text{Kac}}^\mathbb{A} = \mathcal{D}_{\text{sc}}^\mathbb{A}$ by replacing Λ_{sc} and Λ_{sc}^\vee with two finite-index subgroups Λ_{\min} and Λ_{\min}^\vee and restricting the elements $(c_i)_{i \in I}$ and $(h_i)_{i \in I}$ to these submodules. Hence, the Cartan subalgebras $\mathfrak{h}_{\mathcal{D}_{\text{sc}}^\mathbb{A}}(\mathbb{K})$ and $\mathfrak{h}_{\mathcal{D}_{\min}^\mathbb{A}}(\mathbb{K})$ have the same dimension and the Lie algebra structures of $\mathfrak{g}_{\mathcal{D}_{\text{sc}}^\mathbb{A}}(\mathbb{K})$ and $\mathfrak{g}_{\mathcal{D}_{\min}^\mathbb{A}}(\mathbb{K})$ coincide as the restrictions are undone by the tensor products as well.

However, cotorsion-freeness is an important notion in the study of Kac-Moody groups to be discussed in Chapter 3. There, it allows to distinguish groups defined by different root data whose associated Kac-Moody algebras are isomorphic, even though the groups are only defined over fields. A very simple instance of this phenomenon occurs for the GCM $\mathbb{A}^0 = (2)$. Given any field \mathbb{K} , the Kac-Moody algebras of type $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$ and $\mathcal{D}_{\min}^{\mathbb{A}^0}$ over \mathbb{K} are both isomorphic to $\mathfrak{sl}_2(\mathbb{K})$, whereas the corresponding Kac-Moody group over \mathbb{K} is isomorphic to $\text{SL}_2(\mathbb{K})$ in the simply connected case and to $\text{PGL}_2(\mathbb{K})$ in the minimal adjoint case. We treat this example in detail in Section 3.3.

- (iv) If \mathcal{D} is free and cofree, the character and cocharacter maps $c : Q \rightarrow \Lambda$ and $h : Q^\vee \rightarrow \Lambda^\vee$ are injective, so we may view Q and Q^\vee as subsets of Λ and Λ^\vee and, in particular, identify the simple coroots $\Phi_0^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ with elements of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and the simple roots $\Phi_0 = \{\alpha_1, \dots, \alpha_n\}$ with elements of its dual space \mathfrak{h}^* . As discussed in Remark 2.2.2, a root datum can only have this property if its rank is at least $2n - \ell$ and this minimal value is attained by $\mathcal{D}_{\text{Kac}}^\mathbb{A}$. Thus, it gives rise to the configuration below which is the classical approach towards defining Kac-Moody algebras taken in [Kac90] and [Mar18] in the case $\mathbb{K} = \mathbb{C}$.

Definition 2.3.4. Let $\mathbb{A} = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$ be a GCM of rank $\ell \leq n$ and let \mathbb{K} be a field. A \mathbb{K} -realization of \mathbb{A} is a triple $(\mathfrak{h}, \Phi_0, \Phi_0^\vee)$ with the following properties:

- (i) \mathfrak{h} is a \mathbb{K} -vector space of dimension $2n - \ell$.
- (ii) The sets $\Phi_0 = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Phi_0^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are linearly independent.
- (iii) $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$ for all $i, j \in I$.

The following simple result shows that the Kac-Moody algebras constructed from free and cofree Kac-Moody root data differ from those defined by \mathbb{K} -realizations only in a minor way. In its formulation, we use the notation suggested by Remark 2.3.3 (iv).

Lemma 2.3.5. Let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a GCM of rank $\ell \leq n$ and let \mathcal{D} be a free and cofree Kac-Moody root datum of rank $d \geq 2n - \ell$ associated to \mathbb{A} . Given any field \mathbb{K} , denote by $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ the Kac-Moody algebra of type \mathcal{D} over \mathbb{K} , by $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ its Cartan subalgebra and define the linear subspace $\mathfrak{h}' := \bigoplus_{i \in I} \mathbb{K}\alpha_i^\vee \subset \mathfrak{h}$.

- (i) There exists a linear subspace $\tilde{\mathfrak{h}} \subset \mathfrak{h}$ of dimension $2n - \ell$ containing \mathfrak{h}' and a complementary subspace $\mathfrak{h}_0 \subset \mathfrak{h}$ of dimension $d - 2n + \ell$ such that $(\tilde{\mathfrak{h}}, \Phi_0|_{\tilde{\mathfrak{h}}}, \Phi_0^\vee)$ is a \mathbb{K} -realization of \mathbb{A} and $\alpha_i|_{\mathfrak{h}_0} = 0$ for all $i \in I$.
- (ii) The subalgebra $\tilde{\mathfrak{g}}$ of \mathfrak{g} that is generated by $\tilde{\mathfrak{h}}$ and all Chevalley generators $(e_i)_{i \in I}, (f_i)_{i \in I}$ is isomorphic to the Kac-Moody algebra of type $\mathcal{D}_{\text{Kac}}^\mathbb{A}$ over \mathbb{K} and $\mathfrak{g} \cong \tilde{\mathfrak{g}} \oplus \mathfrak{h}_0$ is a trivial central extension of $\tilde{\mathfrak{g}}$.

Proof. As in 2.1.1, we may w.l.o.g. assume that $\mathbb{A} = \begin{pmatrix} A_1 & A_2 \end{pmatrix}$, where A_1 is an $(n \times \ell)$ -matrix of rank ℓ . Using that \mathcal{D} is free and cofree, we view Φ_0^\vee and Φ_0 as linearly independent subsets of \mathfrak{h} and \mathfrak{h}^* . In particular, we may choose an (ordered) \mathbb{K} -basis $\mathcal{B}_1 := \{\alpha_1^\vee, \dots, \alpha_n^\vee, h_{n+1}, \dots, h_d\} \subset \mathfrak{h}$ containing the simple coroots and define the following linear map:

$$f : \mathfrak{h} \rightarrow \mathbb{K}^n$$

$$h \mapsto (\langle \alpha_1, h \rangle, \dots, \langle \alpha_n, h \rangle)^T$$

Since the simple roots are linearly independent on \mathfrak{h} , the map f is surjective. Its matrix representation with respect to \mathcal{B}_1 and the standard basis of \mathbb{K}^n is the $(n \times d)$ -matrix $B_1 := \begin{pmatrix} A_1^T & C \end{pmatrix}$, where C is some matrix of size $n \times (d - n)$. As $\text{rank}(B_1) = n$ and $\text{rank}(A_1^T) = \ell$, it follows that $\text{rank}(C) = n - \ell$ and $\dim(\ker(C)) = (d - n) - (n - \ell) = d - 2n + \ell$. By performing column operations, we can transform C into $\begin{pmatrix} C' & 0 \end{pmatrix}$, where C' is some matrix of size $n \times (n - \ell)$ and rank $n - \ell$ and the zero matrix has size $n \times (d - 2n + \ell)$. This process replaces the last $d - n$ basis vectors $h_{n+1}, \dots, h_d \in \mathcal{B}_1$ by other linearly independent vectors $h'_{n+1}, \dots, h'_d \in \mathfrak{h}$ that also span a complement to \mathfrak{h}' in \mathfrak{h} . Thus, $\mathcal{B}_2 := \{\alpha_1^\vee, \dots, \alpha_n^\vee, h'_{n+1}, \dots, h'_d\}$ is a basis of \mathfrak{h} as well and the matrix representation of f with respect to \mathcal{B}_2 and the standard basis of \mathbb{K}^n is

$$B_2 := \begin{pmatrix} A_1^T & C' & 0 \end{pmatrix} = \begin{pmatrix} A_1^T & C'_1 & 0 \\ A_2^T & C'_2 & 0 \end{pmatrix}, \quad \text{where } C' = \begin{pmatrix} C'_1 \\ C'_2 \end{pmatrix}.$$

The assumption that $\text{rank}(A_1^T) = \ell$ implies that we can add suitable linear combinations of the columns of A_1^T to those of C'_1 to reduce C'_1 to the zero matrix. At the same time, this turns the square matrix C'_2 of size $(n - \ell) \times (n - \ell)$ into another such matrix C''_2 of rank $n - \ell$. Using further column operations, C''_2 can therefore be transformed to $\mathbb{I}_{n-\ell}$. These two steps amount to replacing the $n - \ell$ vectors $h'_{n+1}, \dots, h'_{2n-\ell} \in \mathcal{B}_2$ by other vectors $h''_{n+1}, \dots, h''_{2n-\ell} \in \mathfrak{h}$ with the property that $\mathcal{B}_3 := \{\alpha_1^\vee, \dots, \alpha_n^\vee, h''_{n+1}, \dots, h''_{2n-\ell}, h'_{2n-\ell+1}, \dots, h'_d\}$ is also a basis of \mathfrak{h} and the matrix representation of f with respect to \mathcal{B}_3 and the standard basis of \mathbb{K}^n is

$$B_3 := \begin{pmatrix} A_1^T & 0 & 0 \\ A_2^T & \mathbb{I}_{n-\ell} & 0 \end{pmatrix}.$$

We now define the linear subspaces

$$\tilde{\mathfrak{h}} := \bigoplus_{i \in I} \mathbb{K} \alpha_i^\vee \oplus \bigoplus_{r=n+1}^{2n-\ell} \mathbb{K} h''_r \quad \text{and} \quad \mathfrak{h}_0 := \bigoplus_{r=2n-\ell+1}^d \mathbb{K} h'_r.$$

Then $\mathfrak{h} = \tilde{\mathfrak{h}} \oplus \mathfrak{h}_0$, every simple root vanishes on \mathfrak{h}_0 and $(\tilde{\mathfrak{h}}, \Phi_0|_{\tilde{\mathfrak{h}}}, \Phi_0^\vee)$ is a \mathbb{K} -realization of \mathbb{A} , which proves the first assertion and (KMA2) shows that $\mathfrak{g} \cong \tilde{\mathfrak{g}} \oplus \mathfrak{h}_0$ is a trivial central extension. Finally, by comparing B_3 with the matrix B in the construction of $\mathcal{D}_{\text{Kac}}^\mathbb{A}$ in Example 2.2.3 (i), it follows that the Lie algebra structures of $\tilde{\mathfrak{g}}$ and $\mathfrak{g}_{\mathcal{D}_{\text{Kac}}^\mathbb{A}}(\mathbb{K})$ are identical which establishes the desired isomorphism. \square

It is also not difficult to decompose the subalgebra $\tilde{\mathfrak{g}} \subset \mathfrak{g}$ in the previous proof further according to the indecomposable components of \mathbb{A} , but we omit the precise statement. Instead, we return to the situation where \mathcal{D} is an arbitrary Kac-Moody root datum, not necessarily free or cofree.

Notation 2.3.6. Let $J \subset I$ be any subset and consider the J -restriction \mathcal{D}_J of \mathcal{D} . Given a field \mathbb{K} , the Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}_J}(\mathbb{K})$ of type \mathcal{D}_J over \mathbb{K} can be naturally identified with the subalgebra \mathfrak{g}_J of \mathfrak{g} generated by \mathfrak{h} and the Chevalley generators $(e_j)_{j \in J}, (f_j)_{j \in J}$. It has the same Cartan subalgebra as \mathfrak{g} and its Chevalley involution is the restriction of $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ to \mathfrak{g}_J . The set of roots $\Phi_J^\infty := \Phi^\infty(\mathbb{A}_J)$ coincides with the set of all $\alpha \in \Phi^\infty(\mathbb{A})$ with $I(\alpha) \subset J$. The corresponding set of simple roots will be denoted by $\Phi_{J,0} := \{\alpha_j \mid j \in J\}$ and we define all related symbols, such as $\Phi_{J,0}^\vee, Q_J, Q_J^\vee, \Phi_{J,\pm}^\infty, \mathfrak{u}_{J,\pm}$ and so on, in the obvious ways. Hence, the root space decomposition and the triangular decomposition of \mathfrak{g}_J take the following form:

$$\mathfrak{g}_J = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_J^\infty} \mathfrak{u}_\alpha = \mathfrak{u}_{J,+} \oplus \mathfrak{h} \oplus \mathfrak{u}_{J,-} \quad (2.3.11)$$

Example 2.3.7. Parallel to the classification of indecomposable GCM's into finite, affine and indefinite type, Kac-Moody algebras also fall into three classes with wildly different properties.

- (i) If \mathbb{A} is a GCM of finite type (or, more generally, spherical), it is the Cartan matrix of a finite-dimensional, complex, (semi)simple Lie algebra $\bar{\mathfrak{g}}$. Let $\bar{\mathfrak{h}}$ be a Cartan subalgebra of $\bar{\mathfrak{g}}$ in the usual sense of Lie theory, then $\Phi^\infty(\mathbb{A})$ can be identified with the set of roots of the finite root system $\Phi(\bar{\mathfrak{g}}, \bar{\mathfrak{h}}) \subset \bar{\mathfrak{h}}^*$ arising from the adjoint action of $\bar{\mathfrak{h}}$ on $\bar{\mathfrak{g}}$. In the case of the Kac-Moody root datum $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A} = \mathcal{D}_{\text{sc}}^\mathbb{A}$, the Kac-Moody algebra $\mathfrak{g}_\mathcal{D}(\mathbb{K})$ of type \mathcal{D} over any field \mathbb{K} is the split \mathbb{K} -form of $\bar{\mathfrak{g}}$. Hereby, we mean the tensor product $\bar{\mathfrak{g}}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{K}$, where $\bar{\mathfrak{g}}_\mathbb{Z}$ is the canonical \mathbb{Z} -form of $\bar{\mathfrak{g}}$ constructed from a ‘‘Chevalley basis’’ (cf. [Ste68, §1]). For example, if

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{with} \quad \Gamma(\mathbb{A}) = \bullet \text{---} \bullet \text{---} \bullet$$

is of finite type C_3 and $\mathcal{D} = \mathcal{D}_{\text{sc}}^\mathbb{A}$, then $\bar{\mathfrak{g}} = \mathfrak{sp}_6(\mathbb{C})$ and $\mathfrak{g}_\mathcal{D}(\mathbb{K}) = \mathfrak{sp}_6(\mathbb{K})$ is the set of (6×6) -matrices that are skew-symmetric with respect to a non-degenerate alternating bilinear form on \mathbb{K}^6 and the same is true for $\mathcal{D} = \mathcal{D}_{\text{min}}^\mathbb{A}$ (cf. Remark 2.3.3 (iii)). Given any other root datum \mathcal{D} associated to \mathbb{A} , which is necessarily free and cofree, it follows from Lemma 2.3.5 that $\mathfrak{g}_\mathcal{D}(\mathbb{K})$ is a trivial central extension of $\bar{\mathfrak{g}}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{K}$ and therefore a \mathbb{K} -split reductive Lie algebra.

- (ii) Let $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ be a GCM of ‘‘untwisted’’ affine type, so that its Dynkin diagram $\Gamma(\mathbb{A})$ is one of the graphs in Table 2.2. Its reduced matrix $\mathring{\mathbb{A}} \in \mathbb{Z}^{n \times n}$ is the Cartan matrix of a finite-dimensional, complex, simple Lie algebra $\mathring{\mathfrak{g}}$, so the subset $\mathring{\Phi} := \Phi^\infty(\mathring{\mathbb{A}}) \subset \Phi^\infty(\mathbb{A})$ is finite. Let $a_0, \dots, a_n \in \mathbb{N}$ denote the numerical labels of the vertices in $\Gamma(\mathbb{A})$ and set $\delta := \sum_{i \in I} a_i \alpha_i \in Q_+$, then it can be shown that

$$\Phi^\infty(\mathbb{A}) = \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\} \cup \{k\delta + \beta \mid k \in \mathbb{Z}, \beta \in \mathring{\Phi}\}$$

and such a root is positive if and only if $k > 0$ or $k = 0$ and $\beta \in \mathring{\Phi}_+$. For every ring R , we set $\mathring{\mathfrak{g}}(R) := \mathring{\mathfrak{g}}_\mathbb{Z} \otimes_\mathbb{Z} R$, where $\mathring{\mathfrak{g}}_\mathbb{Z}$ is the canonical \mathbb{Z} -form of $\mathring{\mathfrak{g}}$. Given an arbitrary Kac-Moody root datum \mathcal{D} associated to \mathbb{A} and any field \mathbb{K} , the Kac-Moody algebra $\mathfrak{g}_\mathcal{D}(\mathbb{K})$ of type \mathcal{D} over \mathbb{K} is closely related to the ‘‘loop algebra’’ $\mathring{\mathfrak{g}}(\mathbb{K}[t, t^{-1}])$. Here, we denote by $\mathbb{K}[t, t^{-1}]$ the ring of Laurent polynomials with coefficients in \mathbb{K} and the fact that this is not a field is one of the main reasons why the theory of Kac-Moody algebras is usually developed over arbitrary rings. For instance, if \mathcal{D} is the unique extension of $\mathcal{D}_{\text{sc}}^\mathbb{A}$ to a Kac-Moody root datum associated to \mathbb{A} (cf. Example 2.2.5), which has rank $n = |I|$ and is neither free nor cofree, then $\mathfrak{g}_\mathcal{D}(\mathbb{K}) \cong \mathring{\mathfrak{g}}(\mathbb{K}[t, t^{-1}])$. If $\mathcal{D} = \mathcal{D}_{\text{sc}}^\mathbb{A}$, which has rank $n + 1$ and is cofree but not free, then $\mathfrak{g}_\mathcal{D}(\mathbb{K})$ is a one-dimensional central extension of $\mathring{\mathfrak{g}}(\mathbb{K}[t, t^{-1}])$. If $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A}$, which is free and cofree with rank $n + 2$, then $\mathfrak{g}_\mathcal{D}(\mathbb{K})$ is another one-dimensional semidirect extension of the previous Lie algebra. We refer to [Kac90, Chapter 6 and 7] and [Mar18, Section 5.3] for a much more detailed discussion of untwisted affine Kac-Moody algebras and proofs of all the assertions made here.

While the ‘‘smallest’’ Kac-Moody algebra associated to \mathbb{A} , i.e. the loop algebra $\mathring{\mathfrak{g}}(\mathbb{K}[t, t^{-1}])$ constructed from the finite-dimensional, complex, simple Lie algebra $\mathring{\mathfrak{g}}$, can still be realized as a matrix algebra over a suitable ring, this is no longer true for the extensions just described. Thus, for computations in affine Kac-Moody algebras defined by free or cofree root data, it is necessary to understand how the extensions alter the Lie algebra structure of the underlying loop algebra. For the root datum $\mathcal{D}_{\text{Kac}}^\mathbb{A}$, this can be described explicitly and the formula can be found in [Kac90, (7.2.2)].

- (iii) We briefly sketch the simplest example of the construction in (ii). Let $\mathbb{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ be of untwisted affine type $A_1^{(1)}$, then $\mathring{\mathbb{A}} = \mathbb{A}^0 = (2)$ is the Cartan matrix of $\mathfrak{sl}_2(\mathbb{C})$, its canonical \mathbb{Z} -form is $\mathfrak{sl}_2(\mathbb{Z})$ and its set of roots is $\mathring{\Phi} = \Phi^\infty(\mathring{\mathbb{A}}) = \{\alpha_1, -\alpha_1\}$. The two vertices in $\Gamma(\mathbb{A})$ are both labelled by 1, so we have $\delta = \alpha_0 + \alpha_1$ and

$$\Phi^\infty(\mathbb{A}) = \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\} \cup \{k\delta + \alpha_1, k\delta - \alpha_1 \mid k \in \mathbb{Z}\}.$$

If we take \mathcal{D}^+ to be the Kac-Moody root datum of rank 1 constructed in Example 2.2.5, i.e. the unique extension of $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ to a root datum associated to \mathbb{A} , the Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}^+}(\mathbb{K})$ is isomorphic to the loop algebra $\mathfrak{sl}_2(\mathbb{K}[t, t^{-1}])$. An explicit isomorphism can be obtained by sending $v = h_1 \in \mathfrak{h}_{\mathcal{D}^+}(\mathbb{K})$, which is a basis for the Cartan subalgebra of $\mathfrak{g}_{\mathcal{D}^+}(\mathbb{K})$, to the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and mapping the Chevalley generators as follows:

$$e_0 \mapsto \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix} \quad e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f_0 \mapsto \begin{pmatrix} 0 & t^{-1} \\ 0 & 0 \end{pmatrix} \quad f_1 \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Under this identification, the Chevalley involution of $\mathfrak{g}_{\mathcal{D}^+}(\mathbb{K})$ corresponds to the automorphism $X(t) \mapsto -X(t^{-1})^T$ of $\mathfrak{sl}_2(\mathbb{K}[t, t^{-1}])$ and the Cartan subalgebra and root spaces of $\mathfrak{g}_{\mathcal{D}^+}(\mathbb{K})$ coincide with the following sets of matrices:

$$\begin{aligned} \mathfrak{h}_{\mathcal{D}^+}(\mathbb{K}) &= \left\{ \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \mid z \in \mathbb{K} \right\} & \mathfrak{u}_{k\delta+\alpha_1} &= \left\{ \begin{pmatrix} 0 & zt^k \\ 0 & 0 \end{pmatrix} \mid z \in \mathbb{K} \right\} \\ \mathfrak{u}_{k\delta} &= \left\{ \begin{pmatrix} zt^k & 0 \\ 0 & -zt^k \end{pmatrix} \mid z \in \mathbb{K} \right\} & \mathfrak{u}_{k\delta-\alpha_1} &= \left\{ \begin{pmatrix} 0 & 0 \\ -zt^k & 0 \end{pmatrix} \mid z \in \mathbb{K} \right\} \end{aligned}$$

In particular, \mathfrak{u}_+ (resp. \mathfrak{u}_-) is the subalgebra of $\mathfrak{sl}_2(\mathbb{K}[t])$ (resp. $\mathfrak{sl}_2(\mathbb{K}[t^{-1}])$) of all matrices that are strictly upper-triangular (resp. lower-triangular) modulo t (resp. t^{-1}):

$$\begin{aligned} \mathfrak{u}_+ &= \left\{ \begin{pmatrix} tP(t) & Q(t) \\ tR(t) & -tP(t) \end{pmatrix} \mid P, Q, R \in \mathbb{K}[t] \right\} \subset \mathfrak{sl}_2(\mathbb{K}[t]) \\ \mathfrak{u}_- &= \left\{ \begin{pmatrix} t^{-1}P(t^{-1}) & t^{-1}Q(t^{-1}) \\ R(t^{-1}) & -t^{-1}P(t^{-1}) \end{pmatrix} \mid P, Q, R \in \mathbb{K}[t] \right\} \subset \mathfrak{sl}_2(\mathbb{K}[t^{-1}]) \end{aligned}$$

- (iv) For the GCM's of "twisted" affine type corresponding to the Dynkin diagrams in Table 2.3, the associated Kac-Moody algebras can be constructed as fixed-point sets of automorphisms of "untwisted" affine Kac-Moody algebras. In particular, they are related to loop algebras as well. The interested reader is referred to [Kac90, Chapter 8] or [Mar18, Section 5.3] for more precise statements.

The affine type is special in Kac-Moody theory for two reasons: First, an affine GCM is never invertible which is the cause of many untypical phenomena. Second, the structure of affine Kac-Moody algebras and their root systems is particularly well-understood and their close connection to loop algebras allows for explicit computations and arguments that exploit these matrix realizations. Furthermore, affine Kac-Moody algebras have numerous applications in other fields of mathematics and are also important in theoretical physics. In contrast, the "generic" indefinite case has hitherto remained largely mysterious. It is currently not known whether indefinite Kac-Moody algebras possess any reasonable sort of "concrete realization" and their general theory is much less developed than in the affine case.

Given any field \mathbb{K} , it follows from Lemma 2.3.5 and Example 2.3.7 (i) that every Kac-Moody algebra over \mathbb{K} associated to $\mathbb{A}^0 = (2)$ is isomorphic to a trivial central extension of $\mathfrak{sl}_2(\mathbb{K})$. More generally, if $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ is a Kac-Moody root datum associated to an arbitrary GCM \mathbb{A} and $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ the Kac-Moody algebra of type \mathcal{D} over \mathbb{K} , then for every $i \in I$, the subalgebra

$$\mathfrak{g}_i := \mathbb{K}e_i \oplus \mathbb{K}h_i \oplus \mathbb{K}f_i \subset \mathfrak{g} \tag{2.3.12}$$

is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. An explicit isomorphism $\varphi'_i : \mathfrak{sl}_2(\mathbb{K}) \rightarrow \mathfrak{g}_i \subset \mathfrak{g}$ is given by the map which sends $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to ze_i and $\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$ to zf_i and, consequently, $\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$ to zh_i for all $z \in \mathbb{K}$. This property sometimes allows to investigate the Kac-Moody algebra \mathfrak{g} by exploiting the well-developed theory of $\mathfrak{sl}_2(\mathbb{K})$. A particularly important application of this mechanism concerns the representation theory of \mathfrak{g} to be discussed in the next section.

2.4 Integrable modules

To prepare for later applications, we need to collect some basic notions and results from the representation theory of Kac-Moody algebras. Throughout this section, let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum, \mathbb{K} a field, $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ the Kac-Moody algebra of type \mathcal{D} over \mathbb{K} and $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ its Cartan subalgebra. A representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ of \mathfrak{g} on a (not necessarily finite-dimensional) \mathbb{K} -vector space V will be denoted by (V, ρ) , or simply by V if the map ρ is clear from the context. Occasionally, we may also call (V, ρ) a \mathfrak{g} -module. For $x \in \mathfrak{g}$ and $v \in V$, we are often going to simplify the notation by writing $x \cdot v$ or $x(v)$ instead of $\rho(x)v$. The following terminology will be useful.

First, an element $f \in \mathfrak{gl}(V)$, i.e. a \mathbb{K} -linear map $f : V \rightarrow V$, is said to be *locally nilpotent* if for every $v \in V$, there exists some $N \in \mathbb{N}$ such that $f^N(v) = 0$. Second, a representation (V, ρ) of \mathfrak{g} is \mathfrak{h} -diagonalizable if all the operators $(\rho(h))_{h \in \mathfrak{h}} \in \mathfrak{gl}(V)$ are simultaneously diagonalizable. Equivalently, V decomposes into the direct sum of the joint eigenspaces for the action of \mathfrak{h} on V and the eigenvalues of $\rho(h)$ depend linearly on $h \in \mathfrak{h}$, so V has the *weight space decomposition*

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}, \quad \text{where } V_{\mu} := \{v \in V : \rho(h)v = \langle \mu, h \rangle v \ \forall h \in \mathfrak{h}\}. \quad (2.4.1)$$

A functional $\mu \in \mathfrak{h}^*$ is called a *weight* of (V, ρ) if $V_{\mu} \neq \{0\}$. In this case, V_{μ} is the *weight space* and the dimension of V_{μ} is the *multiplicity* of μ in (V, ρ) . The set of weights of (V, ρ) will be denoted by $\text{wt}(V, \rho)$, or simply by $\text{wt}(V)$ when ρ is clear. The defining relations of \mathfrak{g} imply that

$$\rho(u_{\alpha})(V_{\mu}) \subset V_{\mu+c_{\alpha}} \quad \text{for all } \alpha \in \Phi^{\infty} \cup \{0\} \text{ and } \mu \in \text{wt}(V, \rho). \quad (2.4.2)$$

Definition 2.4.1. A \mathfrak{g} -module (V, ρ) is said to be *integrable* if it is \mathfrak{h} -diagonalizable and the operators $\rho(e_i), \rho(f_i) \in \mathfrak{gl}(V)$ are locally nilpotent for every $i \in I$.

Remark 2.4.2. If the Kac-Moody root datum \mathcal{D} is not free, one sometimes adds an additional technical condition in the definition of an integrable \mathfrak{g} -module (cf. [Mar18, Definition 7.21]). It requires such a representation to have a certain “abstract gradation” that refines the weight space decomposition (2.4.1). We omit this requirement on purpose for three reasons: First, while representation-theoretic aspects will be essential for the proofs of two of our main theorems in Chapter 5, they will only appear in situations where \mathcal{D} is assumed to be free and cofree. Second, this additional condition is satisfied for the most important examples of \mathfrak{g} -modules, namely the adjoint representation and the irreducible highest-weight modules defined below. Third, it is not necessary for the fundamental properties of integrable representations and the relevant arguments of [Kac90, Chapter 3 and 9] and [Mar18, Section 4.1] apply equally well to all root data upon incorporating the character map $c : Q \rightarrow \Lambda$ accordingly.

The notion of integrability essentially allows to “reduce” the study of representations of \mathfrak{g} to the representation theory of $\mathfrak{sl}_2(\mathbb{K})$ which is well-understood and plays a prominent role in the structure theory of finite-dimensional semisimple Lie algebras. As discussed after Example 2.3.7, for every $i \in I$, the subalgebra $\mathfrak{g}_i = \mathbb{K}e_i \oplus \mathbb{K}h_i \oplus \mathbb{K}f_i \subset \mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. If (V, ρ) is an integrable \mathfrak{g} -module considered as a \mathfrak{g}_i -module, then (V, ρ) decomposes into a direct sum of finite-dimensional, irreducible, \mathfrak{h} -invariant \mathfrak{g}_i -submodules. Using the well-known structure of irreducible representations of $\mathfrak{sl}_2(\mathbb{K})$, this result leads to the following consequences:

- a) If $\mu \in \mathfrak{h}^*$ is a weight of (V, ρ) and $i \in I$, there exist $p, q \in \mathbb{N}_0 \cup \{\infty\}$ such that the set $\{t \in \mathbb{Z} : \mu + tc_i \in \text{wt}(V, \rho)\}$ is the unbroken string of integers $[-p, q] \cap \mathbb{Z}$. If $p, q < \infty$, then $p - q = \langle \mu, h_i \rangle$. If $\dim(V_{\mu}) < \infty$, then $p, q < \infty$. In all cases, we have $\langle \mu, h_i \rangle \in \mathbb{Z}$.
- b) The map $t \mapsto \dim(V_{\mu+tc_i})$ is symmetric with respect to $t_0 := -\frac{1}{2}\langle \mu, h_i \rangle$ and monotonically increasing for $t < t_0$. If $t < t_0$, then $\rho(e_i) : V_{\mu+tc_i} \rightarrow V_{\mu+(t+1)c_i}$ is injective.
- c) If μ and $\mu + c_i$ are weights of (V, ρ) , then $\rho(e_i)(V_{\mu}) \neq \{0\}$.
- d) If μ is a weight of (V, ρ) but $\mu + c_i$ is not, then $\langle \mu, h_i \rangle \geq 0$. Moreover, $\mu - \langle \mu, h_i \rangle c_i$ is also a weight of (V, ρ) with the same multiplicity as μ .

For example, it is not very difficult to conclude from the defining relations of \mathfrak{g} that the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ is integrable. Its set of weights is $\text{wt}(\mathfrak{g}, \text{ad}) = \{c_\alpha \mid \alpha \in \Phi^\infty \cup \{0\}\}$ and the weight space decomposition is given by (2.3.7). In particular, if we choose \mathcal{D} to be free (e.g. by taking $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A}$) and identify Q with a subset of Λ via the injective character map, the non-zero weights of the adjoint representation are precisely the roots of \mathbb{A} . For our purposes, a particular class of representations will be important, where we denote by $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of \mathfrak{g} .

Definition 2.4.3. A \mathfrak{g} -module V is called a *highest-weight module* with highest weight $\lambda \in \mathfrak{h}^*$ if there exists a vector $v_\lambda \in V \setminus \{0\}$, called a *highest-weight vector*, with the following properties:

- (i) $\mathfrak{u}_+(v_\lambda) = 0$.
- (ii) $h(v_\lambda) = \langle \lambda, h \rangle v_\lambda$ for all $h \in \mathfrak{h}$.
- (iii) V is generated by v_λ as a $\mathcal{U}(\mathfrak{g})$ -module, i.e. $\mathcal{U}(\mathfrak{g})(v_\lambda) = V$.

By the *Poincaré-Birkhoff-Witt Theorem* (cf. [Kna02, Theorem 3.8]), the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is isomorphic to the tensor product $\mathcal{U}(\mathfrak{u}_+) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{u}_-)$. Together with (2.4.2), this implies that if V and λ are as in Definition 2.4.3, then V is \mathfrak{h} -diagonalizable, every weight of V can be written as $\lambda - c_\alpha$ for some $\alpha \in Q_+$, every weight space is finite-dimensional and $\dim(V_\lambda) = 1$. For every $\lambda \in \mathfrak{h}^*$, there exists a highest-weight module $M(\lambda)$ with highest weight λ which is “universal” in the sense that every other such module is a quotient of $M(\lambda)$ by a suitable submodule. In particular, it is unique up to isomorphism and called the *Verma module* for λ . It can be constructed as follows: The action of \mathfrak{g} on its universal enveloping algebra by left-multiplication gives $\mathcal{U}(\mathfrak{g})$ the structure of a \mathfrak{g} -module. The left-ideal $J_\lambda \subset \mathcal{U}(\mathfrak{g})$ generated by the set $\mathfrak{u}_+ \cup \{h - \langle \lambda, h \rangle 1 \mid h \in \mathfrak{h}\} \subset \mathcal{U}(\mathfrak{g})$ is \mathfrak{g} -stable, so \mathfrak{g} acts on the quotient $\mathcal{U}(\mathfrak{g})/J_\lambda$ and the image of $1 \in \mathcal{U}(\mathfrak{g})$ is a highest-weight vector with weight λ . It is not difficult to verify that this quotient has the required universal property. The Verma module $M(\lambda)$ contains a unique maximal proper submodule $M'(\lambda)$, so the quotient $L(\lambda) := M(\lambda)/M'(\lambda)$ is an irreducible highest-weight module with highest weight λ . From the universality of $M(\lambda)$, it also follows that $L(\lambda)$ is unique up to isomorphism. It is now a natural question whether this representation is integrable.

Definition 2.4.4. A linear functional $\lambda \in \mathfrak{h}^*$ is called a *dominant integral weight* if it satisfies $\langle \lambda, h_i \rangle \in \mathbb{N}_0$ for all $i \in I$. In this case, we further say that λ is *proper* if it is contained in $\Lambda \subset \mathfrak{h}^*$.

With this notion at hand, the irreducible highest-weight module $L(\lambda)$ can be shown to be integrable if and only if $\lambda \in \mathfrak{h}^*$ is a dominant integral weight. In the following rather degenerate situation, it even takes a particularly simple form.

Lemma 2.4.5. *If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda, h_i \rangle = 0$ for all $i \in I$, then the irreducible highest-weight module $L(\lambda)$ is one-dimensional. The subalgebras \mathfrak{u}_+ and \mathfrak{u}_- of \mathfrak{g} act trivially on $L(\lambda)$ and $h \in \mathfrak{h}$ acts by multiplication with $\langle \lambda, h \rangle \in \mathbb{K}$.*

Proof. If $\langle \lambda, h_i \rangle = 0$ for all $i \in I$, the defining relations (KMA1)-(KMA4) imply that we may construct a representation $\rho_\lambda : \mathfrak{g} \rightarrow \mathbb{K} \cong \mathfrak{gl}(\mathbb{K})$ of \mathfrak{g} on the one-dimensional vector space \mathbb{K} by setting $\rho_\lambda(e_i) = \rho_\lambda(f_i) := 0$ for all $i \in I$ and $\rho_\lambda(h) := \langle \lambda, h \rangle$ for all $h \in \mathfrak{h}$. This clearly defines an irreducible representation of \mathfrak{g} with highest weight λ and highest-weight vector $1 \in \mathbb{K}$. Since $L(\lambda)$ is unique up to isomorphism, it follows that $L(\lambda)$ is one-dimensional and isomorphic to \mathbb{K} under the map $v_\lambda \mapsto 1$. The required properties are satisfied by definition. \square

Remark 2.4.6. For the representation theory of Kac-Moody groups to be discussed in Section 3.2, the notion of a “dominant integral weight” in the sense of Definition 2.4.4 is not quite sufficient. In this context, the functional λ has to be contained in the \mathbb{Z} -module $\Lambda \subset \mathfrak{h}^*$ in order to lift the corresponding irreducible highest-weight module $L(\lambda)$ to the group level. This condition ensures that $\langle \lambda, h_i \rangle \in \mathbb{Z}$ for all $i \in I$, but is stronger than that. On the level of Kac-Moody algebras, this additional assumption is not necessary. This is the reason why we have also introduced the notion of a “proper” dominant integral weight. In the literature on Kac-Moody theory, such weights are sometimes called “integral”, but we have chosen to avoid this terminology because expressions like

“integral dominant integral weight” might lead to confusion. Depending on the Kac-Moody root datum \mathcal{D} , there does not necessarily exist a non-zero dominant integral weight. For example, the root datum \mathcal{D}^+ in Example 2.2.5 has this property. If \mathcal{D} is cofree, however, then $(h_i)_{i \in I}$ is contained in a \mathbb{K} -basis of \mathfrak{h} and the corresponding dual basis of \mathfrak{h}^* consists of dominant integral weights. These weights need not be proper, but if \mathcal{D} is cofree and cotorsion-free, then $(h_i)_{i \in I}$ can even be extended to a \mathbb{Z} -basis of Λ^\vee and the elements of the resulting dual basis of Λ are proper dominant integral weights.

2.5 The Weyl group

With the principal notions of Kac-Moody theory at hand, we now incorporate the language of Chapter 1 into this setting. A key property to investigate the structure and representation theory of Kac-Moody algebras is the fact that every GCM naturally gives rise to a Coxeter group that acts on the Cartan subalgebra of any Kac-Moody algebra associated to it.

Definition 2.5.1. Let $\mathbb{A} = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$ be a GCM. The *Weyl group* of \mathbb{A} is the group $W(\mathbb{A})$ generated by the abstract set $S := \{s_i \mid i \in I\}$ subject to the relations $(s_i s_j)^{m_{ij}} = 1$ for all $i, j \in I$, where m_{ij} is equal to 1 if $i = j$ and determined by the following table if $i \neq j$:

$$\begin{array}{c|c|c|c|c} a_{ij}a_{ji} & 0 & 1 & 2 & 3 \\ \hline m_{ij} & 2 & 3 & 4 & 6 \end{array} \quad \begin{array}{c} \geq 4 \\ \infty \end{array} \quad (2.5.1)$$

If the GCM \mathbb{A} is clear from the context, we will simply write W instead of $W(\mathbb{A})$. By definition, (W, S) is a Coxeter system with Coxeter matrix $M = (m_{ij})_{i,j \in I}$. Its Coxeter diagram $\Delta(W, S)$ can be obtained from the Dynkin diagram $\Gamma(\mathbb{A})$ by removing all arrows and multiple edges and replacing the edge labels according to (2.5.1). In particular, two vertices are adjacent in $\Delta(W, S)$ if and only if they are adjacent in $\Gamma(\mathbb{A})$, so the two graphs have the same connected components. Thus, the irreducible factors of (W, S) are in bijective correspondence with the indecomposable components of \mathbb{A} and indexed by the same subsets of I .

Whenever $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ is a Kac-Moody root datum associated to \mathbb{A} , the Weyl group $W := W(\mathbb{A})$ operates on Q and Λ , where for every $i \in I$, the action of the generator $s_i \in S$ is given by the following maps:

$$\begin{array}{ll} Q \rightarrow Q & \Lambda \rightarrow \Lambda \\ \beta \mapsto \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i & \lambda \mapsto \lambda - \langle \lambda, h_i \rangle c_i \end{array} \quad (2.5.2)$$

Likewise, W also acts on Q^\vee and Λ^\vee by the respective dual actions such that the dual pairings between Q and Q^\vee (resp. Λ and Λ^\vee) are W -invariant⁵ and the character and cocharacter maps $c : Q \rightarrow \Lambda$ and $h : Q^\vee \rightarrow \Lambda^\vee$ are W -equivariant. Whereas the W -actions on Q and Q^\vee are faithful, this is not necessarily true for the actions on Λ and Λ^\vee . Examples of this phenomenon, which can only occur if \mathbb{A} has an indecomposable component of affine type, will be discussed in the next section.

Let $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ be the Kac-Moody algebra of type \mathcal{D} over a field \mathbb{K} and $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{K}) = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}$ its Cartan subalgebra. By extension of scalars, the W -actions on Λ and Λ^\vee induce \mathbb{K} -linear representations of W on \mathfrak{h} and its dual space \mathfrak{h}^* . The subgroups of $\mathrm{GL}(\mathfrak{h})$ and $\mathrm{GL}(\mathfrak{h}^*)$ generated by the W -actions are contragredient linear groups which are isomorphic to W if the action is faithful. Let (V, ρ) be an integrable \mathfrak{g} -module, then it follows from d) in Section 2.4 that the set of weights of (V, ρ) is W -invariant. More precisely, for every $\lambda \in \mathrm{wt}(V, \rho)$, the functional $w(\lambda) \in \mathfrak{h}^*$ is also a weight of (V, ρ) with the same multiplicity as λ . In particular, if \mathcal{D} is free, applying this

⁵Although Q and Q^\vee are not considered to be dual \mathbb{Z} -modules in their own right, we may view them as subsets of the dual \mathbb{Z} -modules Λ_{Kac} and $\Lambda_{\mathrm{Kac}}^\vee$ of the free and cofree Kac-Moody root datum $\mathcal{D}_{\mathrm{Kac}}^\mathbb{A}$ for which the character and cocharacter maps are injective. Thus, we still have a natural dual pairing between Q and Q^\vee , so an expression of the form $\langle \beta, \alpha^\vee \rangle$ with $\beta \in Q$ and $\alpha^\vee \in Q^\vee$ is a well-defined integer. Moreover, the W -action on Q (resp. Q^\vee) is simply the restriction of the W -action on Λ_{Kac} (resp. $\Lambda_{\mathrm{Kac}}^\vee$) to the W -invariant submodule Q (resp. Q^\vee).

fact to the adjoint representation of \mathfrak{g} shows that the set of roots $\Phi^\infty \subset Q$ is W -invariant. Hence, every W -translate of a simple root is a root as well and the subset of Φ^∞ constructed in this way will be of central importance.

Definition 2.5.2. The subset $\Phi := W \cdot \Phi_0 = \{w(\alpha_i) \mid w \in W, i \in I\} \subset \Phi^\infty$ is called the set of *real roots* and its complement $\Phi^{\text{im}} := \Phi^\infty \setminus \Phi$ is the set of *imaginary roots* of \mathbb{A} .

Since Φ^∞ depends only on \mathbb{A} and not on the root datum \mathcal{D} (cf. Remark 2.3.3), the same is true for Φ and Φ^{im} , so the suffix “of \mathbb{A} ” in the definition is justified and we may denote these sets by $\Phi(\mathbb{A})$ and $\Phi^{\text{im}}(\mathbb{A})$ if the GCM might not be clear. As root spaces belonging to roots in the same W -orbit have the same dimension and $\dim(\mathfrak{u}_i) = 1$ for all $i \in I$, all root spaces $\mathfrak{u}_\alpha \subset \mathfrak{g}$ with $\alpha \in \Phi$ are one-dimensional. The set of imaginary roots will not play a substantial role in our discussion. We will focus our attention on the real roots which are intimately related to the structure of the Coxeter system (W, S) .

In fact, it follows from (2.5.2) that $-\alpha_i = s_i(\alpha_i) \in \Phi$ for every $i \in I$, so the negative of a real root is also real. This implies that $\Phi = \Phi_+ \sqcup \Phi_-$ and $\Phi_+ = -\Phi_-$, where $\Phi_\pm := \Phi \cap Q_\pm$ is the set of *positive* (resp. *negative*) *real roots*. If $\alpha \in \Phi$ is written as $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in I$, the element $\alpha^\vee := w(\alpha_i^\vee) \in Q^\vee$ is independent of the chosen expression. Thus, the set $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ is well-defined and called the set of *real coroots*. The canonical map $\Phi \rightarrow \Phi^\vee$, $\alpha \mapsto \alpha^\vee$, is a W -equivariant bijection. Moreover, if $\alpha = w(\alpha_i) \in \Phi$, we can associate to α the reflection $s_\alpha := ws_iw^{-1} \in R \subset W$ which is independent of the chosen expression as well. It acts on Q and Λ by the following formulas:

$$\begin{aligned} Q &\rightarrow Q & \Lambda &\rightarrow \Lambda \\ \beta &\mapsto \beta - \langle \beta, \alpha^\vee \rangle \alpha & \lambda &\mapsto \lambda - \langle \lambda, h_\alpha \rangle c_\alpha \end{aligned} \quad (2.5.3)$$

We clearly have $s_\alpha = s_{-\alpha}$ for all $\alpha \in \Phi$. Restricted to positive roots, the map $\alpha \mapsto s_\alpha$ defines a bijection between Φ_+ and the set of reflections of the Coxeter system (W, S) . Exactly as the real roots, the set of real coroots is partitioned into two opposite subsets $\Phi_\pm^\vee := \Phi^\vee \cap Q_\pm^\vee$ of *positive* and *negative real coroots* and the canonical W -equivariant bijection $\Phi \rightarrow \Phi^\vee$ maps Φ_\pm to Φ_\pm^\vee , respectively. The W -actions on Φ and Φ^\vee , however, do not preserve positivity. More precisely, the following criteria hold for all $w \in W$ and $i \in I$:

$$\begin{aligned} w(\alpha_i) \in \Phi_+ &\iff l(w) < l(ws_i) \\ w(\alpha_i^\vee) \in \Phi_+^\vee &\iff l(w) < l(ws_i) \end{aligned} \quad (2.5.4)$$

In particular, this shows that the W -actions on Φ and Φ^\vee (and hence on Q and Q^\vee) are faithful and suggests to associate to any $w \in W$ the set

$$\Phi_w := \{\alpha \in \Phi_+ : w(\alpha) \in \Phi_-\}. \quad (2.5.5)$$

More explicitly, if $w = s_{i_1} \dots s_{i_k}$ is a reduced decomposition, then $|\Phi_w| = k = l(w)$ and

$$\Phi_w = \{s_{i_k} \dots s_{i_{j+1}}(\alpha_{i_j}) \mid 1 \leq j \leq k\}. \quad (2.5.6)$$

For the imaginary roots, the analogous partition $\Phi^{\text{im}} = \Phi_+^{\text{im}} \sqcup \Phi_-^{\text{im}}$ holds. In contrast to the real roots, the sets $\Phi_\pm^{\text{im}} := \Phi^{\text{im}} \cap Q_\pm$ of *positive* and *negative imaginary roots* are W -invariant, so the definition of Φ_w would not change after replacing Φ_\pm by Φ_\pm^{im} in (2.5.5). They do not correspond to any elements of the Weyl group W and can only be “seen” as root spaces in the Kac-Moody algebra \mathfrak{g} . Hence, the real roots can be constructed directly from the W -action on Q which is independent of \mathfrak{g} , but the imaginary roots are “invisible” without the Kac-Moody context.

Whenever we view W as a group acting on the Cartan subalgebra \mathfrak{h} of a Kac-Moody algebra \mathfrak{g} , the W -action on \mathfrak{h} can be “extended” to all of \mathfrak{g} in the sense that for every $w \in W$, there exists a canonical Lie algebra automorphism $w^* \in \text{Aut}(\mathfrak{g})$ such that $w^*|_{\mathfrak{h}} = w$. It is constructed in two steps: First, if $w = s_i$ for some $i \in I$ is a simple reflection, we set

$$s_i^* := \exp(\text{ad}(f_i)) \exp(\text{ad}(e_i)) \exp(\text{ad}(f_i)) \in \text{GL}(\mathfrak{g}) \quad (2.5.7)$$

which is well-defined since the adjoint representation is integrable. Second, if $w \in W$ is arbitrary, we choose a reduced decomposition $w = s_{i_1} \dots s_{i_k}$ and define $w^* := s_{i_1}^* \dots s_{i_k}^*$. It can be shown that w^* is independent of this choice and defines an automorphism of \mathfrak{g} such that $w^*|_{\mathfrak{h}} = w$. Moreover, it commutes with the Chevalley involution of \mathfrak{g} and permutes the root spaces as follows:

$$w^*(\mathfrak{u}_\alpha) = \mathfrak{u}_{w(\alpha)} \quad \text{for all } \alpha \in \Phi^\infty \quad (2.5.8)$$

A stronger property even holds for the real roots: Suppose that $\alpha = w(\alpha_i) \in \Phi$ for some $w \in W$ and $i \in I$, then the set $E_\alpha := \{w^*(e_i), -w^*(e_i)\} \subset \mathfrak{u}_\alpha$ does not depend on the chosen expression of α . Both vectors in E_α can serve as a basis for the one-dimensional space $\mathfrak{u}_\alpha \subset \mathfrak{g}$, which is why E_α is often called a *double basis* of \mathfrak{u}_α . In every such set, we choose a vector $e_\alpha \in E_\alpha$ in the following way: For every $i \in I$, we take $e_{\alpha_i} := e_i$ and $e_{-\alpha_i} := f_i$. For every other positive real root $\alpha \in \Phi_+ \setminus \Phi_0$, we arbitrarily choose some $w_\alpha \in W$ and $i \in I$ such that $\alpha = w_\alpha(\alpha_i)$ and set $e_\alpha := w_\alpha^*(e_i)$, $e_{-\alpha} := w_\alpha^*(f_i)$ and $f_\alpha := e_{-\alpha}$. As $w_\alpha^* \in \text{Aut}(\mathfrak{g})$ and $w_\alpha^*|_{\mathfrak{h}} = w_\alpha$, we have $[e_\alpha, f_\alpha] = w_\alpha^*([e_i, f_i]) = -w_\alpha(h_i) = -h_\alpha$, so the subalgebra

$$\mathfrak{g}_\alpha := \mathbb{K}e_\alpha \oplus \mathbb{K}h_\alpha \oplus \mathbb{K}f_\alpha = w_\alpha^*(\mathfrak{g}_i) \subset \mathfrak{g} \quad (2.5.9)$$

is isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. Explicitly, the isomorphism

$$\varphi'_\alpha := w_\alpha^* \circ \varphi'_i : \mathfrak{sl}_2(\mathbb{K}) \rightarrow \mathfrak{g}_\alpha \subset \mathfrak{g} \quad (2.5.10)$$

is the map which sends $\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}$ to ze_α and $\begin{pmatrix} 0 & 0 \\ -z & 0 \end{pmatrix}$ to zf_α and, consequently, $\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$ to zh_α for all $z \in \mathbb{K}$. We write $\mathcal{B}_\Phi := \{e_\alpha \mid \alpha \in \Phi\}$ and denote by W^* the subgroup of $\text{Aut}(\mathfrak{g})$ generated by $\{w^* \mid w \in W\}$. Hence, we have

$$w^*(e_\alpha) = \pm e_{w(\alpha)} \quad \text{for all } w \in W \text{ and } \alpha \in \Phi, \quad (2.5.11)$$

where the sign depends on the chosen vectors in \mathcal{B}_Φ . It can be determined by computations inside the universal enveloping algebra of \mathfrak{g} . Restriction to $\mathfrak{h} \subset \mathfrak{g}$ defines a surjective homomorphism $W^* \rightarrow W$ whose kernel is the subgroup $D^* := \langle (s_i^*)^2 \mid i \in I \rangle$. For every $i \in I$, the automorphism $(s_i^*)^2 \in W^*$ preserves the root space decomposition of \mathfrak{g} , where the action is trivial on $\mathfrak{h} = \mathfrak{u}_0$ and

$$(s_i^*)^2(x) = (-1)^{\langle c_\alpha, h_i \rangle} x = (-1)^{\langle \alpha, \alpha_i^\vee \rangle} x \quad \text{for all } \alpha \in \Phi^\infty \text{ and } x \in \mathfrak{u}_\alpha. \quad (2.5.12)$$

Note that $s_i^2 = 1$ in W for all $i \in I$, but it may or may not happen that $(s_i^*)^2 = 1$ in W^* . However, this depends only on the GCM \mathbb{A} and not on the Kac-Moody root datum \mathcal{D} .

Remark 2.5.3. The construction of the group W^* only required that the adjoint representation is integrable. More generally, if (V, ρ) is any integrable \mathfrak{g} -module, it is possible to replace the adjoint representation by ρ in (2.5.7) and associate to every $w \in W$ a canonical linear operator $w^\rho \in \text{GL}(V)$ that permutes the weight spaces of (V, ρ) in a way similar to (2.5.8) and an appropriate analogue of (2.5.12) holds. Also, there is an equation relating w^ρ and w^* which shows that for every $\alpha \in \Phi$, the operator $\rho(e_\alpha) \in \mathfrak{gl}(V)$ is locally nilpotent. We refer to [Mar18, Proposition 4.18] for this more general statement. For our purposes, it will mostly suffice to consider the adjoint representation. In particular, the chosen set \mathcal{B}_Φ will be important for the definition of Kac-Moody groups in Chapter 3.

Notation 2.5.4. Given any subset $J \subset I$, the Weyl group $W(\mathbb{A}_J)$ of the submatrix \mathbb{A}_J of \mathbb{A} indexed by J coincides with the standard subgroup $W_J \subset W$. The real roots of \mathbb{A}_J will be denoted by $\Phi_J := W_J \cdot \Phi_{J,0}$ and we define the sets $\Phi_{J,\pm}, \Phi_{J,\pm}^{\text{im}}, \Phi_J^\vee, \Phi_{J,\pm}^\vee$ in the obvious ways. Each of these objects can be viewed as the subset of its analogue with respect to \mathbb{A} consisting of those elements that are supported in $J \subset I$. Furthermore, we define $\Phi_{J,\pm}^\perp := \Phi_\pm \setminus \Phi_{J,\pm}$ which is exactly the set of positive (resp. negative) real roots $\alpha \in \Phi_\pm$ with $I(\alpha) \not\subset J$.

2.6 The \mathcal{D} -representation

Having defined the Weyl group, we are now in a position to introduce one of the fundamental concepts of this thesis which will be studied extensively in Chapter 4. Throughout this section, we fix an arbitrary Kac-Moody root datum $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ of rank $d \in \mathbb{N}$ and write $W := W(\mathbb{A})$ for the Weyl group of \mathbb{A} . Moreover, we denote by $\mathbb{A}_1, \dots, \mathbb{A}_m$ the indecomposable components of \mathbb{A} with index sets $I_1, \dots, I_m \subset I$, by $\mathcal{D}_1, \dots, \mathcal{D}_m$ the indecomposable components of \mathcal{D} and by $(W_1, S_1), \dots, (W_m, S_m)$ the irreducible factors of (W, S) .

By extension of scalars to the field of real numbers, the W -action on the free \mathbb{Z} -module Λ^\vee induces a real representation of W on $\mathfrak{a} := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$. Computing the dual of the second map in (2.5.2) shows that it is characterized by the following equation:

$$s_i(h) = h - \langle c_i, h \rangle h_i \quad \text{for all } i \in I \text{ and } h \in \mathfrak{a} \quad (2.6.1)$$

Note that we may view \mathfrak{a} as the Cartan subalgebra $\mathfrak{h}_{\mathcal{D}}(\mathbb{R})$ of the real Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}}(\mathbb{R})$ of type \mathcal{D} but for the moment, it suffices to consider the representation without this context.

Definition 2.6.1. Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum of rank $d \in \mathbb{N}$. The real representation of $W = W(\mathbb{A})$ on the d -dimensional real vector space $\mathfrak{a} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ defined by (2.6.1) is called the \mathcal{D} -representation of W .

To investigate the \mathcal{D} -representation, we require a couple of definitions. First of all, we introduce the linear subspace

$$\mathfrak{a}' := \sum_{i \in I} \mathbb{R} h_i \subset \mathfrak{a} \quad (2.6.2)$$

which is clearly W -invariant. Second, we define the following two sets:

$$\begin{aligned} C &:= \{h \in \mathfrak{a} : \langle c_i, h \rangle > 0 \ \forall i \in I\} \\ \overline{C} &:= \{h \in \mathfrak{a} : \langle c_i, h \rangle \geq 0 \ \forall i \in I\} \end{aligned} \quad (2.6.3)$$

We call C (resp. \overline{C}) the *open* (resp. *closed*) *fundamental chamber* of the \mathcal{D} -representation of W . The following observations are immediate:

- a) If \mathcal{D} is a trivial root datum, then $C = \mathfrak{a}$ since the defining condition is vacuous.
- b) If $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is the union of two root data, then $C = C_1 \cap C_2$, where for $k \in \{1, 2\}$, C_k denotes the open fundamental chamber of the \mathcal{D}_k -representation of $W_k = W(\mathbb{A}_k)$ on \mathfrak{a} .
- c) If $\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2$ is the direct sum of two root data, then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ and $C = C_1 \times C_2$, where for $k \in \{1, 2\}$, $\mathfrak{a}_k := \Lambda_k^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ and C_k denotes the open fundamental chamber of the \mathcal{D}_k -representation of $W_k = W(\mathbb{A}_k)$ on \mathfrak{a}_k .

If C is not empty, it follows from simple topological arguments that \overline{C} is the closure of C with respect to the standard Euclidean topology on \mathfrak{a} and that C is the topological interior of \overline{C} . However, there exist root data with $C = \emptyset$ and they can be characterized by a simple condition. In the following, given any $k \in \{1, \dots, m\}$, we denote by $C_k \subset \mathfrak{a}$ the open fundamental chamber of the \mathcal{D}_k -representation of $W_k = W(\mathbb{A}_k)$.

Lemma 2.6.2. *The open fundamental chamber $C \subset \mathfrak{a}$ of the \mathcal{D} -representation of W is empty if and only if the following condition is satisfied:*

$$\text{If } \mathbb{A}_{p_1}, \dots, \mathbb{A}_{p_s} \text{ denote all indecomposable components of } \mathbb{A} \text{ that are of affine type, then } \bigcap_{r=1}^s C_{p_r} = \emptyset. \quad (\text{DAC})$$

Moreover, if $s = 1$, then (DAC) holds if and only if \mathcal{D}_{p_1} is not free.

Proof. We have $\mathcal{D} = \bigcup_{k=1}^m \mathcal{D}_k$ and thus $C = \bigcap_{k=1}^m C_k \subset \bigcap_{r=1}^s C_{p_r}$, so (DAC) implies that $C = \emptyset$. Conversely, suppose that there exists a vector $u_{\text{aff}} \in \bigcap_{r=1}^s C_{p_r}$. We define $I_{\text{aff}} := I_{p_1} \sqcup \dots \sqcup I_{p_s}$ and $Y := \{1, \dots, m\} \setminus \{p_1, \dots, p_s\}$. For every $k \in Y$, we choose a vector $u_k \in C_k$ as follows:

- If \mathbb{A}_k is of finite type, the same is true for \mathbb{A}_k^T , so if we list its index set as $I_k = \{i_1, \dots, i_t\}$ for some $t \in \{1, \dots, n\}$ and $i_1, \dots, i_t \in I$, there exists a vector $x = (x_1, \dots, x_t)^T \in \mathbb{R}^t$ such that $x > 0$ and $\mathbb{A}_k^T x > 0$. We define $u_k := \sum_{a=1}^t x_a h_{i_a} \in \mathfrak{a}$, so that $u_k \in C_k$ because the following holds for all $b \in \{1, \dots, t\}$:

$$\langle c_{i_b}, u_k \rangle = \sum_{a=1}^t x_a \langle c_{i_b}, h_{i_a} \rangle = \sum_{a=1}^t x_a a_{i_a i_b} = \sum_{a=1}^t (\mathbb{A}_k^T)_{ba} x_a = (\mathbb{A}_k^T)_b > 0$$

- If \mathbb{A}_k is of indefinite type and $t := |I_k|$, there exists a vector $x \in \mathbb{R}^t$ such that $x < 0$ and $\mathbb{A}_k^T x > 0$ and the argument proceeds as in the finite case.

Having chosen these vectors, we set $h := u_{\text{aff}} + \sum_{k \in Y} u_k \in \mathfrak{a}$. By construction of the second summand, we have $\langle c_i, h \rangle = \langle c_i, u_{\text{aff}} \rangle > 0$ for every $i \in I_{\text{aff}}$. Moreover, if $i \in I \setminus I_{\text{aff}}$, then $i \in I_k$ for a unique $k \in Y$ and $\langle c_i, h \rangle = \langle c_i, u_k \rangle + \langle c_i, u_{\text{aff}} \rangle$. Here, the first term is positive, but it is not clear whether the second is non-negative. However, since u_{aff} is fixed and C_k is invariant under multiplication by positive scalars, we may simply replace u_k by a sufficiently large scalar multiple to ensure that $\langle c_i, h \rangle > 0$. Doing this for all $i \in I \setminus I_{\text{aff}}$ yields a vector in C .

Finally, suppose that $s = 1$, so that $C = \emptyset$ if and only if $C_{p_1} = \emptyset$. If \mathcal{D}_{p_1} is free, the functionals $(c_i)_{i \in I_{p_1}}$ are linearly independent in \mathfrak{a}^* , so there exist vectors $(v_i)_{i \in I_{p_1}} \in \mathfrak{a}$ with $\langle c_i, v_j \rangle = \delta_{ij}$ for all $i, j \in I_{p_1}$. Hence, their sum $v := \sum_{i \in I_{p_1}} v_i \in \mathfrak{a}$ is contained in C_{p_1} . Conversely, if \mathcal{D}_{p_1} is not free, we have seen in Example 2.2.5 that $\sum_{i \in I_{p_1}} a_i c_i = 0$, where for every $i \in I_{p_1}$, the positive integer $a_i \in \mathbb{N}$ is the label of the vertex i in the Dynkin diagram $\Gamma(\mathbb{A}_{I_{p_1}})$. For every $h \in \mathfrak{a}$, we therefore have $\sum_{i \in I_{p_1}} a_i \langle c_i, h \rangle = 0$. But if there existed a point $h \in C_{p_1}$, the left-hand side of this equation would be strictly positive, a contradiction. \square

Definition 2.6.3. We say that a Kac-Moody root datum \mathcal{D} has *disjoint affine chambers* if (DAC) is satisfied. Otherwise, we call \mathcal{D} *admissible*.

Example 2.6.4. Given a Kac-Moody root datum \mathcal{D} , it is desirable to find necessary and sufficient conditions for admissibility. From Lemma 2.6.2, we conclude the following:

- (i) \mathcal{D} is always admissible if no indecomposable component of \mathbb{A} is of affine type.
- (ii) If \mathbb{A} contains exactly one indecomposable component \mathbb{A}_{p_1} of affine type, then \mathcal{D} is admissible if and only if the I_{p_1} -restriction \mathcal{D}_{p_1} of \mathcal{D} is free. For example, if $\mathcal{D}_{p_1} = \mathcal{D}_{\text{sc}}^{\mathbb{A}_{p_1}}$, then \mathcal{D} is not admissible.

However, as soon as \mathbb{A} contains at least two indecomposable components $\mathbb{A}_{p_1}, \dots, \mathbb{A}_{p_s}$ of affine type, the situation becomes much more complicated. In this case, it is clear from the previous proof that if \mathcal{D} is admissible, then for every $r \in \{1, \dots, s\}$, the I_{p_r} -restriction \mathcal{D}_{p_r} of \mathcal{D} is free, but the converse to this assertion is false.

- (iii) Suppose that $\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_2$ where $\mathbb{A}_1 = \mathbb{A}_2 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is of untwisted affine type $\mathbb{A}_1^{(1)}$. Let Λ and Λ^\vee be dual free \mathbb{Z} -modules of rank 4 with dual bases $\{v_1, \dots, v_4\} \subset \Lambda^\vee$ and $\{v_1^*, \dots, v_4^*\} \subset \Lambda$ and define the following elements:

$$\begin{aligned} h_1 &:= v_1 & c_1 &:= 2v_1^* - 2v_2^* \\ h_2 &:= v_2 & c_2 &:= -2v_1^* + 2v_2^* + v_3^* \\ h_3 &:= -v_1 - v_2 + v_4 & c_3 &:= -v_3^* + 2v_4^* \\ h_4 &:= -v_1 - v_2 - v_4 & c_4 &:= -v_3^* - 2v_4^* \end{aligned}$$

It is straightforward to verify that $\mathcal{D} = (I, \Lambda, \mathbb{A}, (c_i)_{i \in I}, (h_i)_{i \in I})$ defines a Kac-Moody root datum associated to \mathbb{A} and that its restrictions to $I_1 = \{1, 2\}$ and $I_2 = \{3, 4\}$ are free. Hence, $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a union of two free root data. However, \mathcal{D} itself is not free since $2c_1 + 2c_2 + c_3 + c_4 = 0$. As all coefficients in this sum are positive, it follows that the open fundamental chamber $C \subset \mathfrak{a}$ of the \mathcal{D} -representation of W is empty although, for $k \in \{1, 2\}$,

the open fundamental chamber $C_k \subset \mathfrak{a}$ of the \mathcal{D}_k -representation of $W_k = W(\mathbb{A}_k)$ is not. More precisely, we have

$$\begin{aligned} C_1 &= \{xv_1 + yv_2 + zv_3 + uv_4 \in \mathfrak{a} \mid x, y, z, u \in \mathbb{R}, 2y < 2x < 2y + z\}, \\ C_2 &= \{xv_1 + yv_2 + zv_3 + uv_4 \in \mathfrak{a} \mid x, y, z, u \in \mathbb{R}, z < 2u < -z\}, \end{aligned}$$

and it is not difficult to see that $\overline{C} = \mathbb{R} \cdot (v_1 + v_2)$ and $C = C_1 \cap C_2 = \emptyset$. Thus, \mathcal{D} is not admissible even though \mathcal{D}_1 and \mathcal{D}_2 are free.

Therefore, freeness of the affine components $\mathcal{D}_{p_1}, \dots, \mathcal{D}_{p_s}$ of \mathcal{D} does not imply admissibility of \mathcal{D} if $s \geq 2$. On the other hand, freeness of the entire root datum \mathcal{D} clearly does, but the converse to this statement is not true either.

- (iv) If we modify the Kac-Moody root datum in (iii) by replacing c_3, c_4, h_3, h_4 by their negatives c'_3, c'_4, h'_3, h'_4 , we obtain another root datum $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}'_2$ associated to \mathbb{A} . This time, we have $2c_1 + 2c_2 - c'_3 - c'_4 = 0$, so \mathcal{D}' is not free either, but the open fundamental chamber of the \mathcal{D}'_2 -representation of W_2 now satisfies $C'_2 = -C_2$ which intersects C_1 non-trivially. Hence, \mathcal{D}' is admissible but not free.

To conclude this example, it seems difficult to formulate a very precise criterion for admissibility. Not only is it necessary that the affine components $\mathcal{D}_{p_1}, \dots, \mathcal{D}_{p_s}$ of \mathcal{D} are free, but their open fundamental chambers also have to be “oriented” in such a way that they overlap. An interesting sufficient condition will be proved in Corollary 2.6.12.

Whereas the open fundamental chamber $C \subset \mathfrak{a}$ of the \mathcal{D} -representation of W may be empty, the closed fundamental chamber $\overline{C} \subset \mathfrak{a}$ at least contains the origin. It is a closed convex cone that is bounded by the linear hyperplanes $H_1, \dots, H_n \subset \mathfrak{a}$ defined by $H_i := \ker(c_i)$ for $i \in I$. They are called the *walls* of \overline{C} and coincide with the fixed-point sets of the actions of the simple reflections $s_1, \dots, s_n \in S$ on \mathfrak{a} . Given any $J \subset I$, the *cell of type J of \overline{C}* is the set

$$D_J := \{h \in \mathfrak{a} : \langle c_j, h \rangle = 0 \ \forall j \in J, \ \langle c_i, h \rangle > 0 \ \forall i \in I \setminus J\}. \quad (2.6.4)$$

Hence, the closed fundamental chamber has a cell decomposition $\overline{C} = \coprod_{J \subset I} D_J$. Every translate $w(C)$ (resp. $w(\overline{C})$) by some $w \in W$ is called an *open* (resp. *closed*) *Weyl chamber* and the union

$$X_+ := \bigcup_{w \in W} w(\overline{C}) \subset \mathfrak{a} \quad (2.6.5)$$

is the *Tits cone* of the \mathcal{D} -representation. The W -translates $\{w(H_i) \mid w \in W, i \in I\}$ are the *walls* of X_+ and $\{w(D_J) \mid w \in W, J \subset I\}$ are its *cells*. Every closed Weyl chamber is bounded by walls and inherits a cell decomposition from \overline{C} in the obvious way. If $\alpha = w(\alpha_i) \in \Phi$ for some $w \in W$ and $i \in I$, then $H_\alpha := w(H_i) = \ker(c_\alpha)$ is independent of the chosen expression of α and coincides with the fixed-point set of the action of the reflection $s_\alpha = ws_iw^{-1} \in W$ on \mathfrak{a} . Note that it is possible for cells of the Tits cone to be empty if \mathcal{D} is not free. Moreover, it is not a priori clear whether the non-empty cells “fit together” to form a partition of X_+ . However, the next result shows that this is indeed the case and establishes the most important properties of the objects just defined. Given any $h \in \mathfrak{a}$, we denote the stabilizer of h under the W -action by W_h .

Proposition 2.6.5. *The \mathcal{D} -representation of W has the following properties:*

- (i) *If $J \subset I$ and $h \in D_J$ are arbitrary, then W_h coincides with the standard subgroup $W_J \subset W$. In particular, it only depends on the cell of \overline{C} containing h .*
- (ii) *The closed fundamental chamber is a strict fundamental domain for the W -action on X_+ . More precisely, every Weyl group orbit $W \cdot h \subset \mathfrak{a}$ with $h \in X_+$ intersects \overline{C} in a unique point. Hence, if \mathcal{D} is admissible, then W acts simply transitively on the set of open Weyl chambers and the \mathcal{D} -representation is faithful in this case.*
- (iii) *Let $w_1, w_2 \in W$ and $J, K \subset I$ be arbitrary. If the cells $w_1(D_J)$ and $w_2(D_K)$ are non-empty, they are either equal or disjoint.*

(iv) The Tits cone is a convex cone that can equivalently be described as

$$X_+ = \{h \in \mathfrak{a} : \langle c_\alpha, h \rangle < 0 \text{ only for finitely many } \alpha \in \Phi_+^\infty\}. \quad (2.6.6)$$

In addition, (2.6.6) also holds if Φ_+^∞ is replaced by Φ_+ .

(v) The closed fundamental chamber is given by

$$\overline{C} = \left\{ h \in \mathfrak{a} \mid \forall w \in W : h - w(h) = \sum_{i \in I(w)} x_i h_i \text{ with } x_i \in \mathbb{R}_{\geq 0} \text{ for all } i \in I(w) \right\}. \quad (2.6.7)$$

(vi) The following conditions are all equivalent:

- a) $|W| < \infty$ b) \mathbb{A} is spherical c) $X_+ = \mathfrak{a}$ d) $|\Phi^\infty| < \infty$ e) $|\Phi| < \infty$ f) $\Phi = \Phi^\infty$
(vii) A point $h \in X_+$ lies in the topological interior X_+° of X_+ if and only if W_h is finite.

If $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A}$, this proposition is a standard result in Kac-Moody theory and largely proved in [Kac90, Proposition 3.12]. For the reader's convenience, we provide a self-contained and slightly more detailed proof that is adjusted to the use of an arbitrary root datum, but our arguments are almost identical to the classical case. The only difference is that we need to incorporate the character and cocharacter maps $c : Q \rightarrow \Lambda$ and $h : Q^\vee \rightarrow \Lambda^\vee$ which need not be injective.

Proof. Before turning to the core of the proof, we claim that if $h \in \overline{C}$ and $w(h) \in \overline{C}$ for some $w \in W$, then $w(h) = h$. This is proved as follows: Let $w = s_{i_1} \dots s_{i_k}$ be a reduced decomposition, then on the one hand, we have $\langle w(c_{i_k}), w(h) \rangle = \langle c_{i_k}, h \rangle \geq 0$ since $h \in \overline{C}$. On the other hand, we also have $l(ws_{i_k}) < l(w)$ and thus $w(\alpha_{i_k}) \in \Phi_-$ by (2.5.4), so that $w(c_{i_k}) = c_{w(\alpha_{i_k})}$ is a \mathbb{Z} -linear combination of $(c_i)_{i \in I}$ with non-positive coefficients. As $w(h) \in \overline{C}$, this implies that $\langle w(c_{i_k}), w(h) \rangle \leq 0$ and hence $\langle c_{i_k}, h \rangle = 0$ or, equivalently, $s_{i_k}(h) = h$. By induction on $k = l(w)$, it follows that every simple factor of w fixes h , so that $w(h) = h$, as claimed.

- (i) If $h \in D_J$, then $\langle c_j, h \rangle = 0$ and thus $s_j(h) = h$ for all $j \in J$, which proves that $W_J \subset W_h$. Conversely, if $w \in W_h$, then $w(h) = h \in \overline{C}$ and the proof of the previous claim shows that every simple factor of w fixes h , so that $I(w) \subset J$ and $w \in W_J$.
- (ii) The first assertion is an immediate consequence of the claim above. The second follows from (i) since if \mathcal{D} is admissible, open Weyl chambers are non-empty and their stabilizers under the W -action are trivial.
- (iii) Suppose that $w_1(D_J) \cap w_2(D_K) \neq \emptyset$, then $w_2^{-1}w_1(h) \in D_K \subset \overline{C}$ for some $h \in D_J \subset \overline{C}$, so that $w_2^{-1}w_1(h) = h$ by the initial claim. It now follows from (i) that $w_2^{-1}w_1 \in W_h = W_J$ fixes the entire cell D_J point-wise, implying that $J = K$ and $w_1(D_J) = w_2(D_J) = w_2(D_K)$.
- (iv) Let X' denote the set on the right-hand side of (2.6.6). It is clear that $\overline{C} \subset X'$ and we claim that X' is W -invariant. To see this, let $i \in I$ be arbitrary, then the set $\Phi_+^\infty \setminus \{\alpha_i\}$ is s_i -invariant by (2.5.6) and $s_i(\alpha_i) = -\alpha_i$. Consequently, whenever $h \in X'$ and $\alpha \in \Phi_+^\infty$ are such that $0 > \langle c_\alpha, s_i(h) \rangle = \langle c_{s_i(\alpha)}, h \rangle$, then either $\alpha = \alpha_i$ or $\alpha \neq \alpha_i$ and $\langle c_{s_i(\alpha)}, h \rangle < 0$ with $s_i(\alpha) \in \Phi_+^\infty$. As $h \in X'$, there are only finitely many $\alpha \in \Phi_+^\infty$ satisfying the second condition, so that $s_i(h) \in X'$ as well. Hence, X' is s_i -invariant for every $i \in I$ and therefore W -invariant. We conclude that $w(\overline{C}) \subset X'$ for all $w \in W$, so that $X_+ \subset X'$.

Conversely, let $h \in X'$ and consider the finite set $M_h := \{\alpha \in \Phi_+^\infty : \langle c_\alpha, h \rangle < 0\}$. If $M_h = \emptyset$, then $h \in \overline{C} \subset X_+$. Otherwise, we have $\langle c_i, h \rangle < 0$ and thus $\alpha_i \in M_h$ for some $i \in I$. Since $\Phi_+^\infty \setminus \{\alpha_i\}$ is s_i -invariant, it is straightforward to verify that $M_{s_i(h)} = s_i(M_h \setminus \{\alpha_i\})$, which implies that $|M_{s_i(h)}| < |M_h|$. Iterating this process leads to some $w \in W$ such that $M_{w(h)} = \emptyset$, which is equivalent to $w(h) \in \overline{C}$ and $h \in w^{-1}(\overline{C}) \subset X_+$.

The second statement is proved by the same argument upon replacing Φ_+^∞ by Φ_+ . In fact, for every $\alpha \in \Phi_+^{\text{im}}$, the functional $c_\alpha \in \mathfrak{a}^*$ is non-negative on \overline{C} and hence on all of X_+ as Φ_+^{im} is W -invariant. That X_+ is a convex cone is also a direct consequence of (2.6.6).

- (v) The inclusion “ \supset ” in (2.6.7) follows from (2.6.1) by setting $w = s_i$ for any $i \in I$. Conversely, let $h \in \overline{C}$ and $w \in W$ be arbitrary and choose a reduced decomposition $w = s_{i_1} \dots s_{i_k}$. We argue by induction on $k = l(w)$. If $k = 1$, then $h - w(h) = h - s_{i_1}(h) = \langle c_{i_1}, h \rangle h_{i_1}$, where the coefficient is non-negative as $h \in \overline{C}$. Now suppose that $k > 1$ and set $w' := s_{i_1} \dots s_{i_{k-1}}$.

Then $I(w) = I(w') \cup \{i_k\}$ and the induction hypothesis implies that for every $i \in I(w')$, there exists some $x'_i \in \mathbb{R}_{\geq 0}$ such that

$$h - w(h) = h - w'(h) + w'(h - s_{i_k}(h)) = \sum_{i \in I(w')} x'_i h_i + \langle c_{i_k}, h \rangle w'(h_{i_k}).$$

Since $l(w') < l(w) = l(w' s_{i_k})$, we have $w'(\alpha_{i_k}^\vee) \in \Phi_{I(w'),+}^\vee$ by (2.5.4), so $w'(h_{i_k}) = h_{w'(\alpha_{i_k})}$ is a \mathbb{Z} -linear combination of $(h_i)_{i \in I(w')}$ with non-negative coefficients. As $h \in \overline{C}$, we also have $\langle c_{i_k}, h \rangle \geq 0$ and the inclusion “ \subset ” follows.

- (vi) a) \iff b): A direct proof of this equivalence can be found in [Kac90, Proposition 4.9]. In addition, it also follows by comparing the list of Dynkin diagrams of the GCM's of finite type (cf. Table 2.1) with the Coxeter diagrams of the irreducible finite Coxeter groups (cf. [AB08, Section 1.3 and 1.5.6]).
- b) \implies c): Let $h \in \mathfrak{a}$ be arbitrary and consider the orbit $W \cdot h \subset \mathfrak{a}$ which is finite by a). As \mathbb{A} is spherical and thus invertible, it follows that \mathcal{D} is cofree, so for every $w \in W$, there exist unique $x_1, \dots, x_n \in \mathbb{R}$ such that $h - w(h) = \sum_{j=1}^n x_j h_j$. We choose $w_0 \in W$ for which the sum $\sum_{j=1}^n x_j \in \mathbb{R}$ is minimal and claim that $h' := w_0(h) \in \overline{C}$. Assume for a contradiction that $\langle c_i, h' \rangle < 0$ for some $i \in I$, then the point $h'' := s_i(h') \in W \cdot h$ would satisfy

$$h - h'' = h - h' + \langle c_i, h' \rangle h_i = \sum_{j \neq i} x_j h_j + (x_i + \langle c_i, h' \rangle) h_i,$$

where the last coefficient would be strictly smaller than x_i , contradicting the choice of w_0 . It follows that $h = w_0^{-1}(h') \in w_0^{-1}(\overline{C}) \subset X_+$.

- c) \implies d): As \mathfrak{a} is a Baire space w.r.t. its standard Euclidean topology and the Tits cone is a countable union of closed Weyl chambers, its topological interior is non-empty if and only if \overline{C} has this property, so if $X_+ = \mathfrak{a}$, then \mathcal{D} must be admissible. For an arbitrary point $h \in C$, we have $\langle c_\alpha, -h \rangle < 0$ for all $\alpha \in \Phi_+^\infty$, but $-h$ is also contained in $\mathfrak{a} = X_+$, so that $|\Phi^\infty| = 2|\Phi_+^\infty| < \infty$ by (iv).
- d) \implies e): This is clear by definition.
- e) \implies a): The action of W on the set Φ defines a homomorphism $W \rightarrow \text{Sym}(\Phi)$, where the latter denotes the group of all permutations of Φ . This map is injective since if $w \in W \setminus \{1\}$, then (2.5.6) implies that there exists some $\alpha \in \Phi_+$ with $w(\alpha) \in \Phi_-$. If Φ is finite, then so is $\text{Sym}(\Phi)$ and therefore also W .
- b) \iff f): This is proved in [Kac90, Theorem 5.6].
- (vii) To prepare for the proof, we first claim that we may w.l.o.g. assume that $d = \dim(\mathfrak{a}) \geq 2$. In fact, we must have $d \geq \ell$ and it is easy to see that the rank of a GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$ cannot be 1 if $n \geq 3$. We therefore only have the following three possibilities if $d = 1$: Either \mathcal{D} is a root datum associated to the GCM $\mathbb{A}^0 = (2)$ of finite type A_1 in which case $W \cong \mathbb{Z}_2$ and $X_+ = \mathfrak{a}$. Here, the assertion is correct and applies to all points of \mathfrak{a} . Otherwise, \mathcal{D} is a root datum of rank 1 associated to one of the two GCM's of affine type $A_1^{(1)}$ or $A_2^{(2)}$ whose Weyl groups are both isomorphic to the infinite dihedral group. In these cases, \mathcal{D} is not free and a \mathbb{Z} -linear dependence between $c_0, c_1 \in \Lambda$ is necessarily given by $c_0 + c_1 = 0$ if $\mathbb{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and $2c_0 + c_1 = 0$ if $\mathbb{A} = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, so that $X_+ = \overline{C} = \{0\}$ has empty interior (cf. Example 2.2.5). At the same time, the reflections $s_0, s_1 \in W$ both induce the map $h \mapsto -h$ on \mathfrak{a} , so the infinite subgroup of $W \cong D_\infty$ generated by $s_0 s_1$ acts trivially on \mathfrak{a} . Thus, no point in \mathfrak{a} has finite stabilizer under the W -action and the statement is vacuously true. From now on, we may therefore suppose that $d \geq 2$.

Let $h \in X_+$ be arbitrary. Upon replacing h by a suitable W -translate, it can be assumed that $h \in \overline{C}$. Moreover, by the proof of (vi), the origin lies in the interior of X_+ if and only if $X_+ = \mathfrak{a}$, which is equivalent to $|W| = |W_0| < \infty$, so we may further suppose that $h \neq 0$. Let D_J be the unique cell of \overline{C} containing h , so that $W_h = W_J$ by (i). It is trivial if and only if $h \in D_\emptyset = C \subset X_+^\circ$, so we may assume that $J \neq \emptyset$. We set $\mathfrak{a}_h := \mathfrak{a}/\mathbb{R}h$ and denote by $p : \mathfrak{a} \rightarrow \mathfrak{a}_h$ the canonical projection. Observe that $\dim(\mathfrak{a}_h) = d - 1 > 0$ by assumption.

For every $j \in J$, we have $\langle c_j, h \rangle = 0$, so $c_j \in \mathfrak{a}^*$ descends to a well-defined linear functional $\hat{c}_j \in \mathfrak{a}_h^*$ and the W_J -action on \mathfrak{a} factors to \mathfrak{a}_h , which allows us to define

$$\overline{C}_h := \{\hat{h} \in \mathfrak{a}_h : \langle \hat{c}_j, \hat{h} \rangle \geq 0 \ \forall j \in J\} \quad \text{and} \quad X_h := \bigcup_{w \in W_J} w(\overline{C}_h) \subset \mathfrak{a}_h.$$

Set $\hat{h}_j := p(h_j)$ for all $j \in J$, let Λ_h^\vee be a \mathbb{Z} -form of \mathfrak{a}_h containing $(\hat{h}_j)_{j \in J}$ and Λ_h its dual. By definition of these representations, the induced action of W_J on \mathfrak{a}_h coincides with the \mathcal{D}' -representation of W_J belonging to the Kac-Moody root datum

$$\mathcal{D}' := (J, \mathbb{A}_J, \Lambda_h, (\hat{c}_j|_{\Lambda_h^\vee})_{j \in J}, (\hat{h}_j)_{j \in J}).$$

Likewise, \overline{C}_h is its closed fundamental chamber and X_h its Tits cone. Note that $p(\overline{C}) \subset \overline{C}_h$ and (2.6.6) implies that $p(X_+) \subset X_h$. By the proof of (vi), we have $|W_J| < \infty$ if and only if $X_h = \mathfrak{a}_h$ and, in turn, this condition is equivalent to the property that there exists an open neighbourhood \hat{U} of $0 \in \mathfrak{a}_h$ that is contained in X_h .

Suppose now that $h \in \overline{C}$ lies in the interior of X_+ and choose an open neighbourhood U of h contained in X_+ . By definition of the quotient topology, the projection $p : \mathfrak{a} \rightarrow \mathfrak{a}_h$ is an open map, so $p(U)$ is an open neighbourhood of $p(h) = 0 \in \mathfrak{a}_h$ contained in $p(X_+) \subset X_h$. Hence, it follows that $X_h = \mathfrak{a}_h$ and $|W_J| < \infty$.

Conversely, if $|W_J| < \infty$ and $X_h = \mathfrak{a}_h$, we construct an open neighbourhood of h that is contained in X_+ . Upon extending $h \neq 0$ to an (ordered) basis $\{v_1, \dots, v_{d-1}, h\}$ of \mathfrak{a} , we may identify \mathfrak{a} with \mathbb{R}^d , \mathfrak{a}_h with \mathbb{R}^{d-1} and $p : \mathfrak{a} \rightarrow \mathfrak{a}_h$ with the map $(x_1, \dots, x_d) \mapsto (x_1, \dots, x_{d-1})$. Let $\hat{U} \subset \mathfrak{a}_h$ be an open ball of some radius $r \in \mathbb{R}_{>0}$ around 0 (w.r.t. the standard Euclidean metric on $\mathfrak{a}_h \cong \mathbb{R}^{d-1}$), then under this identification, the set $p^{-1}(\hat{U}) \subset \mathfrak{a} \cong \mathbb{R}^d$ is the infinite open cylinder $B_r(0) \times \mathbb{R}$, where $B_r(0)$ denotes the open Euclidean ball of radius r around the origin in \mathbb{R}^{d-1} . It contains $h \equiv (0, \dots, 0, 1)$ where c_i is positive if $i \in I \setminus J$ and zero if $i \in J$. For every $w \in W_J$ and $i \in I \setminus J$, we have $i \notin I(w)$ and therefore $w^{-1}(c_i) = c_i + \sum_{j \in J} k_j c_j$ with $k_j \in \mathbb{N}_0$ for all $j \in J$, which is positive at h . By choosing $r, s \in \mathbb{R}_{>0}$ small enough, we may ensure that each of the finitely many functionals $\{w^{-1}(c_i) \mid w \in W_J, i \in I \setminus J\}$ is positive on the finite open cylinder $U := B_r(0) \times (1-s, 1+s)$. This is an open neighbourhood of h in $\mathfrak{a} \cong \mathbb{R}^d$ and we claim that $U \subset X_+$. Let $h' \in U$ be arbitrary and consider the finite orbit $W_J \cdot h' \subset \mathfrak{a}$. By the same argument as in the proof of the implication b) \implies c) of (vi), there is some $w_0 \in W_J$ such that $\langle c_i, w_0(h') \rangle \geq 0$ for all $i \in J$. In addition, for $i \in I \setminus J$, we have $\langle c_i, w_0(h') \rangle = \langle w_0^{-1}(c_i), h' \rangle > 0$ by construction of U , so we conclude that $w_0(h') \in \overline{C}$ and thus $h' \in w_0^{-1}(\overline{C}) \subset X_+$. \square

Consequently, the non-empty cells of X_+ indeed define a partition of the Tits cone and every such cell D can be written as $D = w(D_J)$ for some $w \in W$, where the set $J \subset I$ is unique and $w \in W$ is unique up to right-multiplication by some element of W_J . Hence, D is uniquely determined by the standard coset $wW_J \subset W$ and we call the set $J \subset I$ the *type* of D . With this definition, W acts on X_+ by type-preserving cellular automorphisms. The stabilizer of $D = w(D_J)$ is the parabolic subgroup $wW_Jw^{-1} \subset W$ and it fixes D point-wise. Since W_J is finite if and only if \mathbb{A}_J is spherical, we also conclude that the notion of a “spherical subset” $J \subset I$ in the sense of Definition 2.1.5 is consistent with Definition 1.1.5, which motivates the following definition:

Definition 2.6.6. A cell of X_+ is said to be *spherical* if its stabilizer under the W -action is finite.

Hence, a cell is spherical if and only if its type is a spherical subset of I . The topological interior $X_+^\circ \subset X_+$ is partitioned by the (non-empty) spherical cells and the W -action on X_+ preserves X_+° . The entire discussion of this section applies verbatim to the set $X_- := -X_+ = \bigcup_{w \in W} w(-\overline{C})$. We will sometimes refer to X_+ (resp. X_-) as the *positive* (resp. *negative*) *Tits cone*. By (2.6.6), the intersection $X_+ \cap X_-$ consists of all $h \in \mathfrak{a}$ such that $\langle c_\alpha, h \rangle = 0$ for all but finitely many $\alpha \in \Phi_+^\infty$. Using that $|\Phi_+^\infty| < \infty$ if and only if \mathbb{A} is spherical, we arrive at a simple but useful result:

Corollary 2.6.7. *If \mathbb{A} is not spherical, then $X_+ \cap -C = \emptyset$ and $X_+^\circ \cap X_-^\circ = \emptyset$.* \square

Moreover, given any subset $J \subset I$, our arguments apply equally well to the J -restriction \mathcal{D}_J of \mathcal{D} for which we establish the following notation:

Notation 2.6.8. For every subset $J \subset I$, the \mathcal{D}_J -representation of $W_J = W(\mathbb{A}_J)$ coincides with the restriction of the \mathcal{D} -representation of W to the subgroup W_J . We denote by C_J (resp. $\overline{C_J}$) the open (resp. closed) fundamental chamber of the \mathcal{D}_J -representation and by $X_{J,+}$ (resp. $X_{J,-}$) its positive (resp. negative) Tits cone with topological interior $X_{J,+}^\circ$ (resp. $X_{J,-}^\circ$). We do not introduce a uniform notation for the cells of $X_{J,\pm}$ since this will rarely be necessary.

Corollary 2.6.9. *For every $J \subset I$, it holds that $C \subset C_J$, $\overline{C} \subset \overline{C_J}$, $X_\pm \subset X_{J,\pm}$ and $X_\pm^\circ \subset X_{J,\pm}^\circ$.*

Proof. The first two assertions are clear from the definitions, the third is an immediate consequence of (2.6.6) and the natural inclusion $\Phi_{J,+}^\infty \subset \Phi_+^\infty$ and the last one follows from the third. \square

Remark 2.6.10. Up to now, all formulations in this section have included the possibility that some cells of the Tits cone may be empty. Let us add some remarks on this subtlety.

- (i) If \mathcal{D} is free, then $d \geq n$ and the functionals $(c_i)_{i \in I}$ are linearly independent in \mathfrak{a}^* , so there exist vectors $(v_i)_{i \in I}$ in \mathfrak{a} such that $\langle c_i, v_j \rangle = \delta_{ij}$ for all $i, j \in I$. Given any subset $J \subset I$, we therefore have $v_J := \sum_{i \in I \setminus J} v_i \in D_J$, so no cell of X_+ is empty in this case. More precisely, D_J is a convex cone of dimension $d - |J|$. The intersection of all walls of \overline{C} , i.e. the cell D_I , is a $(d - n)$ -dimensional linear subspace of \mathfrak{a} .
In this situation, the cellular structure of the Tits cone is closely related to that of the Coxeter system (W, S) . First of all, the map $wW_J \mapsto w(D_J)$ defines a bijection between the standard left cosets of (W, S) and the cells of X_+ . In particular, the open Weyl chambers correspond to the elements of W . Second, the walls of X_+ are in bijective correspondence with the reflections of (W, S) or, equivalently, the positive real roots of \mathbb{A} in the sense that every wall of X_+ can be written as $H_\alpha = \ker(c_\alpha)$ for a unique $\alpha \in \Phi_+$ or, equivalently, it can be characterized as the fixed-point set of a unique reflection $s_\alpha \in R \subset W$.
- (ii) If W is finite, then \mathbb{A} is spherical and hence invertible, so that \mathcal{D} is free and every cell of X_+ is non-empty. Moreover, there are only finitely many cells and walls. Therefore, empty cells can only exist if W is infinite and a spherical cell cannot be empty.
- (iii) If \mathcal{D} is not free, it may happen that some cells of the Tits cone are empty. For instance, this is the case whenever \mathcal{D} is not admissible. Note that Corollary 2.6.7 is trivially true in this situation. However, even if \mathcal{D} is admissible or, equivalently, open Weyl chambers are non-empty, it is possible that other cells of X_+ are empty. An example of this phenomenon is provided by the admissible root datum $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}'_2$ of Example 2.6.4 (iv). Using the notation introduced there, the closed fundamental chamber of the \mathcal{D}' -representation is

$$\overline{C} = \{xv_1 + yv_2 + zv_3 + uv_4 \in \mathfrak{a} \mid x, y, z, u \in \mathbb{R}, 2y \leq 2x \leq 2y + z, -z \leq 2u \leq z\}.$$

On the one hand, the open fundamental chamber $C = D_\emptyset$ is non-empty, e.g. it contains the point $\frac{5}{4}v_1 + v_2 + v_3 \in \mathfrak{a}$. On the other hand, the cells $D_{\{1,2\}}$ and $D_{\{3,4\}}$ are both empty. In fact, we have $\ker(c_1) \cap \ker(c_2) \subset \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_4$, but the restrictions of c'_3 and c'_4 to this subspace are negatives of each other, so there does not exist a point in that space where c'_3 and c'_4 are both positive, proving that $D_{\{1,2\}} = \emptyset$. A similar argument applies to $D_{\{3,4\}}$.

We will study the geometry of Weyl group orbits under the \mathcal{D} -representation much more closely in Chapter 4, where we content ourselves with the free and cofree case, but for later applications that do not require this level of detail, it will be useful to have introduced the \mathcal{D} -representation in the current framework. In particular, the notion of admissibility will be relevant to various results in Chapter 6.

To conclude this section, we prove an interesting sufficient condition for admissibility and discuss the question of faithfulness of the \mathcal{D} -representation. Although these results will not be needed anywhere else in the text, they might be of independent interest. Our proof of the next proposition relies on a lemma that we are only going to establish in Chapter 4, but we present the argument here in order not to distract from the main line of reasoning later.

Proposition 2.6.11. *A cell of the Tits cone is non-empty in each of the following situations:*

- (i) *Let $J_1, \dots, J_l \subset I$ and $J := \bigcap_{k=1}^l J_k$. If $D_{J_k} \neq \emptyset$ for all $k \in \{1, \dots, l\}$, then $D_J \neq \emptyset$.*
- (ii) *Let $\mathbb{A}_{p_1}, \dots, \mathbb{A}_{p_s}$ denote all indecomposable components of \mathbb{A} of affine type. If $J = \bigcup_{r=1}^t I_{q_r}$ for some $q_1, \dots, q_t \in \{1, \dots, m\} \setminus \{p_1, \dots, p_s\}$, then $D_{I \setminus J} \neq \emptyset$.*
- (iii) *Let $J_1 \subset I$ be such that $D_{J_1} \neq \emptyset$. For every $k \in \{1, \dots, m\}$ with $I_k \not\subset J_1$, there exists a subset $J_2 \subset J_1$ that satisfies $J_2 \cap I_k = \emptyset$ and $D_{J_2} \neq \emptyset$.*

In the statement of (ii), the set $I \setminus J$ can equivalently be characterized as a union of connected components of I that contains all the affine components I_{p_1}, \dots, I_{p_s} . In order to establish (iii), we will make use of Lemma 4.3.16 which is proved under the global assumption of Chapter 4 that \mathcal{D} is free and cofree. By inspecting the argument, however, it is clear that this is not necessary and that it can easily be adapted to the case of an arbitrary Kac-Moody root datum upon incorporating the character and cocharacter maps. In that part of the proof, we further employ the terminology about colourings of Coxeter diagrams introduced in Section 1.2.

Proof. (i) Choose a point $h_k \in D_{J_k}$ for every $k \in \{1, \dots, l\}$, then it is clear from the definition of D_J in (2.6.4) that $\sum_{k=1}^l h_k \in D_J$.

(ii) Given any $r \in \{1, \dots, t\}$, the indecomposable component \mathbb{A}_{q_r} of \mathbb{A} is of finite or indefinite type and we have seen in the proof of Lemma 2.6.2 that $C_{q_r} \neq \emptyset$. More precisely, the vector $h_{q_r} \in C_{q_r}$ constructed there is a linear combination of $(h_i)_{i \in I_{q_r}}$ and thus contained in the cell $D_{I \setminus I_{q_r}} \subset \overline{C}$. Hence, $D_{I \setminus I_{q_r}} \neq \emptyset$ for all $r \in \{1, \dots, t\}$ and so $D_{I \setminus J} \neq \emptyset$ by (i).

(iii) Suppose that the Coxeter diagram $\Delta(W, S)$ is equipped with the colouring defined by J_1 , i.e. the vertices in J_1 are black and those in $I \setminus J_1$ are white. Since $I_k \not\subset J_1$, it is visible with respect to this colouring. If $J_1 \cap I_k = \emptyset$, the claim is trivially satisfied by taking $J_2 := J_1$. Otherwise, let $h \in D_{J_1}$ be arbitrary, then the proof of Lemma 4.3.16 shows that, by forming a convex combination of appropriate W_k -translates of h , one can construct a point that is contained in D_{J_2} for $J_2 := J_1 \setminus (J_1 \cap I_k)$. \square

The assertions of Proposition 2.6.11 essentially state that whenever a cell of the Tits cone is not empty, then a “larger” cell, i.e. one whose type is a smaller subset of I , is also not empty, and this process can only be “blocked” by the affine components. In particular, we recover the consequence of Lemma 2.6.2 that if no indecomposable component of \mathbb{A} is of affine type, then \mathcal{D} is admissible. The following result sharpens this statement:

Corollary 2.6.12. *If \mathcal{D} is not admissible, there exists an indecomposable component \mathbb{A}_{p_r} of \mathbb{A} of affine type whose index set $I_{p_r} \subset I$ has the property that, given any subset $J \subset I$, the cell D_J can only be non-empty if $I_{p_r} \subset J$. Moreover, the corresponding irreducible factor (W_{p_r}, S_{p_r}) of (W, S) fixes X_+ point-wise.*

Proof. If we set $I_{\text{aff}} := I_{p_1} \sqcup \dots \sqcup I_{p_s}$, then it follows from Proposition 2.6.11 (ii) that $D_{I_{\text{aff}}} \neq \emptyset$. Let $r \in \{1, \dots, s\}$ be arbitrary and suppose that there exists a set $J_r \subset I$ such that $I_{p_r} \not\subset J_r$ and $D_{J_r} \neq \emptyset$. By (i), it can be assumed to be contained in I_{aff} and (iii) implies that $D_{J_r'} \neq \emptyset$ for $J_r' := J_r \setminus (J_r \cap I_{p_r})$. If this argument could be applied to every $r \in \{1, \dots, s\}$, then using the fact that $\bigcap_{r=1}^s J_r' \subset \bigcap_{r=1}^s (I_{\text{aff}} \setminus I_{p_r}) = \emptyset$, we would conclude from (i) that $C = D_\emptyset \neq \emptyset$, i.e. that \mathcal{D} is admissible, a contradiction. Thus, for some $r \in \{1, \dots, s\}$, the asserted property holds. The stabilizer of every non-empty cell of \overline{C} therefore contains the standard subgroup $W_{p_r} \subset W$, so the irreducible factor (W_{p_r}, S_{p_r}) fixes every non-empty cell of \overline{C} point-wise and hence also X_+ . \square

Remark 2.6.13. On the one hand, if \mathcal{D} is admissible, we have seen in Proposition 2.6.5 (ii) that the \mathcal{D} -representation of W is faithful. On the other hand, if \mathcal{D} is not admissible, Corollary 2.6.12 shows that some irreducible factor of (W, S) , necessarily the Weyl group of an affine indecomposable component of \mathbb{A} , acts trivially on the Tits cone. However, it does not claim that this factor acts trivially on all of \mathfrak{a} and, indeed, this assertion would not be true. Intuitively, the difference can be understood by observing that if \mathcal{D} is not admissible, the Tits cone has empty interior, so an element of W can fix X_+ point-wise without acting trivially on \mathfrak{a} . The following two examples illustrate that admissibility is not a necessary condition for faithfulness of the \mathcal{D} -representation:

- (i) The entire discussion of this section can also be applied to the dual root datum \mathcal{D}^* . If \mathcal{D}^* is admissible, then W acts faithfully on $\mathfrak{a}^* = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ via the \mathcal{D}^* -representation and the corresponding dual action coincides with its \mathcal{D} -representation. In general, admissibility of \mathcal{D}^* is not equivalent to that of \mathcal{D} . For example, if \mathbb{A} is an indecomposable GCM of affine type and \mathcal{D} is cofree but not free, e.g. in the case $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$, then \mathcal{D} is not admissible but \mathcal{D}^* is, which implies that the W -actions on \mathfrak{a} and \mathfrak{a}^* are both faithful.
- (ii) There exist root data that are neither admissible nor the dual of an admissible root datum, but for which the \mathcal{D} -representation of W is still faithful. An example is provided by the root datum \mathcal{D} in Example 2.6.4 (iii). It is not difficult to verify that neither \mathcal{D} nor \mathcal{D}^* are admissible. However, the admissible root datum \mathcal{D}' in Example 2.6.4 (iv) was obtained from \mathcal{D} by replacing c_3, c_4, h_3, h_4 by their negatives, so the \mathcal{D} - and \mathcal{D}' -representations of W coincide and the latter is faithful since \mathcal{D}' is admissible.

The second example may appear somewhat artificial since \mathcal{D} may simply be viewed as a union of two admissible root data that are “oriented in the wrong way”. In the indecomposable case, the next result shows that such a phenomenon is impossible. In its formulation and proof, we need the following fact from [Kac90, §6.5]: If $W = W(\mathbb{A})$ is the Weyl group of an indecomposable GCM $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ of affine type, then W has the structure of a semidirect product $W \cong \dot{W} \ltimes T$, where $\dot{W} = W(\dot{\mathbb{A}})$ is the finite Weyl group of the reduced matrix $\dot{\mathbb{A}} \in \mathbb{Z}^{n \times n}$ and $T \cong \mathbb{Z}^n$ is a free abelian subgroup of rank n that is normalized by \dot{W} , called the *translation subgroup* of W .

Lemma 2.6.14. *If \mathbb{A} is an indecomposable GCM of affine type and \mathcal{D} a Kac-Moody root datum associated to \mathbb{A} that is neither free nor cofree, then the \mathcal{D} -representation of W is not faithful. More precisely, in the semidirect product structure $W \cong \dot{W} \ltimes T$, the finite Weyl group \dot{W} acts faithfully and the translation subgroup T acts trivially on \mathfrak{a} .*

Proof. To prove the statement, we recall from [Kac90, §6.5] how the semidirect product structure $W \cong \dot{W} \ltimes T$ is constructed. Let $a_0, \dots, a_n \in \mathbb{N}$ (resp. $a_0^\vee, \dots, a_n^\vee \in \mathbb{N}$) be the labels of the vertices in the Dynkin diagram $\Gamma(\mathbb{A})$ (resp. $\Gamma(\mathbb{A}^T)$) and define

$$\delta := \sum_{i \in I} a_i \alpha_i \in Q \quad \text{and} \quad \theta := \delta - a_0 \alpha_0 = \sum_{i \in \dot{I}} a_i \alpha_i \in \dot{Q}.$$

It can be shown that θ is a root of $\dot{\mathbb{A}}$, i.e. $\theta \in \dot{\Phi} := \Phi^\infty(\dot{\mathbb{A}})$, which is finite and only consists of real roots by Proposition 2.6.5 (vi), and that its corresponding real coroot $\theta^\vee \in \dot{\Phi}^\vee$ is given by

$$\theta^\vee = \frac{1}{a_0} (K - a_0^\vee \alpha_0^\vee) = \frac{1}{a_0} \sum_{i \in \dot{I}} a_i^\vee \alpha_i^\vee \quad \text{where} \quad K := \sum_{i \in I} a_i^\vee \alpha_i^\vee \in Q^\vee.$$

The associated reflection s_θ is contained in \dot{W} and we set $t_\theta := s_0 s_\theta$. The translation subgroup T is defined as the subgroup of W generated by the set $\{w t_\theta w^{-1} \mid w \in \dot{W}\}$. To prove the first statement, it suffices to show that t_θ acts trivially on \mathfrak{a} . Since \mathcal{D} is neither free nor cofree, we have $\sum_{i \in I} a_i c_i = 0$ and $\sum_{i \in I} a_i^\vee h_i = 0$ or, equivalently, $c(\delta) = 0$ and $h(K) = 0$ (cf. Example 2.2.5). Using that $a_0^\vee = 1$ (cf. Convention 2.1.2), we obtain the following for every $h \in \mathfrak{a}$:

$$s_\theta(h) = h - \langle c_\theta, h \rangle h_\theta = h - \langle -a_0 c_0, h \rangle \left(-\frac{1}{a_0} h_0 \right) = h - \langle c_0, h \rangle h_0 = s_0(h)$$

On the one hand, this shows that the actions of s_0 and s_θ coincide, so that t_θ acts trivially on \mathfrak{a} . On the other hand, the \dot{I} -restriction $\dot{\mathcal{D}}$ of \mathcal{D} is free and cofree and, in particular, admissible as $\dot{\mathbb{A}}$ is invertible, so \dot{W} acts faithfully on \mathfrak{a} via the $\dot{\mathcal{D}}$ -representation which coincides with the restriction of the \mathcal{D} -representation to $\dot{W} \subset W$. \square

We have already discussed in Example 2.2.5 that for most GCM's of affine type, it is possible to construct root data that are neither free nor cofree. In general, given a Kac-Moody root datum \mathcal{D} associated to an arbitrary GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$, the question of faithfulness of the \mathcal{D} -representation is fully decided by the affine indecomposable components of \mathcal{D} . We conclude this section with a satisfying criterion which also explains the phenomena in Remark 2.6.13.

Corollary 2.6.15. *Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be an arbitrary Kac-Moody root datum. The following conditions are equivalent:*

- (i) *The \mathcal{D} -representation of W is faithful.*
- (ii) *If $\mathbb{A}_{p_1}, \dots, \mathbb{A}_{p_s}$ denote all indecomposable components of \mathbb{A} that are of affine type, then each of the components $\mathcal{D}_{p_1}, \dots, \mathcal{D}_{p_s}$ of \mathcal{D} is free or cofree.*

Proof. Suppose first that (ii) is violated. If $r \in \{1, \dots, s\}$ is such that \mathcal{D}_{p_r} is neither free nor cofree, then Lemma 2.6.14 shows that the \mathcal{D}_{p_r} -representation of W_{p_r} is not faithful. By definition, this is simply the restriction of the \mathcal{D} -representation of W to the subgroup W_{p_r} , which is therefore not faithful either.

Conversely, assume that (ii) is satisfied and set $I_{\text{aff}} := I_{p_1} \sqcup \dots \sqcup I_{p_s}$. By Proposition 2.6.11 (ii), there exists a point $h \in D_{I_{\text{aff}}}$. If $w \in W$ lies in the kernel of the \mathcal{D} -representation, then $w(h) = h$, so Proposition 2.6.5 (i) implies that $w \in W_{I_{\text{aff}}}$. Let u_1, \dots, u_s be the unique elements such that $u_r \in W_{p_r}$ for all $r \in \{1, \dots, s\}$ and $w = u_1 \dots u_s$. In order to conclude the proof, we need to show that $u_r = 1$ for any arbitrarily fixed $r \in \{1, \dots, s\}$. To this end, our goal is to find a suitable subspace of \mathfrak{a} or \mathfrak{a}^* where each factor $u_{r'}$ with $r' \neq r$ acts trivially. We distinguish the following two cases:

- If \mathcal{D}_{p_r} is free, we define $\Lambda_r := \bigoplus_{i \in I_{p_r}} \mathbb{Z}c_i \subset \Lambda$ and let Λ_r^\vee be its dual \mathbb{Z} -module. By viewing the elements $(h_i)_{i \in I_{p_r}} \in \Lambda^\vee$ as functionals on Λ and restricting them to the submodule Λ_r , we may define a Kac-Moody root datum

$$\mathcal{D}'_{p_r} := (I_{p_r}, \mathbb{A}_{p_r}, \Lambda_r, (c_i)_{i \in I_{p_r}}, (h_i|_{\Lambda_r})_{i \in I_{p_r}}).$$

By construction, \mathcal{D}'_{p_r} is free and thereby admissible, so W_{p_r} acts faithfully on $\mathfrak{a}_r := \Lambda_r^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ via the \mathcal{D}'_{p_r} -representation and hence on $\mathfrak{a}_r^* := \Lambda_r \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{i \in I_{p_r}} \mathbb{R}c_i$ via the dual action, which is simply the restriction of the dual action of the \mathcal{D}_{p_r} -representation of W_{p_r} to the W_{p_r} -invariant subspace $\mathfrak{a}_r^* \subset \mathfrak{a}^*$. If $r' \neq r$, then $\langle c_i, h_j \rangle = 0$ for all $i \in I_{p_r}$ and $j \in I_{p_{r'}}$, which implies that $W_{p_{r'}}$ fixes \mathfrak{a}_r^* point-wise under the dual of the \mathcal{D} -representation of W . Because w acts trivially on \mathfrak{a}^* by assumption and every factor $u_{r'}$ with $r' \neq r$ acts trivially on \mathfrak{a}_r^* , we conclude that $u_r \in W_{p_r}$ acts trivially on \mathfrak{a}_r^* as well, so that $u_r = 1$.

- If \mathcal{D}_{p_r} is cofree, the argument is very similar: Here, we set $\Lambda_r^\vee := \bigoplus_{i \in I_{p_r}} \mathbb{Z}h_i \subset \Lambda^\vee$ and let Λ_r be its dual \mathbb{Z} -module. By restricting the functionals $(c_i)_{i \in I_{p_r}} \in \Lambda$ to the submodule Λ_r^\vee , we obtain another Kac-Moody root datum

$$\mathcal{D}''_{p_r} := (I_{p_r}, \mathbb{A}_{p_r}, \Lambda_r, (c_i|_{\Lambda_r^\vee})_{i \in I_{p_r}}, (h_i)_{i \in I_{p_r}}).$$

As \mathcal{D}''_{p_r} is cofree, its dual root datum is free and thus admissible, so W_{p_r} acts faithfully on $\mathfrak{a}_r := \Lambda_r^\vee \otimes_{\mathbb{Z}} \mathbb{R} = \bigoplus_{i \in I_{p_r}} \mathbb{R}h_i$ via the \mathcal{D}''_{p_r} -representation, which is the restriction of the \mathcal{D}_{p_r} -representation of W_{p_r} to the W_{p_r} -invariant subspace $\mathfrak{a}_r \subset \mathfrak{a}$. By the same argument as above, w acts trivially on \mathfrak{a} and every factor $u_{r'}$ with $r' \neq r$ acts trivially on \mathfrak{a}_r , so that $u_r \in W_{p_r}$ also acts trivially on \mathfrak{a}_r , proving that $u_r = 1$. \square

2.7 Some assumptions and conventions

In order to prepare for later applications, we conclude this chapter by introducing four assumptions and some terminology that will be used freely in Part II. Each of these assumptions gives rise to additional structure on some of the objects defined in this chapter. We continue to denote by $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ an arbitrary Kac-Moody root datum of rank $d \in \mathbb{N}$ and by \mathbb{K} a field.

Assumption 2.7.1. The field \mathbb{K} is either \mathbb{R} or \mathbb{C} .

Our main theorems in Chapter 5 and 6 will all be formulated for real and complex Kac-Moody groups. As we shall see, it will mostly be possible to treat the real and complex case simultaneously, so the following convention will be useful:

Convention 2.7.2. If the Kac-Moody root datum \mathcal{D} is clear from the context and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ either the real or the complex Kac-Moody algebra of type \mathcal{D} . Distinguishing these Lie algebras will rarely be necessary. However, it will be very important to distinguish the real Cartan subalgebra from the complex one, so we are going to write $\mathfrak{a} := \mathfrak{h}_{\mathcal{D}}(\mathbb{R}) = \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{C}) = \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}$. The reason for this notation is that in the complex case, we still often want to consider the real Cartan subalgebra instead of the complex one and notation like $\mathfrak{h}_{\mathcal{D}}(\mathbb{R}) \subset \mathfrak{g}_{\mathcal{D}}(\mathbb{C})$ might lead to confusion. In fact, the real vector space \mathfrak{a} will be used to construct the “Iwasawa decomposition” of real and complex Kac-Moody groups & algebras in Section 3.4. If the field is irrelevant, we write $\mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ more briefly as \mathfrak{u}_0 .

In addition, we denote by $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ the involutive automorphism obtained from pre-composing the Chevalley involution $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ with complex conjugation. More precisely, if $\mathbb{K} = \mathbb{R}$, then θ coincides with the Chevalley involution $\omega_{\mathbb{R}}$ of $\mathfrak{g}_{\mathcal{D}}(\mathbb{R})$ and if $\mathbb{K} = \mathbb{C}$, then θ is the unique extension of $\omega_{\mathbb{R}}$ to a \mathbb{C} -antilinear automorphism of $\mathfrak{g}_{\mathcal{D}}(\mathbb{C})$. In both cases, we call θ the *compact involution* and its fixed-point set \mathfrak{k} the *compact form* of \mathfrak{g} .

Assumption 2.7.3. The Kac-Moody root datum \mathcal{D} is free and cofree.

Under this assumption, the character and cocharacter maps $c : Q \rightarrow \Lambda$ and $h : Q^{\vee} \rightarrow \Lambda^{\vee}$ are injective. This allows us to view Q and Q^{\vee} as subsets of Λ and Λ^{\vee} and employ the following more common notation that was already suggested in Remark 2.3.3 (iv):

Convention 2.7.4. If \mathcal{D} is free and cofree, we will suppress the inclusions $c : Q \hookrightarrow \Lambda$ and $h : Q^{\vee} \hookrightarrow \Lambda^{\vee}$ from the notation. In particular, for every $i \in I$, we will write α_i (resp. α_i^{\vee}) instead of c_i (resp. h_i). For example, this will be applied to the definition of the cells of the Tits cone of the \mathcal{D} -representation of $W(\mathbb{A})$ on $\mathfrak{a} = \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ in (2.6.4). Furthermore, we may extend the linearly independent subset $\Phi_0^{\vee} \subset Q^{\vee} \subset \Lambda^{\vee} \subset \mathfrak{a}$ to an \mathbb{R} -basis $\mathcal{B} := \{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}, h_{n+1}, \dots, h_d\}$ of \mathfrak{a} and we denote by $\mathcal{B}^* := \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\}$ the corresponding dual basis of \mathfrak{a}^* . The functionals $\omega_1, \dots, \omega_n \in \mathfrak{a}^*$ will be called the *fundamental weights*.

The fundamental weights will play a crucial role in the study of convex hulls of Weyl group orbits in Chapter 4. Recall that \mathcal{D} can only be free and cofree if it has rank $d \geq 2n - \ell \geq n$. By definition, the fundamental weights satisfy $\langle \omega_i, \alpha_j^{\vee} \rangle = \delta_{ij}$ for all $i, j \in I$. If $d > n$, they are not uniquely determined by this condition, but their restrictions to the subspace $\mathfrak{a}' = \bigoplus_{i \in I} \mathbb{R} \alpha_i^{\vee} \subset \mathfrak{a}$ are independent of the chosen extension of Φ_0^{\vee} to an \mathbb{R} -basis of \mathfrak{a} . This slight ambiguity will not lead to any significant complications. Freeness of \mathcal{D} further implies that no cell of the Tits cone $X_+ \subset \mathfrak{a}$ of the \mathcal{D} -representation is empty and, in particular, that \mathcal{D} is admissible.

Assumption 2.7.5. The Kac-Moody root datum \mathcal{D} is free, cofree and cotorsion-free.

This strengthening of Assumption 2.7.3 will be important in the proofs of Kostant’s Convexity Theorems for Kac-Moody groups in Chapter 5 as it will enable us to combine the results of Chapter 4 with representation-theoretic properties of Kac-Moody groups. In this situation, we also apply Convention 2.7.4, but instead of only working in the real vector space \mathfrak{a} , we may already extend $\Phi_0^{\vee} \subset Q^{\vee} \subset \Lambda^{\vee}$ to a \mathbb{Z} -basis $\mathcal{B} = \{\alpha_1^{\vee}, \dots, \alpha_n^{\vee}, h_{n+1}, \dots, h_d\}$ of Λ^{\vee} . Hence, the corresponding dual basis $\mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\}$ defines a \mathbb{Z} -basis of Λ and we have $\mathfrak{a}' = \Lambda^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ with $\Lambda^{\vee} / \Lambda'^{\vee} \cong \mathbb{Z}^{d-n}$, where $\Lambda'^{\vee} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i^{\vee} \subset \Lambda^{\vee}$.

Assumption 2.7.6. The GCM \mathbb{A} is symmetrizable.

The relevance of this assumption to Kac-Moody theory in general has already been mentioned in Remark 2.3.2. It ensures that the Gabber-Kac Theorem holds. For our purposes, it yields the existence of a very useful additional structure in highest-weight representations of real or complex Kac-Moody algebras. To explain this, we apply Convention 2.7.2.

Let $\lambda \in \mathfrak{a}^* \subset \mathfrak{h}^*$ be a dominant integral weight and $L(\lambda)_{\mathbb{C}}$ the complex irreducible highest-weight $\mathfrak{g}_{\mathcal{D}}(\mathbb{C})$ -module with highest weight λ . By [Kac90, Proposition 9.4 and Lemma 11.5], there exists a non-degenerate Hermitian form $H : L(\lambda)_{\mathbb{C}} \times L(\lambda)_{\mathbb{C}} \rightarrow \mathbb{C}$ which is *contravariant* in the sense that

$$H(x \cdot v, w) = -H(v, \theta(x) \cdot w) \quad \text{for all } x \in \mathfrak{g}_{\mathcal{D}}(\mathbb{C}) \text{ and } v, w \in L(\lambda)_{\mathbb{C}}. \quad (2.7.1)$$

It is unique up to real scalar multiples and weight spaces of $L(\lambda)_{\mathbb{C}}$ belonging to different weights are H -orthogonal. While the construction of H does not require symmetrizability, it is shown in [Kac90, Theorem 11.7] that if \mathbb{A} is symmetrizable and H is normalized such that $H(v_\lambda, v_\lambda) = 1$ for some highest-weight vector $v_\lambda \in L(\lambda)_{\mathbb{C}}$, then H is positive-definite.

Restriction to $\mathfrak{g}_{\mathcal{D}}(\mathbb{R})$ turns $L(\lambda)_{\mathbb{C}}$ into a complex $\mathfrak{g}_{\mathcal{D}}(\mathbb{R})$ -module and the assumption that $\lambda|_{\mathfrak{a}}$ is real-valued implies that the $\mathcal{U}(\mathfrak{g}_{\mathcal{D}}(\mathbb{R}))$ -submodule $L(\lambda)_{\mathbb{R}} \subset L(\lambda)_{\mathbb{C}}$ generated by v_λ is an \mathbb{R} -form of $L(\lambda)_{\mathbb{C}}$ that coincides with the real irreducible highest-weight $\mathfrak{g}_{\mathcal{D}}(\mathbb{R})$ -module with highest weight λ . Moreover, the Hermitian form H restricts to a positive-definite symmetric bilinear form on $L(\lambda)_{\mathbb{R}}$ which is also denoted by H and contravariant in the sense of (2.7.1) with \mathbb{C} replaced by \mathbb{R} . Note that if \mathcal{D} is free and cofree, the functionals in the dual basis $\mathcal{B}^* \subset \mathfrak{a}^*$ defined above are dominant integral weights and if \mathcal{D} is also cotorsion-free, they are even proper.

Convention 2.7.7. As in the case of the Kac-Moody algebras, it will usually not be necessary to distinguish the real highest-weight modules from the complex ones. Given a dominant integral weight $\lambda \in \mathfrak{a}^* \subset \mathfrak{h}^*$, we will therefore denote by $L(\lambda)$ either the real or the complex irreducible highest-weight \mathfrak{g} -module with highest weight λ and by H the corresponding positive-definite, contravariant, symmetric bilinear or Hermitian form on $L(\lambda)$.

Throughout Part II, we will combine the assumptions introduced in this section in various ways. In some situations, they will be required to hold simultaneously. For ease of reference, we therefore summarize their most important consequences in a final proposition. Its proof is immediate from the previous discussion and Lemma 2.4.5.

Proposition 2.7.8. *With the notational conventions introduced in Section 2.7, let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a symmetrizable GCM of rank $\ell \leq n$, let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a free, cofree and cotorsion-free Kac-Moody root datum of rank $d \in \mathbb{N}$ and denote by \mathfrak{g} either the real or complex Kac-Moody algebra of type \mathcal{D} . Then $d \geq 2n - \ell$ and the following assertions hold:*

- (i) *The dual basis $\mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\} \subset \Lambda$ consists of proper dominant integral weights. Each of the corresponding real or complex irreducible highest-weight \mathfrak{g} -modules $L(\omega_1), \dots, L(\omega_n), L(\phi_{n+1}), \dots, L(\phi_d)$ can be endowed with a positive-definite, contravariant, symmetric bilinear or Hermitian form H with the property that weight spaces belonging to different weights are H -orthogonal.*
- (ii) *If $i \in \{n+1, \dots, d\}$, then the irreducible highest-weight \mathfrak{g} -module $L(\phi_i)$ is one-dimensional. The subalgebras \mathfrak{u}_+ and \mathfrak{u}_- of \mathfrak{g} act trivially on $L(\phi_i)$ and an element $h \in \mathfrak{u}_0$ acts on $L(\phi_i)$ by multiplication with $\langle \phi_i, h \rangle$.*
- (iii) *The open fundamental chamber $C \subset \mathfrak{a}$ of the \mathcal{D} -representation of $W(\mathbb{A})$ is not empty. \square*

At this point, we have gathered enough information about Kac-Moody algebras and established a convenient setup that will turn out to be well-adapted to formulate and prove our main results. Before, however, we still have to introduce Kac-Moody groups in this framework, where we will encounter many definitions and properties that have close analogues on the algebra level.

Chapter 3

Kac-Moody Groups

The way in which Kac-Moody algebras successfully generalize finite-dimensional semisimple Lie algebras naturally suggests trying to construct analogues of classical semisimple Lie groups in the Kac-Moody setting. These *Kac-Moody groups* can be defined in many different ways and varying levels of generality, depending on whether algebraic or analytic methods are preferred. A common approach that was introduced by J. Tits – motivated by the classification of split reductive algebraic groups over arbitrary fields and the scheme-theoretic construction of the Chevalley groups associated to the classical Cartan matrices – is to define a Kac-Moody group over a ring R as the group $\mathcal{G}(R)$ of R -points of a group-valued functor $\mathcal{G} : \mathbb{Z}\text{-alg} \rightarrow \text{Grp}$ satisfying a certain list of axioms. However, Tits also shows that over a field \mathbb{K} , this functor is uniquely determined (up to some non-degeneracy assumption) and the group $\mathcal{G}(\mathbb{K})$ has a concrete presentation in terms of generators and relations (cf. [Tit87, Section 2, 3.6 and 3.9, Theorem 1]). For our purposes, it will suffice to consider Kac-Moody groups over fields defined via this presentation which is called the *constructive Tits functor* in [Mar18, Section 7.4] and [Rém02, Section 8.3]. In particular, we will view them purely as abstract groups without any topological or analytic properties. Before the work of Tits, Kac-Moody groups were already introduced by other authors using different procedures that are typically not suitable for our setting (cf. e.g. the construction of Kac & Peterson in [KP85]). Another noteworthy approach towards Kac-Moody groups that we will not follow was developed by Kumar (cf. [Kum02]). It uses the notion of “ind-varieties” which captures the spirit of classical algebraic geometry more than the functorial perspective.

To a large extent, the material in Section 3.1, 3.2 and 3.6 is contained in [Mar18, Chapter 7] which is based on [Tit87] and [Rém02]. The results of Section 3.3 and 3.4 are mostly well-known facts of Kac-Moody theory but scattered through the literature with few precise proofs, so for the reader’s convenience, we usually provide self-contained arguments there. In Section 3.5, we carry out a number of simple but technical computations for which the author is not aware of an explicit reference. They will be relevant to the proofs of Kostant’s Convexity Theorems for Kac-Moody groups in Chapter 5 and further applications in Chapter 6.

3.1 The constructive Tits functor

As in the previous chapter, we denote by $\mathbb{A} \in \mathbb{Z}^{n \times n}$ a GCM with Weyl group $W := W(\mathbb{A})$, by $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ a Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to \mathbb{A} and by \mathbb{K} a field with multiplicative subgroup $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$. The definition of Kac-Moody groups via the constructive Tits functor involves two groups that will be introduced in the following paragraphs.

The \mathbb{K} -split torus

The first object is an abelian group constructed from the free \mathbb{Z} -module Λ^\vee that is dual to Λ . By viewing \mathbb{K}^* as a \mathbb{Z} -module, we may define another \mathbb{Z} -module by forming the tensor product

$T_\Lambda(\mathbb{K}) := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ which is a group that is isomorphic to $(\mathbb{K}^*)^d$ and called the *(\mathbb{K} -split) torus*. If $\{v_1, \dots, v_d\}$ is a \mathbb{Z} -basis of Λ^\vee , then $T_\Lambda(\mathbb{K})$ is generated by the set $\{v_i \otimes r \mid 1 \leq i \leq d, r \in \mathbb{K}^*\}$. Under the isomorphism $T_\Lambda(\mathbb{K}) \cong (\mathbb{K}^*)^d$, such a generator $v_i \otimes r$ corresponds to the vector in $(\mathbb{K}^*)^d$ having r in the i -th coordinate and 1's everywhere else. The canonical action of the Weyl group W on Λ^\vee defined by the dual of the second map in (2.5.2) induces an action of W on $T_\Lambda(\mathbb{K})$ by group homomorphisms by acting on the first factor, i.e.

$$w(h \otimes r) := w(h) \otimes r \quad \text{for all } w \in W, h \in \Lambda^\vee \text{ and } r \in \mathbb{K}^*. \quad (3.1.1)$$

Equivalently, it is possible to identify $T_\Lambda(\mathbb{K})$ with the group $\text{Hom}_{\text{Grp}}(\Lambda, \mathbb{K}^*)$ of homomorphisms from the free abelian group $\Lambda \cong \mathbb{Z}^d$ into the multiplicative group \mathbb{K}^* . More precisely, there exists an isomorphism $T_\Lambda(\mathbb{K}) \rightarrow \text{Hom}_{\text{Grp}}(\Lambda, \mathbb{K}^*)$ which maps a simple tensor of the form $h \otimes r \in T_\Lambda(\mathbb{K})$ with $h \in \Lambda^\vee$ and $r \in \mathbb{K}^*$ to the following homomorphism:

$$\begin{aligned} r^h : \Lambda &\rightarrow \mathbb{K}^* \\ \lambda &\mapsto r^h(\lambda) := r^{\langle \lambda, h \rangle} \end{aligned} \quad (3.1.2)$$

This is well-defined as $\langle \lambda, h \rangle \in \mathbb{Z}$. In this picture, if $\{v_1, \dots, v_d\}$ is a \mathbb{Z} -basis of Λ^\vee , then $T_\Lambda(\mathbb{K})$ is generated by the homomorphisms $\{r^{v_i} \mid 1 \leq i \leq d, r \in \mathbb{K}^*\}$. Moreover, the W -action on $\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ on the first factor translates into the dual action of W on $\text{Hom}_{\text{Grp}}(\Lambda, \mathbb{K}^*)$, i.e.

$$w(t) := t \circ w^{-1} \quad \text{for } w \in W \text{ and } t \in \text{Hom}_{\text{Grp}}(\Lambda, \mathbb{K}^*). \quad (3.1.3)$$

Indeed, this follows easily by comparing (3.1.1) with (3.1.3) in the case where $t = r^h$ for some $h \in \Lambda^\vee$ and $r \in \mathbb{K}^*$. We will mostly work with the identification $T_\Lambda(\mathbb{K}) \cong \text{Hom}_{\text{Grp}}(\Lambda, \mathbb{K}^*)$. In particular, given any $t \in T_\Lambda(\mathbb{K})$ and $\lambda \in \Lambda$, the expression $t(\lambda)$ is a well-defined element of \mathbb{K}^* .

The root groups

The second ingredient is a family of groups $(U_\alpha(\mathbb{K}))_{\alpha \in \Phi}$ indexed by the set Φ of real roots of \mathbb{A} that are each isomorphic to the additive group of \mathbb{K} and satisfy certain “commutator relations”. For every $\alpha \in \Phi$ and $z \in \mathbb{K}$, we introduce an abstract symbol $x_\alpha(z)$. The *root group* associated to α is the group $U_\alpha(\mathbb{K})$ generated by the set $\{x_\alpha(z) \mid z \in \mathbb{K}\}$ subject to the relations

$$x_\alpha(z_1)x_\alpha(z_2) = x_\alpha(z_1 + z_2) \quad \text{for all } z_1, z_2 \in \mathbb{K}. \quad (3.1.4)$$

Hence, the map $z \mapsto x_\alpha(z)$ defines an isomorphism between the additive group of \mathbb{K} and $U_\alpha(\mathbb{K})$. In particular, there is a natural action of \mathbb{K}^* on $U_\alpha(\mathbb{K})$ that is given by $r \cdot x_\alpha(z) := x_\alpha(rz)$ for $r \in \mathbb{K}^*$ and $z \in \mathbb{K}$. Observe that the root groups arise in pairs in the sense that for every $\alpha \in \Phi$, the group $U_{-\alpha}(\mathbb{K})$ is also defined. They are assembled into a bigger group whose defining relations require an auxiliary notion.

Definition 3.1.1. Let $\alpha, \beta \in \Phi$ be two real roots with $\alpha \neq \pm\beta$. The set $\{\alpha, \beta\} \subset \Phi$ is said to be a *prenilpotent pair* if there exist $w_1, w_2 \in W$ that satisfy $w_1(\{\alpha, \beta\}) \subset \Phi_+$ and $w_2(\{\alpha, \beta\}) \subset \Phi_-$. In this case, the *open interval* between α and β is defined as $] \alpha, \beta[_{\mathbb{N}} := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Phi$.

If $\{\alpha, \beta\}$ is a prenilpotent pair and $w_1, w_2 \in W$ are as in Definition 3.1.1, then $w_1(] \alpha, \beta[_{\mathbb{N}}) \subset \Phi_+$ is contained in the finite set $\Phi_{w_2 w_1^{-1}}$ defined in (2.5.5), so $] \alpha, \beta[_{\mathbb{N}}$ is finite as well.

With this notion, we can formulate the desired relations between the root groups. Recall from the end of Section 2.5 that for every $\alpha \in \Phi$, we have chosen a vector e_α from the double basis $E_\alpha \subset \mathfrak{u}_\alpha$ and defined the set $\mathcal{B}_\Phi = \{e_\alpha \mid \alpha \in \Phi\}$. The *Steinberg group* $\text{St}(\mathbb{K})$ is the group generated by all root groups $(U_\alpha(\mathbb{K}))_{\alpha \in \Phi}$ subject to the following commutator relations:

$$[x_\alpha(z_1), x_\beta(z_2)] = \prod_{\gamma = i\alpha + j\beta \in] \alpha, \beta[_{\mathbb{N}}} x_\gamma(C_{ij}^{\alpha\beta} z_1^i z_2^j) \quad (3.1.5)$$

Here, the brackets on the left-hand side denote the commutator $x_\alpha(z_1)x_\beta(z_2)x_\alpha(z_1)^{-1}x_\beta(z_2)^{-1}$ and these relations are imposed for all $z_1, z_2 \in \mathbb{K}$ whenever $\{\alpha, \beta\} \subset \Phi$ is a prenilpotent pair. The product on the right is taken in an arbitrarily chosen order on the finite set $]\alpha, \beta[_{\mathbb{N}}$. The symbols $C_{ij}^{\alpha\beta}$ denote integers that depend on α and β , the vectors $e_\alpha, e_\beta \in \mathcal{B}_\Phi$ (which are unique up to sign) and the chosen order on $]\alpha, \beta[_{\mathbb{N}}$. They can be determined explicitly by computations inside the universal enveloping algebra of $\mathfrak{g}_{\mathcal{D}}(\mathbb{K})$. A complete explanation of (3.1.5), which will not be necessary for the subsequent discussion, can be found in [Mar18, Section 7.4.3].

Remark 3.1.2. In general, the commutator relations (3.1.5) may be very complicated, but they simplify in some interesting situations.

- (i) If no matrix entry of \mathbb{A} is equal to -1 , it follows from the main result of [Mor88] that for all $\alpha, \beta \in \Phi$, the sum $\alpha + \beta$ is either not a root of \mathbb{A} or imaginary. In the former case, the interval $]\alpha, \beta[_{\mathbb{N}}$ is empty, so $U_\alpha(\mathbb{K})$ and $U_\beta(\mathbb{K})$ commute. In the latter, the pair $\{\alpha, \beta\}$ is not prenilpotent, so there are no relations between the corresponding root groups.
- (ii) Let $\mathbb{A}_1, \dots, \mathbb{A}_m$ be the indecomposable components of \mathbb{A} and Φ_1, \dots, Φ_m their sets of real roots. Then $\Phi = \coprod_{k=1}^m \Phi_k$ and whenever $\alpha \in \Phi_k$ and $\beta \in \Phi_l$ for some $k \neq l \in \{1, \dots, m\}$, the pair $\{\alpha, \beta\} \subset \Phi$ is prenilpotent since $s_\alpha \in W$ exchanges $\pm\alpha$ and fixes β while $s_\beta \in W$ exchanges $\pm\beta$ and fixes α . Also, no non-trivial \mathbb{Z} -linear combination of α and β is a root, so the open interval $]\alpha, \beta[_{\mathbb{N}}$ is empty and the root groups $U_\alpha(\mathbb{K})$ and $U_\beta(\mathbb{K})$ commute.

By definition, the Weyl group of \mathbb{A} acts on the set of real roots which gives rise to an action of W on the free product of all root groups by

$$w(x_\alpha(z)) := x_{w(\alpha)}(\pm z) \quad \text{for all } w \in W, \alpha \in \Phi \text{ and } z \in \mathbb{K}, \quad (3.1.6)$$

where the ambiguous sign is the same as in (2.5.11) and depends on the chosen vectors in \mathcal{B}_Φ . This preserves the commutator relations (3.1.5) and therefore induces an action of W on $\text{St}(\mathbb{K})$. For every $\alpha \in \Phi$ and $r \in \mathbb{K}^*$, we define an element of $\text{St}(\mathbb{K})$ by

$$\tilde{s}_\alpha(r) := x_\alpha(r)x_{-\alpha}(r^{-1})x_\alpha(r). \quad (3.1.7)$$

Observe that $\tilde{s}_\alpha(-r) = \tilde{s}_\alpha(r)^{-1}$. We further set $\tilde{s}_\alpha := \tilde{s}_\alpha(1)$. Moreover, given any $i \in I$, we write $U_{\pm i}(\mathbb{K}) := U_{\pm\alpha_i}(\mathbb{K})$ and call $U_+(\mathbb{K})$ (resp. $U_-(\mathbb{K})$) a *positive* (resp. *negative*) *simple root group*. The elements of $U_{\pm i}(\mathbb{K})$ will be denoted by $x_{\pm i}(z)$ instead of $x_{\pm\alpha_i}(z)$ and we write $\tilde{s}_i(r) := \tilde{s}_{\alpha_i}(r)$ and $\tilde{s}_i := \tilde{s}_{\alpha_i}$.

Kac-Moody groups via presentations

The definition of a Kac-Moody group now combines the Steinberg group with the torus and adds conjugation relations between them that essentially turn the natural W -actions on these groups and the action of \mathbb{K}^* on each root group into inner automorphisms.

Definition 3.1.3. Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum and let \mathbb{K} be a field. The *(split, minimal) Kac-Moody group of type \mathcal{D} over \mathbb{K}* is the group $G_{\mathcal{D}}(\mathbb{K})$ generated by the \mathbb{K} -split torus $T_\Lambda(\mathbb{K})$ and the Steinberg group $\text{St}(\mathbb{K})$ subject to the following relations:

$$tx_i(z)t^{-1} = x_i(t(c_i)z) \quad \text{for } i \in I, t \in T_\Lambda(\mathbb{K}) \text{ and } z \in \mathbb{K} \quad (\text{KMG1})$$

$$\tilde{s}_i t \tilde{s}_i^{-1} = s_i(t) \quad \text{for } i \in I \text{ and } t \in T_\Lambda(\mathbb{K}) \quad (\text{KMG2})$$

$$\tilde{s}_i(r^{-1}) = \tilde{s}_i r^{h_i} \quad \text{for } i \in I \text{ and } r \in \mathbb{K}^* \quad (\text{KMG3})$$

$$\tilde{s}_i x_\alpha(z) \tilde{s}_i^{-1} = x_{s_i(\alpha)}(\pm z) \quad \text{for } i \in I, \alpha \in \Phi \text{ and } z \in \mathbb{K} \quad (\text{KMG4})$$

The sign in (KMG4) is determined by (2.5.11). The *rank* of $G_{\mathcal{D}}(\mathbb{K})$ is the cardinality $n = |I|$ of the index set of the GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$. Moreover, we define the following subgroups of $G_{\mathcal{D}}(\mathbb{K})$:

- (i) The *positive* (resp. *negative*) *unipotent subgroup*: $U_\pm(\mathbb{K}) := \langle U_\alpha(\mathbb{K}) \mid \alpha \in \Phi_\pm \rangle$
- (ii) The *positive* (resp. *negative*) *Borel subgroup*: $B_\pm(\mathbb{K}) := \langle T_\Lambda(\mathbb{K}), U_\pm(\mathbb{K}) \rangle$
- (iii) The *extended Weyl group*: $\tilde{W} := \langle \tilde{s}_i \mid i \in I \rangle$

Remark 3.1.4. By Definition 3.1.3, all Kac-Moody groups considered in this thesis are \mathbb{K} -*split* in the sense that, given any field \mathbb{K} , the torus $T_\Lambda(\mathbb{K})$ is isomorphic to $(\mathbb{K}^*)^d$, where $d \in \mathbb{N}$ is the rank of the root datum \mathcal{D} and thus independent of \mathbb{K} . Note that this is not necessarily equal to the rank of $G_{\mathcal{D}}(\mathbb{K})$ which we have defined as $n = |I|$. A theory of Galois descent that provides an appropriate notion of “non-split” Kac-Moody groups has been developed by Rémy (cf. [Rém02]). In addition, our groups are *minimal* in the sense that the definition of $G_{\mathcal{D}}(\mathbb{K})$ only involves root groups that are associated to real roots. It is certainly desirable to incorporate the imaginary roots on the group level as well. However, this is a delicate matter that can be carried out in many non-equivalent ways and all of them require tools that are beyond the scope of this thesis. For a very good survey of the known constructions, we refer to [Mar18, Chapter 8].

If the Kac-Moody root datum \mathcal{D} and the field \mathbb{K} are clear from the context, we will often write $G := G_{\mathcal{D}}(\mathbb{K})$ and $T := T_\Lambda(\mathbb{K})$ and omit the reference to \mathbb{K} in all subgroups that we have defined. With this notation, Definition 3.1.3 leads to the following consequences:

- a) By (KMG2), the extended Weyl group normalizes the torus and conjugation by \tilde{W} induces the W -action on T . Likewise, by (KMG4), it permutes the root groups according to the W -action on Φ . In particular, every root group is conjugate to a simple one by an element of \tilde{W} . More precisely, let $\alpha \in \Phi$ be written as $\alpha = w(\alpha_i)$ for some $w \in W$ and $i \in I$, choose a reduced decomposition $w = s_{i_1} \dots s_{i_k}$ and set $\tilde{w} := \tilde{s}_{i_1} \dots \tilde{s}_{i_k} \in \tilde{W}$, then $\tilde{w}U_i\tilde{w}^{-1} = U_\alpha$. Since $\tilde{s}_i \in \langle U_i, U_{-i} \rangle$ for all $i \in I$, it also follows that G is generated by the torus together with the positive and negative simple root groups.
- b) As the W -action on Φ is faithful, (KMG4) implies that the map $\tilde{s}_i \mapsto s_i$ ($i \in I$) uniquely extends to a surjective homomorphism $\tilde{W} \rightarrow W$ whose kernel is the subgroup $\tilde{D} \subset \tilde{W}$ generated by the set $\{\tilde{s}_i^2 \mid i \in I\}$, where $\tilde{s}_i^2 = (-1)^{h_i} \in T$ for all $i \in I$ by (KMG3). Hence, we have $W \cong \tilde{W}/\tilde{D}$ and \tilde{D} is a finite abelian group where every element has order two. Conversely, given any $w \in W$, it can be shown that the element $\tilde{w} \in \tilde{W}$ defined in a) is independent of the chosen reduced decomposition of w (cf. [Mar18, Proposition 7.57]). We call \tilde{w} the *canonical representative of w in \tilde{W}* . The group \tilde{D} depends on the Kac-Moody root datum since it is possible that $(-1)^{h_i} = 1$ in G for some $i \in I$. We will see in Section 3.2 that this cannot happen if \mathcal{D} is cofree and cotorsion-free, e.g. if $\mathcal{D} = \mathcal{D}_{\text{Kac}}^\mathbb{A}$ or $\mathcal{D} = \mathcal{D}_{\text{sc}}^\mathbb{A}$, implying that $\tilde{D} \cong (\mathbb{Z}_2)^n$ in this case.
- c) Together with a), it follows from (KMG1) and (KMG2) that every root group is normalized by the torus. More precisely, given $\alpha \in \Phi$ and $t \in T$, conjugating the elements of U_α by t corresponds to multiplication by $t(c_\alpha) \in \mathbb{K}^*$ under the isomorphism $U_\alpha \cong \mathbb{K}$, i.e.

$$tx_\alpha(z)t^{-1} = x_\alpha(t(c_\alpha)z) \quad \text{for all } z \in \mathbb{K}. \quad (3.1.8)$$

In particular, T normalizes U_+ and U_- . Using the theory of buildings that we will not discuss in detail (cf. Remark 3.1.5), it can even be proved that $T \cap U_\pm = \{1\}$, so the Borel subgroups split as semidirect products:

$$B_\pm \cong T \ltimes U_\pm \quad (3.1.9)$$

By similar methods, one can also establish that $B_+ \cap B_- = T$ and $U_+ \cap U_- = \{1\}$.

- d) Whenever $t \in T$ satisfies $t(c_i) = 1$ for all $i \in I$, it follows from (3.1.8) that t commutes with all root groups and is therefore contained in the center Z of G . The converse can be shown to be true as well (cf. [Mar18, Corollary 7.73]), so that

$$Z = \{t \in T : t(c_i) = 1 \ \forall i \in I\}. \quad (3.1.10)$$

- e) Let $\mathbb{A}_1, \dots, \mathbb{A}_m$ be the indecomposable components of \mathbb{A} with real roots Φ_1, \dots, Φ_m and $\mathcal{D}_1, \dots, \mathcal{D}_m$ the corresponding components of \mathcal{D} . For every $k \in \{1, \dots, m\}$, the Kac-Moody group $G^k := G_{\mathcal{D}_k}(\mathbb{K})$ of type \mathcal{D}_k over \mathbb{K} can be identified with the subgroup of G generated by T and all U_α with $\alpha \in \Phi_k$. Since root groups belonging to different indecomposable components commute by Remark 3.1.2 (ii), we conclude that

$$G = \prod_{k=1}^m G^k \quad \text{with} \quad G^k \cap G^l = T \text{ for all } k \neq l \in \{1, \dots, m\}. \quad (3.1.11)$$

Remark 3.1.5. Although we do not have occasion to discuss this subject here in great detail, it is important to mention that the tuple $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ associated to a Kac-Moody group is an *RGD system* of type (W, S) as defined in [AB08, Section 8.6.1]. This is a simple consequence of the fact that the set of real roots Φ is in bijective correspondence with the “roots” of the Coxeter complex $\Sigma(W, S)$ in the sense of [AB08, Definition 3.50] and the commutator relations between the root groups that are built into the presentation of G (cf. [Mar18, Theorem 7.69]). From the general theory of RGD systems, it follows that G acts strongly transitively by type-preserving automorphisms on a thick Moufang *twin building* of type (W, S) which is an essential tool to investigate its algebraic structure. Many of the properties of G listed above are consequences of this result and hold more generally for groups admitting an RGD system. The reader is referred to [AB08, Chapter 8] for a detailed exposition of these notions. In particular, the assertions of c) are proved in [AB08, Section 8.8].

Given a Kac-Moody group $G = G_{\mathcal{D}}(\mathbb{K})$, there is an important family of subgroups of G that will appear frequently throughout our discussion. To introduce them, we use Notation 2.3.6 and 2.5.4 and restrict our attention to positive roots for simplicity. For every subset $J \subset I$, we define the following three subgroups of G :

$$\begin{aligned} G_J &:= \langle T, U_\alpha \mid \alpha \in \Phi_J \rangle \\ U_{J,+}^\perp &:= \langle U_\alpha \mid \alpha \in \Phi_{J,+}^\perp \rangle \\ P_J &:= \langle T, U_\alpha \mid \alpha \in \Phi_+ \cup \Phi_J \rangle \end{aligned} \tag{3.1.12}$$

We call P_J the *standard (positive) parabolic subgroup of G of type J* . As T normalizes every root group, it follows from the commutator relations (3.1.5) that G_J contains U_α if and only if $\alpha \in \Phi$ satisfies $I(\alpha) \subset J$ and P_J is obtained from G_J by including all the remaining positive root groups. If $\alpha \in \Phi_J$ and $\beta \in \Phi_{J,+}^\perp$, then $I(\alpha) \subset J$ and $I(\beta) \not\subset J$. Since every root is either positive or negative, it follows that, given any $i, j \in \mathbb{N}$, the linear combination $i\alpha + j\beta$ is either not a root at all or a positive root that is not supported in J . Hence, we also deduce from the commutator relations that G_J normalizes $U_{J,+}^\perp$. Using the theory of RGD systems, it can further be shown that these groups intersect trivially, so P_J is a semidirect product (cf. [Mar18, Proposition 7.75]):

$$P_J \cong G_J \ltimes U_{J,+}^\perp \tag{3.1.13}$$

The splitting (3.1.13) is called the *Levi decomposition* of P_J with *Levi factor* G_J and *unipotent radical* $U_{J,+}^\perp$. By definition, the Levi factor G_J is isomorphic to the Kac-Moody group $G_{\mathcal{D}_J}(\mathbb{K})$, where \mathcal{D}_J denotes the J -restriction of \mathcal{D} . If $J = \emptyset$, then $P_J = B_+$ and the Levi decomposition reduces to (3.1.9). If $J = I$, then $P_J = G$ and $U_{J,+}^\perp = \{1\}$, so the splitting is trivial in this case. Observe that the subgroup $G_J \subset G$ is the counterpart to the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ on the level of Kac-Moody algebras, but in that setting, we have not constructed any “parabolic subalgebra” of \mathfrak{g} that would correspond to the subgroup $P_J \subset G$. Also, on the group level, we have not introduced “negative” parabolic subgroups of G . These choices were simply a matter of convenience and both notions can be defined in the obvious ways.

Example 3.1.6. Similar to the situation on the algebra level, the properties of Kac-Moody groups largely depend on the type of the underlying GCM. The reader may find it instructive to compare the following assertions with Example 2.3.7. We continue to denote by \mathbb{K} an arbitrary field.

- (i) Parallel to the way in which the Kac-Moody algebras over \mathbb{K} associated to spherical GCM’s are finite-dimensional, \mathbb{K} -split, reductive Lie algebras, the corresponding Kac-Moody groups can be viewed as groups of \mathbb{K} -points of connected, \mathbb{K} -split, reductive algebraic \mathbb{K} -groups. In order to explain this in a bit more detail, it is necessary to use standard terminology from the theory of algebraic groups and recall some deep results about reductive groups. Readers unfamiliar with this subject are referred to [Spr09, Chapters 7-10 and 15-17] for a detailed account. A quick introduction to affine group schemes is contained in [Mar18, Appendix A] and a brief overview of the connection between algebraic groups and Kac-Moody groups can be found in [Mar18, Section 7.4.1] or the introduction to [Tit87].

It is known from the work of Chevalley and Demazure that the classification of connected reductive algebraic groups that are defined and split over \mathbb{K} is independent of the field. More precisely, every such group \mathcal{G} (viewed as a group-valued functor on the category of \mathbb{K} -algebras) is uniquely determined by a root datum in the sense of algebraic geometry. In our terminology, this refers to a Kac-Moody root datum associated to a spherical GCM. It is constructed as follows: Let $\mathcal{T} \subset \mathcal{G}$ be a maximal \mathbb{K} -split torus of dimension $d \in \mathbb{N}$ and define a free \mathbb{Z} -module $\Lambda := X(\mathcal{T})$ as the character group of \mathcal{T} :

$$X(\mathcal{T}) := \text{Hom}(\mathcal{T}, \mathbb{G}_m) \cong \mathbb{Z}^d \quad (3.1.14)$$

Here, $d = \dim(\mathcal{T})$ is independent of the choice of \mathcal{T} and called the rank of \mathcal{G} . Let $n \in \mathbb{N}_0$ be the semisimple rank of \mathcal{G} , i.e. the rank of the derived subgroup $\mathcal{G}' := [\mathcal{G}, \mathcal{G}]$, then $d \geq n$ where equality holds if and only if $\mathcal{G} = \mathcal{G}'$ is semisimple. The adjoint action of \mathcal{T} on the Lie algebra of \mathcal{G} gives rise to a finite set of roots $\Phi := \Phi(\mathcal{G}, \mathcal{T}) \subset X(\mathcal{T})$ which is an abstract root system in an n -dimensional subspace of the real vector space $X(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$. Moreover, \mathcal{G} is generated (as an algebraic group) by the torus \mathcal{T} and a family of root groups $(\mathcal{U}_{\alpha})_{\alpha \in \Phi}$ that are each \mathbb{K} -isomorphic to the one-dimensional additive group \mathbb{G}_a and normalized by \mathcal{T} , while the root groups alone generate the derived subgroup $\mathcal{G}' \subset \mathcal{G}$. If $\mathcal{G} = \mathcal{G}'$ is semisimple, then $d = n$ and $\Lambda_r \subset \Lambda \subset \Lambda_w$, where Λ_r denotes the root lattice and Λ_w the weight lattice of Φ . If \mathcal{G} is reductive but not semisimple, then $d > n$ and \mathcal{G} contains a central torus of dimension $d - n > 0$ that is not contained in \mathcal{G}' . If the families $(c_i)_{i \in I}$ and $(h_i)_{i \in I}$ denote some choice of simple roots and coroots for the root system Φ and $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is defined as its Cartan matrix, then Φ can be identified with $\Phi(\mathbb{A})$ and $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ is a Kac-Moody root datum associated to a spherical GCM that determines \mathcal{G} up to isomorphism.

Conversely, every such root datum gives rise to a unique *Chevalley-Demazure group scheme* which is a connected, split, reductive, affine group scheme over \mathbb{Z} and hence, by restriction to the category of \mathbb{K} -algebras, to a connected, \mathbb{K} -split, reductive algebraic \mathbb{K} -group \mathcal{G} . If the rank of \mathcal{D} is equal to n , then \mathcal{G} is semisimple and every lattice Λ satisfying $\Lambda_r \subset \Lambda \subset \Lambda_w$ occurs in such a root datum. For example, the choice $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ corresponds to $\Lambda = \Lambda_w$ and belongs to the simply connected group \mathcal{G}_{sc} associated to \mathbb{A} , whereas $\mathcal{D} = \mathcal{D}_{\text{min}}^{\mathbb{A}}$ is equivalent to $\Lambda = \Lambda_r$ and arises from the adjoint group \mathcal{G}_{ad} . This actually motivates the names of these root data. In general, \mathcal{G} is a central quotient of \mathcal{G}_{sc} in the sense of algebraic varieties, while \mathcal{G}_{ad} is a central quotient of \mathcal{G} and has trivial center.

The groups obtained by evaluating a Chevalley-Demazure group scheme over a field are known as *Chevalley groups*. Whenever $G := \mathcal{G}(\mathbb{K})$ is such a group, it follows from the Bruhat decomposition that G is generated by the groups of \mathbb{K} -points of the torus and the root groups, i.e. by $T := \mathcal{T}(\mathbb{K})$ and all $U_{\alpha} := \mathcal{U}_{\alpha}(\mathbb{K})$ with $\alpha \in \Phi$. Interestingly, including the torus in this generating set is really necessary, even if $\mathcal{G} = \mathcal{G}'$ is semisimple, because in general, the group $G' := \langle U_{\alpha} \mid \alpha \in \Phi \rangle$ is contained in $\mathcal{G}'(\mathbb{K})$, but it may be a proper subgroup if \mathbb{K} is not algebraically closed.¹

A different notion of “Chevalley groups” over a field \mathbb{K} associated to a spherical GCM \mathbb{A} was introduced by Steinberg in [Ste68, §3]. His approach is more explicit and based on constructing a \mathbb{Z} -form of the finite-dimensional, complex, semisimple Lie algebra belonging to \mathbb{A} and “exponentiating” appropriate representations. However, it is shown in [Ste68, §5] that if \mathbb{K} is algebraically closed, the Chevalley groups over \mathbb{K} in the sense of Steinberg are precisely the \mathbb{K} -points of the connected semisimple algebraic \mathbb{K} -groups. Every such group arises as the restriction to \mathbb{K} of a unique semisimple Chevalley-Demazure group scheme and Steinberg’s construction implies that, for any field \mathbb{K} , his Chevalley groups over \mathbb{K} coincide with the groups of the form $G' = \langle U_{\alpha} \mid \alpha \in \Phi \rangle$ obtained from the Chevalley-Demazure group schemes as above. Therefore, it follows from [Ste68, p.41, Corollary 3] that every such group G' has a presentation whose generators $(U_{\alpha})_{\alpha \in \Phi}$ are the \mathbb{K} -points of the root groups and

¹For example, the semisimple group PGL_2 corresponds to the root datum $\mathcal{D}_{\text{min}}^{\mathbb{A}_0}$, but the subgroup of $\text{PGL}_2(\mathbb{K})$ generated by the \mathbb{K} -points of the root groups is $\text{PSL}_2(\mathbb{K})$. The quotient $\text{PGL}_2(\mathbb{K})/\text{PSL}_2(\mathbb{K}) \cong \mathbb{K}^*/(\mathbb{K}^*)^2$ can be identified with the set of square classes in \mathbb{K}^* which is trivial if \mathbb{K} is algebraically closed.

the relations have the same form as in the constructive Tits functor. Given an arbitrary Chevalley-Demazure group scheme \mathcal{G} , his result extends to a presentation for $G := \mathcal{G}(\mathbb{K})$, i.e. a Chevalley group in the sense of schemes, by separately including the torus $T = \mathcal{T}(\mathbb{K})$ which acts on each root group exactly as in (3.1.8).

Hence, it holds by definition that if \mathcal{D} is a Kac-Moody root datum associated to a spherical GCM \mathbb{A} and \mathbb{K} is a field, the Kac-Moody group $G_{\mathcal{D}}(\mathbb{K})$ coincides with the Chevalley group of type \mathcal{D} over \mathbb{K} , i.e. the group of \mathbb{K} -points of the Chevalley-Demazure group scheme belonging to \mathcal{D} . In particular, it can be viewed as a Zariski-closed subgroup of $\mathrm{GL}_N(\mathbb{K})$ for some $N \in \mathbb{N}$. For example, in the case of the smallest GCM $\mathbb{A} = \mathbb{A}^0 = (2)$, the Kac-Moody group of type $\mathcal{D}_{\mathrm{sc}}^{\mathbb{A}^0}$ over \mathbb{K} is $\mathrm{SL}_2(\mathbb{K})$ while $\mathcal{D}_{\mathrm{min}}^{\mathbb{A}^0}$ leads to the group $\mathrm{PGL}_2(\mathbb{K})$. We discuss these examples in more detail in Section 3.3.

More information about Chevalley groups can be found in [Ste68] and [AB08, Section 7.9.2]. However, apart from the short Section 5.5.2, the theory of algebraic groups will not play a major role in this thesis and even there, we will only work with reductive groups over the fields of real or complex numbers where the abstract language of schemes is not necessary. Thus, although the current example is rather imprecise, it does not contain any assertions that are crucial for later arguments. It merely suggests that Kac-Moody groups over a field \mathbb{K} should be thought of as generalizations of groups of \mathbb{K} -points of connected, \mathbb{K} -split, reductive algebraic \mathbb{K} -groups and coincide with them in the spherical case, which is satisfied in many concrete situations.

- (ii) Similar to the connection between affine Kac-Moody algebras over \mathbb{K} and “loop algebras” $\mathfrak{g}(\mathbb{K}[t, t^{-1}])$, where \mathfrak{g} is a \mathbb{Z} -form of a finite-dimensional, complex, semisimple Lie algebra, affine Kac-Moody groups over \mathbb{K} are related to “loop groups” of the form $\hat{\mathcal{G}}(\mathbb{K}[t, t^{-1}])$, where $\hat{\mathcal{G}}$ is a Chevalley-Demazure group scheme in the sense of (i). For example, if \mathbb{A} is a GCM of untwisted affine type with reduced matrix $\hat{\mathbb{A}}$ and $\hat{\mathcal{D}} := \mathcal{D}_{\mathrm{sc}}^{\hat{\mathbb{A}}}$, then the Kac-Moody group of type $\hat{\mathcal{D}}$ over \mathbb{K} can be identified with the group of \mathbb{K} -points of the simply connected Chevalley-Demazure group scheme $\hat{\mathcal{G}} := \mathcal{G}_{\mathrm{sc}}$ belonging to $\hat{\mathbb{A}}$. If \mathcal{D} is the unique extension of $\hat{\mathcal{D}}$ to a Kac-Moody root datum associated to \mathbb{A} (which always exists in the untwisted case), then $G_{\mathcal{D}}(\mathbb{K})$ is isomorphic to the loop group $\hat{\mathcal{G}}(\mathbb{K}[t, t^{-1}])$. If we take $\mathcal{D} = \mathcal{D}_{\mathrm{sc}}^{\mathbb{A}}$, it is a one-dimensional central extension of $\hat{\mathcal{G}}(\mathbb{K}[t, t^{-1}])$ and if $\mathcal{D} = \mathcal{D}_{\mathrm{Kac}}^{\mathbb{A}}$, it is another semidirect extension of the previous group. We will not make these assertions more precise here and refer to [Mar18, Section 7.6] for much more details and proofs of all these identifications. There, the construction of Kac-Moody groups in the twisted affine case is also outlined.
- (iii) Instead, following [Mar18, Example 7.85], we make the simplest case of the construction in (ii) more explicit. Consider the GCM $\mathbb{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ of untwisted affine type $A_1^{(1)}$ and the Kac-Moody root datum \mathcal{D}^+ of rank 1 constructed in Example 2.2.5, i.e. the unique extension of $\mathcal{D}_{\mathrm{sc}}^{\mathbb{A}^0}$ to a root datum associated to \mathbb{A} . In this situation, the underlying simply connected Chevalley-Demazure group scheme is known to be SL_2 , so for any field \mathbb{K} , the Kac-Moody group $G_{\mathcal{D}^+}(\mathbb{K})$ is isomorphic to the loop group $\mathrm{SL}_2(\mathbb{K}[t, t^{-1}])$. An explicit isomorphism is provided by mapping an element of $T_{\mathbb{A}}(\mathbb{K})$, which can be uniquely written as r^v , where $r \in \mathbb{K}^*$ and $v = h_1$ is a basis for the free \mathbb{Z} -module Λ^{\vee} of rank 1, to the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}$ and sending the elements of the four simple root groups to the following matrices, where $z \in \mathbb{K}$ is arbitrary:

$$\begin{aligned} x_0(z) &\mapsto \begin{pmatrix} 1 & 0 \\ -zt & 1 \end{pmatrix} & x_1(z) &\mapsto \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \\ x_{-0}(z) &\mapsto \begin{pmatrix} 1 & zt^{-1} \\ 0 & 1 \end{pmatrix} & x_{-1}(z) &\mapsto \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \end{aligned}$$

Hence, it follows from (3.1.7) that the canonical representatives of $s_0, s_1 \in W \cong D_{\infty}$ in the extended Weyl group \tilde{W} are

$$\tilde{s}_0 = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix} \quad \text{and} \quad \tilde{s}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By explicitly computing the W -action on the simple roots, it is not difficult to conclude that the real roots of \mathbb{A} are given by $\Phi(\mathbb{A}) = \{k\delta + \alpha_1, k\delta - \alpha_1 \mid k \in \mathbb{Z}\}$, where $\delta = \alpha_0 + \alpha_1$, and that the torus and root groups of $G_{\mathcal{D}+}(\mathbb{K})$ correspond to the following sets of matrices:

$$T_{\Lambda}(\mathbb{K}) = \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \mid r \in \mathbb{K}^* \right\} \quad U_{k\delta+\alpha_1} = \left\{ \begin{pmatrix} 1 & zt^k \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{K} \right\}$$

$$U_{k\delta-\alpha_1} = \left\{ \begin{pmatrix} 1 & 0 \\ -zt^k & 1 \end{pmatrix} \mid z \in \mathbb{K} \right\}$$

In particular, U_+ (resp. U_-) is the subgroup of $\mathrm{SL}_2(\mathbb{K}[t])$ (resp. $\mathrm{SL}_2(\mathbb{K}[t^{-1}])$) consisting of all matrices that are upper-triangular (resp. lower-triangular) with 1's on the diagonal modulo t (resp. t^{-1}). Note that $\mathrm{SL}_2(\mathbb{K}[t, t^{-1}])$ is generated by the matrices on the right and, by our definition of “minimal” Kac-Moody groups, there do not exist any root groups associated to the imaginary roots $\Phi^{\mathrm{im}}(\mathbb{A}) = \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\} \subset \Phi^{\infty}(\mathbb{A})$.

- (iv) If \mathbb{A} is of indefinite type, it is not known whether the Kac-Moody groups associated to \mathbb{A} have any reasonable sort of “concrete realization”, but they certainly cannot be identified with matrix groups as for the spherical and affine types. Indeed, it was proved by Caprace that if \mathcal{D} is a Kac-Moody root datum associated to \mathbb{A} and \mathbb{K} is an infinite field, then $G_{\mathcal{D}}(\mathbb{K})$ is not “linear” in the sense that it does not admit a linear representation $G_{\mathcal{D}}(\mathbb{K}) \rightarrow \mathrm{GL}_N(\mathbb{K}')$ with central kernel, where $N \in \mathbb{N}$ and \mathbb{K}' is another field (cf. [Cap09, Theorem 7.1]). As on the algebra level, the absence of such a realization has the consequence that the general theory of indefinite Kac-Moody groups is much less developed than the affine case.

3.2 Representations of Kac-Moody groups

The definition of Kac-Moody groups via the constructive Tits functor is tailored in such a way that many concepts from the theory of Kac-Moody algebras have natural analogues on the group level. This was already visible in some of the basic definitions, but becomes even more apparent in the representation theory of Kac-Moody groups. We keep the notation of the previous section. In addition, we denote by $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ the Kac-Moody algebra of type \mathcal{D} over \mathbb{K} with Cartan subalgebra $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ and consider again the chosen set $\mathcal{B}_{\Phi} = \{e_{\alpha} \mid \alpha \in \Phi\}$ of vectors from the double bases of the real root spaces of \mathfrak{g} . Whenever (V, ρ) is an integrable \mathfrak{g} -module, the operator $\rho(e_{\alpha}) \in \mathfrak{gl}(V)$ is locally nilpotent for every $\alpha \in \Phi$. Hence, given any $z \in \mathbb{K}$ and $v \in V$, the sum

$$\exp(\rho(ze_{\alpha}))v := \sum_{k=0}^{\infty} \frac{z^k}{k!} \rho(e_{\alpha})^k v \quad (3.2.1)$$

is finite, so there exists a well-defined invertible linear map $\hat{\rho}(x_{\alpha}(z)) := \exp(\rho(ze_{\alpha})) \in \mathrm{GL}(V)$ whose action on V is given by (3.2.1). This defines a representation of the root group U_{α} on V . Furthermore, if $\mathrm{wt}(V, \rho) \subset \Lambda$, then as $V = \bigoplus_{\mu \in \mathrm{wt}(V, \rho)} V_{\mu}$ is assumed to be \mathfrak{h} -diagonalizable, we may define a diagonal action of the torus T on V by setting

$$\hat{\rho}(t)v_{\mu} := t(\mu)v_{\mu} \quad \text{for } t \in T, \mu \in \mathrm{wt}(V, \rho) \text{ and } v_{\mu} \in V_{\mu}. \quad (3.2.2)$$

The defining relations of the constructive Tits functor are chosen in such a way that these actions “fit together” to define a representation of G . More precisely, the map $\hat{\rho}$ that is defined on the torus and the root groups by these conditions uniquely extends to a homomorphism $\hat{\rho} : G \rightarrow \mathrm{GL}(V)$, i.e. to a representation of G on V (cf. [Mar18, Theorem 7.48]). Note that if $v = v_{\mu} \in V_{\mu}$ in (3.2.1) is contained in the weight space of some $\mu \in \mathrm{wt}(V, \rho)$ and $N \in \mathbb{N}_0$ is such that $\rho(e_{\alpha})^{N+1}v_{\mu} = 0$, then (2.4.2) implies the following for all $z \in \mathbb{K}$:

$$\hat{\rho}(x_{\alpha}(z))v_{\mu} = \exp(\rho(ze_{\alpha}))v_{\mu} = v_{\mu} + \sum_{k=1}^N (\text{an element of } V_{\mu+k\alpha}) \quad (3.2.3)$$

In particular, if we take $(V, \rho) = (\mathfrak{g}, \text{ad})$ to be the adjoint representation of \mathfrak{g} whose set of weights is $\text{wt}(\mathfrak{g}, \text{ad}) = \{c_\beta \mid \beta \in \Phi^\infty \cup \{0\}\} \subset \Lambda$, we obtain an adjoint representation $\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$ on the group level that is characterized by:

$$\begin{aligned} \text{Ad}(x_\alpha(z)) &:= \exp(\text{ad}(ze_\alpha)) && \text{for } \alpha \in \Phi \text{ and } z \in \mathbb{K} \\ \text{Ad}(t)x_\beta &:= t(c_\beta)x_\beta && \text{for } t \in T, \beta \in \Phi^\infty \cup \{0\} \text{ and } x_\beta \in \mathfrak{u}_\beta \end{aligned} \quad (3.2.4)$$

It can even be shown that $\text{Ad}(G) \subset \text{Aut}(\mathfrak{g})$ and that the kernel of Ad coincides with the center $Z = \{t \in T : t(c_i) = 1 \ \forall i \in I\}$ of G (cf. [Mar18, Corollary 7.54 and 7.73]). By (KMG4), we have $\tilde{s}_i = \tilde{s}_{-i}$ and hence $\text{Ad}(\tilde{s}_i) = s_i^*$ for all $i \in I$, so the surjective homomorphism $\tilde{W} \rightarrow W$ factors through $W^* \subset \text{Aut}(\mathfrak{g})$. Observe that G acts on all root spaces of \mathfrak{g} via the adjoint representation, also the imaginary ones, but only the real roots are incorporated into the definition of G .

Similarly, if $\lambda \in \Lambda \subset \mathfrak{h}^*$ is a proper dominant integral weight, the irreducible highest-weight module $\rho_\lambda : \mathfrak{g} \rightarrow \mathfrak{gl}(L(\lambda))$ is integrable and every weight of $L(\lambda)$ is of the form $\mu = \lambda - c_\beta \in \Lambda$ for some $\beta \in Q_+$. Hence, it can be lifted to a representation $\hat{\rho}_\lambda : G \rightarrow \text{GL}(L(\lambda))$ on the group level that is defined as follows:

$$\begin{aligned} \hat{\rho}_\lambda(x_\alpha(z)) &:= \exp(\rho_\lambda(ze_\alpha)) && \text{for } \alpha \in \Phi \text{ and } z \in \mathbb{K} \\ \hat{\rho}_\lambda(t)v_\mu &:= t(\mu)v_\mu && \text{for } t \in T, \mu \in \text{wt}(L(\lambda)) \text{ and } v_\mu \in L(\lambda)_\mu \end{aligned} \quad (3.2.5)$$

- Remark 3.2.1.* (i) Suppose that the Kac-Moody root datum \mathcal{D} is cofree and cotorsion-free. Let $\mathcal{B} = \{\alpha_1^\vee, \dots, \alpha_n^\vee, h_{n+1}, \dots, h_d\}$ and $\mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\}$ be the dual \mathbb{Z} -bases of Λ^\vee and Λ introduced after Assumption 2.7.5.² For every $i \in I$, the fundamental weight $\omega_i \in \Lambda$ is a proper dominant integral weight, so G acts on the irreducible highest-weight module $L(\omega_i)$. Since $\langle \omega_i, \alpha_i^\vee \rangle = 1$, it follows from (3.1.2) and (3.2.5) that the action of $\tilde{s}_i^2 = (-1)^{\alpha_i^\vee} \in \tilde{D} \subset T$ is non-trivial on the highest-weight space $L(\omega_i)_{\omega_i}$, so \tilde{s}_i^2 is not the neutral element of G and $\tilde{D} \cong (\mathbb{Z}_2)^n$.
- (ii) For an arbitrary root datum \mathcal{D} , a very similar argument using the adjoint representation and a direct sum of irreducible highest-weight modules shows that the torus T and every root group U_α with $\alpha \in \Phi$ embeds into G , i.e. the natural homomorphisms $T \rightarrow G$ and $U_\alpha \rightarrow G$ are injective (cf. [Mar18, Remark 7.52 and Corollary 7.65]).
- (iii) In order to lift the highest-weight \mathfrak{g} -module $L(\lambda)$ of some dominant integral weight $\lambda \in \mathfrak{h}^*$ to the group level, it is really necessary that λ is proper, i.e. that $\lambda \in \Lambda \subset \mathfrak{h}^*$. This is visible in (3.2.5) since we need the expression $t(\lambda)$ to be a well-defined element of \mathbb{K}^* for all $t \in T$. Hence, the representation theory of Kac-Moody groups really depends on the precise root datum and not only on the underlying GCM. For example, as we will see in the next section, for the GCM $\mathbb{A}^0 = (2)$, the Kac-Moody group $G_{\mathcal{D}}(\mathbb{K})$ is isomorphic to $\text{SL}_2(\mathbb{K})$ if $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$ and to $\text{PGL}_2(\mathbb{K})$ if $\mathcal{D} = \mathcal{D}_{\text{min}}^{\mathbb{A}^0}$. However, these groups do not have the same highest-weight representations because e.g. the standard representation of $\text{SL}_2(\mathbb{K}) \subset \text{GL}_2(\mathbb{K})$ on \mathbb{K}^2 does not descend to $\text{PGL}_2(\mathbb{K})$. In fact, this phenomenon is well-known in the theory of reductive algebraic groups.

3.3 Rank-1 groups

Although Kac-Moody groups usually cannot be viewed as matrix groups, it is sometimes possible to reduce arguments to sufficiently small subgroups that do possess such a realization. A particularly important fact is that every Kac-Moody group contains many subgroups that are isomorphic to $\text{SL}_2(\mathbb{K})$ or $\text{PSL}_2(\mathbb{K})$. In order to explain this, we first discuss the structure of an arbitrary rank-1 Kac-Moody group, i.e. a group defined by a root datum associated to the GCM $\mathbb{A}^0 = (2)$. The Weyl group of \mathbb{A}^0 is $W^0 := W(\mathbb{A}^0) = \{1, s_1\} \cong \mathbb{Z}_2$ and its roots are $\Phi^0 := \Phi(\mathbb{A}^0) = \{\alpha_1, -\alpha_1\}$. In this section, our standing assumption that every field \mathbb{K} has characteristic zero implies that $\text{PSL}_2(\mathbb{K})$ is a simple group.

²The existence of these dual bases does not require \mathcal{D} to be free.

Let $\mathcal{D}^0 = (\{1\}, \mathbb{A}^0, \Lambda, c_1, h_1)$ be a Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to $\mathbb{A}^0 = (2)$. If $d = 1$, there are precisely two possibilities: Let $v \in \Lambda^\vee$ and $v^* \in \Lambda$ be dual basis vectors, then $c_1 = kv^*$ and $h_1 = lv$ for some $k, l \in \mathbb{Z}$ that satisfy $2 = \langle c_1, h_1 \rangle = kl$, where we may assume that $k, l > 0$ upon possibly replacing v and v^* by their negatives. Hence, we either have $h_1 = v$ and $c_1 = 2v^*$ or $h_1 = 2v$ and $c_1 = v^*$. Both root data are trivially free and cofree. In the former case, \mathcal{D}^0 is coadjoint but not adjoint and equal to the simply connected root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$. In the latter, \mathcal{D}^0 is adjoint but not coadjoint and coincides with the minimal adjoint root datum $\mathcal{D}_{\text{min}}^{\mathbb{A}^0}$. Also, we observe that $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$ is cotorsion-free but $\mathcal{D}_{\text{min}}^{\mathbb{A}^0}$ is not.

The Kac-Moody groups defined by these root data have simple descriptions. Given any field \mathbb{K} , we claim that $G := G_{\mathcal{D}^0}(\mathbb{K})$ is isomorphic to $\text{SL}_2(\mathbb{K})$ if $\mathcal{D}^0 = \mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$ and to $\text{PGL}_2(\mathbb{K})$ if $\mathcal{D}^0 = \mathcal{D}_{\text{min}}^{\mathbb{A}^0}$. To prove this, we define the following matrices in $\text{GL}_2(\mathbb{K})$:

$$\begin{aligned} x_+(z) &:= \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_-(z) := \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \quad \text{for } z \in \mathbb{K} \\ h(r, s) &:= \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \quad \text{for } r, s \in \mathbb{K}^* \end{aligned} \tag{3.3.1}$$

It is a simple consequence of the Gauss algorithm that the group $\text{SL}_2(\mathbb{K})$ is generated by the set $\{x_+(z), x_-(z) \mid z \in \mathbb{K}\}$ and that $\text{GL}_2(\mathbb{K})$ is generated by $\{h(r, s), x_+(z), x_-(z) \mid r, s \in \mathbb{K}^*, z \in \mathbb{K}\}$. Moreover, the center of $\text{SL}_2(\mathbb{K})$ is $\{\mathbb{I}_2, -\mathbb{I}_2\}$ and that of $\text{GL}_2(\mathbb{K})$ is $\mathbb{K}^*\mathbb{I}_2$. We use square brackets to denote the equivalence classes of matrices in $\text{PGL}_2(\mathbb{K}) = \text{GL}_2(\mathbb{K})/\mathbb{K}^*\mathbb{I}_2$.

- Suppose first that $\mathcal{D}^0 = \mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$. Since $h_1 = v$ is a \mathbb{Z} -basis of $\Lambda^\vee = \Lambda_{\text{sc}}^\vee$, (KMG3) implies that G is generated by the root groups U_1 and U_{-1} . By inspecting the other relations, it also follows that the map which takes $x_+(z)$ to $x_1(z)$ and $x_-(z)$ to $x_{-1}(z)$ for all $z \in \mathbb{K}$ uniquely extends to a homomorphism $\varphi : \text{SL}_2(\mathbb{K}) \rightarrow G$ mapping $h(r, r^{-1})$ to $r^{h_1} \in T$ for every $r \in \mathbb{K}^*$. This is surjective as both root groups of G are contained in the image of φ . As $\text{PSL}_2(\mathbb{K})$ is simple, the only non-trivial proper normal subgroup of $\text{SL}_2(\mathbb{K})$ is its center $\{\mathbb{I}_2, -\mathbb{I}_2\}$, but $-\mathbb{I}_2 = h(-1, -1)$ gets mapped to $(-1)^{h_1}$ which is not the neutral element of G by Remark 3.2.1 (i). We conclude that φ is injective as well and $G \cong \text{SL}_2(\mathbb{K})$.
- Assume now that $\mathcal{D}^0 = \mathcal{D}_{\text{min}}^{\mathbb{A}^0}$. In this case, $h_1 = 2v$ is not a \mathbb{Z} -basis of $\Lambda^\vee = \Lambda_{\text{min}}^\vee$, which implies that G is not generated by the root groups alone, but the homomorphism above can be extended to $\text{GL}_2(\mathbb{K})$. More precisely, the map which takes $x_\pm(z)$ to $x_{\pm 1}(z)$ for $z \in \mathbb{K}$ and $h(r, s)$ to $(rs^{-1})^v \in T$ for $r, s \in \mathbb{K}^*$ uniquely extends to a surjective homomorphism $\hat{\varphi} : \text{GL}_2(\mathbb{K}) \rightarrow G$. Since $r\mathbb{I}_2 = h(r, r)$ is mapped to $1^v = 1$ for all $r \in \mathbb{K}^*$, it descends to a surjective homomorphism $\varphi : \text{PGL}_2(\mathbb{K}) \rightarrow G$. Restricted to the subgroup $\text{PSL}_2(\mathbb{K})$, it must be trivial or injective by simplicity, but the former is impossible because if $z \neq 0$, then $\varphi([x_\pm(z)]) = x_{\pm 1}(z) \neq 1$ by Remark 3.2.1 (ii). Finally, the left cosets of $\text{PSL}_2(\mathbb{K})$ in $\text{PGL}_2(\mathbb{K})$ are represented by elements of the form $[h(r, 1)]$, where $r \in \mathbb{K}^*$ runs over a set of representatives of the square classes of \mathbb{K}^* . If $r \in \mathbb{K}^*$ is not a square, then $\varphi([h(r, 1)]) = r^v$ does not lie in the image of $\text{PSL}_2(\mathbb{K})$ under φ , which implies that φ is injective on every left coset and thus an isomorphism, so $G \cong \text{PGL}_2(\mathbb{K})$.

Hence, the distinct root data $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$ and $\mathcal{D}_{\text{min}}^{\mathbb{A}^0}$ define non-isomorphic Kac-Moody groups, whereas the corresponding Kac-Moody algebras are both isomorphic to $\mathfrak{sl}_2(\mathbb{K})$. This phenomenon is precisely what we alluded to in Remark 2.3.3 (iii) where we claimed that the notion of cotorsion-freeness is only visible on the group level.

If the rank $d \in \mathbb{N}$ is arbitrary, then \mathcal{D}^0 is either cotorsion-free or $h_1 = 2v$ for some $v \in \Lambda^\vee$ that can be extended to a \mathbb{Z} -basis of Λ^\vee . Let v_1 be either h_1 or v , depending on which case occurs, and set $\Lambda_1^\vee := \mathbb{Z}v_1$, then $\mathcal{D}_1 := (\{1\}, \mathbb{A}^0, \Lambda_1, c_1|_{\Lambda_1^\vee}, h_1)$ is a Kac-Moody root datum of rank 1 and the associated Kac-Moody group $G_1 := G_{\mathcal{D}_1}(\mathbb{K})$ is isomorphic to $\text{SL}_2(\mathbb{K})$ or $\text{PGL}_2(\mathbb{K})$. It can be identified with the subgroup of $G := G_{\mathcal{D}^0}(\mathbb{K})$ generated by the root groups U_1 and U_{-1} and the subtorus $T_1 := \Lambda_1^\vee \otimes_{\mathbb{Z}} \mathbb{K}^* \cong \mathbb{K}^*$. If we choose a complement Λ_2^\vee to Λ_1^\vee in Λ^\vee and define $T_2 := \Lambda_2^\vee \otimes_{\mathbb{Z}} \mathbb{K}^* \cong (\mathbb{K}^*)^{d-1}$, then T is a direct product $T \cong T_1 \times T_2$ and G a semidirect product

$$G \cong G_1 \rtimes T_2. \tag{3.3.2}$$

In view of (KMG1), it is a direct product if and only if $t(c_1) = 1$ for all $t \in T_2$ or, equivalently, $c_1|_{\Lambda_2^\vee} \equiv 0$, but it is not necessarily possible to choose a complement with this property.

The previous results can now be transferred to a general Kac-Moody group in the following way: Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be an arbitrary Kac-Moody root datum, let \mathbb{K} be a field and set $G := G_{\mathcal{D}}(\mathbb{K})$. Given any $i \in I$, the same argument as above shows that the elements $c_i \in \Lambda$ and $h_i \in \Lambda^\vee$ give rise to a Kac-Moody root datum of rank 1 associated to \mathbb{A}^0 such that the corresponding Kac-Moody group \hat{G}_i is a subgroup of G isomorphic to $\mathrm{SL}_2(\mathbb{K})$ or $\mathrm{PGL}_2(\mathbb{K})$. More precisely, in either case, there exists a homomorphism $\varphi_i : \mathrm{SL}_2(\mathbb{K}) \rightarrow G$ that is characterized by $\varphi_i(x_\pm(z)) = x_{\pm i}(z)$ for all $z \in \mathbb{K}$. Its image is the subgroup $G_i := \langle U_i, U_{-i} \rangle \subset \hat{G}_i$ of G generated by the root groups U_i and U_{-i} . If $\hat{G}_i \cong \mathrm{SL}_2(\mathbb{K})$, then φ_i is injective and defines an isomorphism onto $G_i = \hat{G}_i$. If $\hat{G}_i \cong \mathrm{PGL}_2(\mathbb{K})$, then the kernel of φ_i is the center $\{\mathbb{I}_2, -\mathbb{I}_2\}$ of $\mathrm{SL}_2(\mathbb{K})$ and its image is isomorphic to $\mathrm{PSL}_2(\mathbb{K})$. In this situation, the index of $G_i \cong \mathrm{PSL}_2(\mathbb{K})$ in $\hat{G}_i \cong \mathrm{PGL}_2(\mathbb{K})$ coincides with the number of square classes in \mathbb{K}^* . Equivalently, φ_i is injective if and only if $\varphi_i(-\mathbb{I}_2) = (-1)^{h_i} \neq 1$. For example, this is always satisfied if \mathcal{D} is cofree and cotorsion-free. Observe that in the non-injective case, we could even define φ_i on all of $\mathrm{GL}_2(\mathbb{K})$, but we will not need this extension.

More generally, let $\alpha = w_\alpha(\alpha_i) \in \Phi_+$ be a positive real root, where $w_\alpha \in W$ and $i \in I$ are the elements that were chosen in Section 2.5 to define the vectors $e_\alpha, f_\alpha \in \mathcal{B}_\Phi$. Denote by $\tilde{w}_\alpha \in \tilde{W}$ the canonical representative of w_α in \tilde{W} and set $\varphi_\alpha := \mathrm{conj}_{\tilde{w}_\alpha} \circ \varphi_i : \mathrm{SL}_2(\mathbb{K}) \rightarrow G$. The image $G_\alpha := \varphi_\alpha(\mathrm{SL}_2(\mathbb{K})) = \langle U_\alpha, U_{-\alpha} \rangle$ is a subgroup of G that is isomorphic to $\mathrm{SL}_2(\mathbb{K})$ or $\mathrm{PSL}_2(\mathbb{K})$ and contains $\tilde{s}_\alpha(r)$ for every $r \in \mathbb{K}^*$. More precisely, the following holds:

$$\tilde{s}_\alpha(r) = x_\alpha(r)x_{-\alpha}(r^{-1})x_\alpha(r) = \varphi_\alpha(x_+(r)x_-(r^{-1})x_+(r)) = \varphi_\alpha \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix} \quad (3.3.3)$$

The group homomorphism $\varphi_\alpha : \mathrm{SL}_2(\mathbb{K}) \rightarrow G$ can be viewed as a “lift” of the corresponding Lie algebra homomorphism $\varphi'_\alpha : \mathfrak{sl}_2(\mathbb{K}) \rightarrow \mathfrak{g}$ that was defined in (2.5.10).

Example 3.3.1. If we identify $\mathrm{SL}_2(\mathbb{K}[t, t^{-1}])$ with a Kac-Moody group as in Example 3.1.6 (iii), the homomorphism $\varphi_0 : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}_2(\mathbb{K}[t, t^{-1}])$ is the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -ct^{-1} \\ -bt & a \end{pmatrix}$$

and $\varphi_1 : \mathrm{SL}_2(\mathbb{K}) \rightarrow \mathrm{SL}_2(\mathbb{K}[t, t^{-1}])$ is the natural inclusion.

Notation 3.3.2. Given an arbitrary Kac-Moody group G over a field \mathbb{K} , we have seen that there exist a lot of homomorphisms from $\mathrm{SL}_2(\mathbb{K})$ into G and in a sense, we may view $\mathrm{SL}_2(\mathbb{K})$ as the “smallest” Kac-Moody group. In order to emphasize this special role, we will often denote $\mathrm{SL}_2(\mathbb{K})$ by G^0 if the field \mathbb{K} is clear from the context and decorate all related symbols (e.g. subgroups) also by a superscript 0 . For example, we will write B_+^0 for the positive Borel subgroup of $\mathrm{SL}_2(\mathbb{K})$. Under the identifications in this section, this coincides with the group of all upper-triangular (2×2) -matrices having determinant 1. Likewise, we will frequently write $\mathfrak{g}^0 := \mathfrak{sl}_2(\mathbb{K})$ and apply the same convention on the Lie algebra level. The subgroups $(G_\alpha)_{\alpha \in \Phi_+}$ of a Kac-Moody group G constructed above will be called the G^0 -images in G .

Remark 3.3.3. If $\alpha = \alpha_i$ for some $i \in I$, the G^0 -image $G_{\alpha_i} = G_i = \varphi_i(G^0) = \langle U_i, U_{-i} \rangle \subset G$ defined in this section should not be confused with the Levi factor $G_{\{i\}}$ of the standard parabolic subgroup $P_{\{i\}}$ of type $\{i\}$. In fact, the latter contains the full torus T , whose dimension coincides with the rank of \mathcal{D} , whereas $G_i \cap T$ is the one-dimensional subtorus $\{r^{h_i} \mid r \in \mathbb{K}^*\} \subset T$. However, this is the only difference between these groups since $G_i \subset G_{\{i\}}$ and $G_{\{i\}} = TG_i$. We therefore must not omit the set brackets when working with $P_{\{i\}}$ and its Levi factor, but this will rarely be necessary. The group G_i will play a much more prominent role in our discussion, so we have chosen to reserve for it this characteristic notation. The subgroup $\hat{G}_i \subset G$, which may be slightly larger than G_i if φ_i is not injective, will not be important. Finally, observe that if G_i happens to be isomorphic to $\mathrm{PSL}_2(\mathbb{K})$, it is not quite a Kac-Moody group in the sense of Definition 3.1.3, which is why we prefer to call it a “ G^0 -image” rather than a “rank-1 subgroup” of G .

3.4 Real and complex Kac-Moody groups

In this section, we specialize to the case $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ which will later be the framework for our main theorems in Chapter 5 and 6. In the following, we fix an arbitrary Kac-Moody root datum $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ of rank $d \in \mathbb{N}$ and set $W := W(\mathbb{A})$. We denote by $G := G_{\mathcal{D}}(\mathbb{K})$ (resp. $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$) either the real or the complex Kac-Moody group (resp. algebra) of type \mathcal{D} and only indicate the field if there is risk of confusion. At the same time, it will be crucial to distinguish the real Cartan subalgebra from the complex one, so we also employ Convention 2.7.2 and write $\mathfrak{a} := \mathfrak{h}_{\mathcal{D}}(\mathbb{R})$ and $\mathfrak{h} := \mathfrak{h}_{\mathcal{D}}(\mathbb{C})$ as well as $\mathfrak{u}_0 := \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$ when this distinction is not necessary.

Working in the real or complex setting allows for a number of new constructions. To begin with, let $\{v_1, \dots, v_d\}$ be a \mathbb{Z} -basis of Λ^\vee and consider the following map:

$$\exp : \mathfrak{h} \rightarrow T_\Lambda(\mathbb{C}) \quad \sum_{i=1}^d z_i v_i \mapsto \prod_{i=1}^d (e^{z_i})^{v_i} \quad (3.4.1)$$

It is not difficult to verify that this definition is independent of the chosen basis and the resulting map $\exp : \mathfrak{h} \rightarrow T_\Lambda(\mathbb{C})$ is called the *exponential map*. Under the natural isomorphisms $\mathfrak{h} \cong \mathbb{C}^d$ and $T_\Lambda(\mathbb{C}) \cong (\mathbb{C}^*)^d$ induced by this basis, it corresponds to the coordinate-wise exponential map $\mathbb{C} \rightarrow \mathbb{C}^*$. Thus, it is a group homomorphism that is surjective but not injective. The Weyl group W acts on $\mathfrak{h} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ and $T_\Lambda(\mathbb{C}) = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{C}^*$ by extension of scalars of the W -action on Λ^\vee and a simple computation using (3.4.1) shows that \exp is W -equivariant. Its restriction to the real form $\mathfrak{a} \subset \mathfrak{h}$ is injective, the image $A := \exp(\mathfrak{a})$ is isomorphic to $(\mathbb{R}_{>0})^d$ and the W -action on \mathfrak{h} (resp. $T_\Lambda(\mathbb{C})$) preserves \mathfrak{a} (resp. A). Therefore, we will usually consider the exponential map as a W -equivariant isomorphism $\exp : \mathfrak{a} \rightarrow A$. Its inverse will be denoted by $\log : A \rightarrow \mathfrak{a}$ and called the *logarithm*. The extended Weyl group normalizes A and conjugation by \tilde{W} induces the natural W -action on A . Since A is contained in $T_\Lambda(\mathbb{R})$, it can be viewed as a subgroup of $G_{\mathcal{D}}(\mathbb{R})$ and $G_{\mathcal{D}}(\mathbb{C})$. In this picture, the polar decomposition in \mathbb{C} and its trivial restriction to \mathbb{R} imply that $T_\Lambda(\mathbb{K}) \cong M \times A$, where M is isomorphic to $(S^1)^d$ if $\mathbb{K} = \mathbb{C}$ and to $(\mathbb{Z}_2)^d$ if $\mathbb{K} = \mathbb{R}$. In addition, the compact involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ has an analogue on the group level. More precisely, it follows directly from the defining relations (KMG1)-(KMG4) of G that there exists a unique involutive automorphism $\Theta : G \rightarrow G$ with the following properties:

$$\begin{aligned} \Theta(x_\alpha(z)) &= x_{-\alpha}(\bar{z}) \quad \text{for } \alpha \in \Phi \text{ and } z \in \mathbb{K} \\ \Theta(r^h) &= \bar{r}^{-h} \quad \text{for } h \in \Lambda^\vee \text{ and } r \in \mathbb{K}^* \end{aligned} \quad (3.4.2)$$

Here, the bars denote complex conjugation which can be omitted if $\mathbb{K} = \mathbb{R}$. Parallel to the situation on the algebra level, we call Θ the *compact involution* of G . By definition, it preserves $T := T_\Lambda(\mathbb{K})$ and A and maps U_α to $U_{-\alpha}$ for all $\alpha \in \Phi$, which implies that $\Theta(U_+) = U_-$ and $\Theta(B_+) = B_-$. The fixed-point set of Θ is denoted by K and called the *compact form* of G .

Example 3.4.1. Let us make these definitions explicit for the Kac-Moody group $G^0 = \text{SL}_2(\mathbb{K})$ which is associated to the Kac-Moody root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$. In the following, we make extensive use of Notation 3.3.2. The construction in Section 3.3 implies that its torus T^0 can be identified with the group of diagonal matrices with determinant 1 and the positive (resp. negative) root group U_+^0 (resp. U_-^0) with the group of upper-triangular (resp. lower-triangular) matrices with 1's on the diagonal. Hence, it is immediate from (3.4.2) that the compact involution $\Theta^0 : G^0 \rightarrow G^0$ is the map $g \mapsto (g^{-1})^* := \overline{(g^{-1})}^T$. Its fixed-point set $K^0 \subset G^0$ is the special orthogonal group $\text{SO}(2)$ if $\mathbb{K} = \mathbb{R}$ and the special unitary group $\text{SU}(2)$ if $\mathbb{K} = \mathbb{C}$. Moreover, the corresponding Kac-Moody algebra can be identified with $\mathfrak{g}^0 = \mathfrak{sl}_2(\mathbb{K})$ and the real Cartan subalgebra \mathfrak{a}^0 with the space of real trace-free diagonal matrices. The exponential map is the usual matrix exponential, so $A^0 = \exp(\mathfrak{a}^0)$ is the group of diagonal matrices having determinant 1 and positive real entries, i.e. $A^0 = \{h(r, r^{-1}) \mid r \in \mathbb{R}_{>0}\}$, both in the real and complex case. The extended Weyl group \tilde{W}^0 has order 4. It is generated by the matrix $\tilde{s}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, whose conjugation action on A^0 exchanges the two diagonal entries, and the kernel of the canonical surjection $\tilde{W}^0 \rightarrow W^0$ is $\tilde{D}^0 = \{\mathbb{I}_2, -\mathbb{I}_2\}$.

Note that K^0 is the subgroup of G^0 of matrices that are orthogonal (resp. unitary) with respect to the standard symmetric bilinear (resp. Hermitian) form on \mathbb{R}^2 (resp. \mathbb{C}^2). Therefore, it is a consequence of the Gram-Schmidt orthonormalization procedure that every $g \in G^0$ can be written as $g = kau$ with $k \in K^0$, $a \in A^0$ and $u \in U_+^0$ and by comparing eigenvalues, it is not difficult to see that this expression is also unique. This is known as the “Iwasawa decomposition” of $\mathrm{SL}_2(\mathbb{K})$. Analogous assertions are true for the group $\mathrm{PGL}_2(\mathbb{K})$, but their precise formulation will not be relevant in the following discussion.

A similar decomposition holds on the Lie algebra level. Here, the compact involution $\theta^0 : \mathfrak{g}^0 \rightarrow \mathfrak{g}^0$ is the map $X \mapsto -X^* := -\overline{X}^T$ and it is easy to verify that $\mathfrak{g}^0 = \mathfrak{k}^0 \oplus \mathfrak{a}^0 \oplus \mathfrak{u}_+^0$ as vector spaces, where \mathfrak{k}^0 is equal to $\mathfrak{so}(2)$ if $\mathbb{K} = \mathbb{R}$ and to $\mathfrak{su}(2)$ if $\mathbb{K} = \mathbb{C}$, and \mathfrak{u}_+^0 is the set of upper-triangular matrices with 0’s on the diagonal. In fact, \mathfrak{k}^0 , \mathfrak{a}^0 and \mathfrak{u}_+^0 are simply the respective Lie algebras of K^0 , A^0 and U_+^0 and θ^0 is the tangent map at the identity of Θ^0 . Furthermore, \mathfrak{k}^0 is precisely the subalgebra of \mathfrak{g}^0 of matrices that are skew-symmetric (resp. skew-Hermitian) with respect to the standard symmetric bilinear (resp. Hermitian) form on \mathbb{R}^2 (resp. \mathbb{C}^2).

Remark 3.4.2. The previous example is prototypical in the following sense: If \mathbb{A} is spherical, we have seen in Example 3.1.6 (i) that the Kac-Moody group $G = G_{\mathcal{D}}(\mathbb{K})$ can be viewed as the group $\mathcal{G}(\mathbb{K})$ of real or complex points of a connected, \mathbb{R} -split, reductive algebraic \mathbb{R} -group \mathcal{G} and hence as a Zariski-closed subgroup of $\mathrm{GL}_N(\mathbb{K})$ for some $N \in \mathbb{N}$. Thus, it is also closed with respect to the standard Euclidean topology of $\mathrm{GL}_N(\mathbb{K})$, which is finer than the Zariski topology, so it has a natural structure of a real or complex Lie group and the fact that it “originated” from an algebraic group gives rise to some remarkable additional properties. For the proofs of all of the following assertions, we refer to [PR94, Section 3.2]. First of all, since \mathcal{G} is connected as an algebraic group, $G = \mathcal{G}(\mathbb{K})$ is also connected in its Lie group topology if $\mathbb{K} = \mathbb{C}$ and has finitely many connected components if $\mathbb{K} = \mathbb{R}$. In addition, there exists a matrix $a \in \mathrm{GL}_N(\mathbb{R})$ such that $aGa^{-1} \subset \mathrm{GL}_N(\mathbb{K})$ is stable under the map $g \mapsto (g^{-1})^*$, that aTa^{-1} is the intersection of aGa^{-1} with the diagonal subgroup of $\mathrm{GL}_N(\mathbb{K})$ and if $\alpha \in \Phi_+$ (resp. $\alpha \in \Phi_-$), then $aU_\alpha a^{-1}$ is contained in the group of upper-triangular (resp. lower-triangular) matrices with 1’s on the diagonal. Upon replacing G by aGa^{-1} , the compact involution coincides with the map $g \mapsto (g^{-1})^*$, so its fixed-point set K is the intersection of G with the orthogonal or unitary group. Moreover, K is a maximal compact subgroup of G in the Lie group topology that intersects every connected component.

The existence of an Iwasawa decomposition for all connected semisimple Lie groups & algebras is a standard result of Lie theory (cf. [Kna02, Theorem 6.46]). An extension that applies to all groups of real or complex points of reductive algebraic \mathbb{R} -groups (and hence includes some Lie groups that are not connected or semisimple) is proved in [PR94, Theorem 3.9]. The main goal of this section is to establish an analogue also for real and complex Kac-Moody groups & algebras. We first “lift” the SL_2 -case in Example 3.4.1 to the general Kac-Moody setting as follows:

Recall from Section 3.3 that for every $\alpha \in \Phi_+$, there exists a group homomorphism $\varphi_\alpha : G^0 \rightarrow G$ that is either injective or its kernel is $\{\mathbb{I}_2, -\mathbb{I}_2\}$. Its image $G_\alpha = \varphi_\alpha(G^0) = \langle U_\alpha, U_{-\alpha} \rangle$ is isomorphic to $G^0 = \mathrm{SL}_2(\mathbb{K})$ or $\mathrm{PSL}_2(\mathbb{K})$, depending on whether or not $(-1)^{h_\alpha} \in T$ coincides with the neutral element of G . Likewise, there is an injective Lie algebra homomorphism $\varphi'_\alpha : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ whose image $\mathfrak{g}_\alpha = \mathbb{K}e_\alpha \oplus \mathbb{K}h_\alpha \oplus \mathbb{K}f_\alpha \subset \mathfrak{g}$ is isomorphic to $\mathfrak{g}^0 = \mathfrak{sl}_2(\mathbb{K})$. The definitions of these homomorphisms and the respective compact involutions imply that the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{g}^0 & \xrightarrow{\theta^0} & \mathfrak{g}^0 \\ \varphi'_\alpha \downarrow & & \downarrow \varphi'_\alpha \\ \mathfrak{g} & \xrightarrow{\theta} & \mathfrak{g} \end{array} \quad \begin{array}{ccc} G^0 & \xrightarrow{\Theta^0} & G^0 \\ \varphi_\alpha \downarrow & & \downarrow \varphi_\alpha \\ G & \xrightarrow{\Theta} & G \end{array} \quad (3.4.3)$$

In particular, it follows that $K_\alpha := \varphi_\alpha(K^0) \subset K$ and $\mathfrak{k}_\alpha := \varphi'_\alpha(\mathfrak{k}^0) \subset \mathfrak{k}$. By definition, we also have $A_\alpha := \varphi_\alpha(A^0) = \{r^{h_\alpha} \mid r \in \mathbb{R}_{>0}\} \subset A$ and $\mathfrak{a}_\alpha := \varphi'_\alpha(\mathfrak{a}^0) = \mathbb{R}h_\alpha \subset \mathfrak{a}$ as well as $\varphi_\alpha(U_+^0) = U_\alpha \subset U_+$ and $\varphi'_\alpha(\mathfrak{u}_+^0) = \mathfrak{u}_\alpha \subset \mathfrak{u}_+$ since $\alpha \in \Phi_+$. In analogy with Notation 3.3.2, the subgroups $(K_\alpha)_{\alpha \in \Phi_+}$ will be called the K^0 -images in K . If $\alpha = \alpha_i$ for some $i \in I$, we again write K_i instead of K_{α_i} and use the same simplification for all the other objects just introduced.

Lemma 3.4.3. *The extended Weyl group \tilde{W} is contained in K .*

Proof. Given any $i \in I$ and $r \in \mathbb{K}^*$, it follows from (3.3.3) and (3.4.3) that $\Theta(\tilde{s}_i(r)) = \tilde{s}_i(\bar{r}^{-1})$. Hence, if $r = \bar{r}^{-1}$ or, equivalently, $|r|^2 = 1$, then $\tilde{s}_i(r) \in K$. In particular, this is the case if $r = 1$, so $\tilde{s}_i = \tilde{s}_i(1) \in K$ for all $i \in I$ and thus $\tilde{W} \subset K$. \square

We are now going to establish analogues of the Iwasawa decomposition for all real and complex Kac-Moody groups & algebras. On the Lie algebra level, the argument is rather simple and works almost exactly as in the well-known finite-dimensional case (cf. [Kna02, Proposition 6.43]).

Proposition 3.4.4. *It holds that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ as vector spaces.*

Proof. The compact involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ acts as $-\text{id}$ on \mathfrak{a} and maps \mathfrak{u}_+ to \mathfrak{u}_- , so $\mathfrak{u}_+ \cap \mathfrak{u}_- = \{0\}$ yields that $\mathfrak{a} \cap \mathfrak{u}_+ = \{0\}$. Similarly, if $x \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{u}_+)$, then $x = \theta(x) \in (\mathfrak{a} \oplus \mathfrak{u}_+) \cap (\mathfrak{a} \oplus \mathfrak{u}_-) = \mathfrak{a}$, so that $x = \theta(x) = -x$, i.e. $x = 0$. It remains to prove that these subspaces span \mathfrak{g} . The root space decomposition (2.3.3) implies that for every $x \in \mathfrak{g}$, there exists a finite set $\Psi \subset \Phi^\infty$, an element $x_0 \in \mathfrak{u}_0$, where $\mathfrak{u}_0 = \mathfrak{a}$ if $\mathbb{K} = \mathbb{R}$ and $\mathfrak{u}_0 = \mathfrak{h}$ if $\mathbb{K} = \mathbb{C}$, and vectors $x_\alpha \in \mathfrak{u}_\alpha$ for all $\alpha \in \Psi$ such that

$$x = x_0 + \sum_{\alpha \in \Psi} x_\alpha.$$

By adding $x_{-\alpha} := 0$ for all $\alpha \in \Psi$ with $-\alpha \notin \Psi$, we may assume that $\Psi = -\Psi$. If we set $\Psi_+ := \Psi \cap \Phi_+^\infty$, the sum can be rewritten as

$$x = \left(\sum_{\alpha \in \Psi_+} (x_{-\alpha} + \theta(x_{-\alpha})) \right) + x_0 + \left(\sum_{\alpha \in \Psi_+} (x_\alpha - \theta(x_{-\alpha})) \right),$$

which is an element of $\mathfrak{k} + \mathfrak{u}_0 + \mathfrak{u}_+$. If $\mathbb{K} = \mathbb{R}$, then $\mathfrak{u}_0 = \mathfrak{a}$ and we are done. If $\mathbb{K} = \mathbb{C}$, then $\mathfrak{u}_0 = \mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$, so that $x_0 = x'_0 + ix''_0$ for some $x'_0, x''_0 \in \mathfrak{a}$. But $i\mathfrak{a} \subset \mathfrak{k}$ because θ is \mathbb{C} -antilinear, so we only need to move ix''_0 to the first sum to finish the proof. \square

Definition 3.4.5. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ established in Proposition 3.4.4 is called the *Iwasawa decomposition* of \mathfrak{g} . The corresponding linear projections onto these three subspaces are the *Iwasawa projections* of \mathfrak{g} and will be denoted by:

$$\pi_{\mathfrak{k}} : \mathfrak{g} \rightarrow \mathfrak{k} \quad \pi_{\mathfrak{a}} : \mathfrak{g} \rightarrow \mathfrak{a} \quad \pi_{\mathfrak{u}_+} : \mathfrak{g} \rightarrow \mathfrak{u}_+ \quad (3.4.4)$$

For $x \in \mathfrak{g}$, the unique elements $x_{\mathfrak{k}} := \pi_{\mathfrak{k}}(x) \in \mathfrak{k}$, $x_{\mathfrak{a}} := \pi_{\mathfrak{a}}(x) \in \mathfrak{a}$, $x_{\mathfrak{u}_+} := \pi_{\mathfrak{u}_+}(x) \in \mathfrak{u}_+$ such that $x = x_{\mathfrak{k}} + x_{\mathfrak{a}} + x_{\mathfrak{u}_+}$ are the *Iwasawa components* of x .

On the group level, however, we cannot proceed as in classical Lie theory where analytic methods are usually employed (cf. [Kna02, Theorem 6.46]). Instead, we first establish a preliminary result whose proof requires some concepts from the theory of buildings that we have not introduced. As this will be the only instance in this thesis where this is necessary, we will not define them here either in order not to distract from the main discussion. Our terminology is entirely standard and follows [AB08] where all undefined notions are carefully explained.

Proposition 3.4.6. $G = KB_+$

Proof. As mentioned in Remark 3.1.5, the tuple $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ is an RGD system of type (W, S) . Following [AB08, Chapter 8], this implies that G acts strongly transitively on a thick Moufang twin building of type (W, S) by type-preserving automorphisms. By definition of that building, its positive half is the simplicial complex $\Delta(G, B_+)$ whose simplices are the left cosets of the standard (positive) parabolic subgroups of G ordered by reverse-inclusion. In particular, its chambers are simply the left cosets of the positive Borel subgroup B_+ which is therefore precisely the stabilizer of the fundamental chamber $C_0 := 1 \cdot B_+ \in \Delta(G, B_+)$ under the G -action. Thus, the assertion that $G = KB_+$ is equivalent to the statement that K acts chamber-transitively on $\Delta(G, B_+)$. To prove this, we will make use of the following auxiliary observation:

Claim. If \mathcal{P} is a panel of $\Delta(G, B_+)$, then the link of \mathcal{P} is isomorphic to (the flag complex of) the projective line over \mathbb{K} .

Proof of Claim. Let $\text{lk}(\mathcal{P})$ denote the link of \mathcal{P} . By definition of the building, the G -action on $\Delta(G, B_+)$ is chamber-transitive and type-preserving, so we may assume that \mathcal{P} is a panel of the fundamental chamber C_0 . Let $i \in I$ be such that the cotype of \mathcal{P} is s_i , then its stabilizer is the standard parabolic subgroup $P_{\{i\}}$ of G of type $\{i\}$ and $\text{lk}(\mathcal{P})$ is a thick building of type $(W_{\{i\}}, S_{\{i\}})$ on which $P_{\{i\}}$ acts strongly transitively. In the Levi decomposition $P_{\{i\}} \cong G_{\{i\}} \ltimes U_{\{i\},+}^\perp$, the unipotent radical $U_{\{i\},+}^\perp$ fixes $\text{lk}(\mathcal{P})$ point-wise by [Mar18, Proposition 7.75], so the Levi factor $G_{\{i\}}$ already acts strongly transitively on $\text{lk}(\mathcal{P})$. By Remark 3.3.3, we have $G_{\{i\}} = TG_i$ where $G_i = \varphi_i(G^0) = \langle U_i, U_{-i} \rangle$. Since T fixes the standard apartment Σ_+ of $\Delta(G, B_+)$ point-wise, the same is true for the apartment $\text{lk}_{\Sigma_+}(\mathcal{P})$ of $\text{lk}(\mathcal{P})$, so it follows that G^0 acts strongly transitively on $\text{lk}(\mathcal{P})$ through the map φ_i and the stabilizer of the fundamental chamber is $\varphi_i^{-1}(B_+)$. As φ_i is either injective or its kernel is $\{\mathbb{I}_2, -\mathbb{I}_2\}$, this is just the subgroup $B_+^0 \subset G^0$ of all upper-triangular (2×2) -matrices with determinant 1. Hence, this strongly transitive action of G^0 on a thick rank-1 building with stabilizer B_+^0 yields the standard BN -pair for G^0 and establishes an isomorphism between the link of \mathcal{P} and the spherical building associated to this BN -pair, which is known to be the flag complex of non-trivial proper subspaces of \mathbb{K}^2 (cf. [AB08, Section 4.3 and 6.5]). As every flag consists of a single one-dimensional subspace, this is simply the set of all one-dimensional subspaces of \mathbb{K}^2 , i.e. the projective line over \mathbb{K} .

Conclusion of the proof. We are now able to show that K acts chamber-transitively on $\Delta(G, B_+)$. Let $g \in G$ be arbitrary, choose a minimal gallery $\Gamma = (C_0, C_1, \dots, C_t)$ from $C_0 = 1 \cdot B_+$ to $C_t := gC_0 = gB_+$ and let $(s_{i_1}, \dots, s_{i_t})$ be its type. For every $r \in \{1, \dots, t\}$, let \mathcal{P}_r be the panel of $\Delta(G, B_+)$ separating C_{r-1} from C_r , so that the cotype of \mathcal{P}_r is s_{i_r} . We claim that for every $r \in \{1, \dots, t\}$, there exists some $k_r \in K$ such that $C_r = k_r C_0$. This will prove that $gC_0 = k_t C_0$ or, equivalently, $k_t^{-1}g \in B_+$, as required. We argue by induction on r .

If $r = 1$, then the link of \mathcal{P}_1 can be identified with the projective line over \mathbb{K} and G^0 acts strongly transitively on $\text{lk}(\mathcal{P}_1)$ through the map $\varphi_{i_1} : G^0 \rightarrow G$. Since $K^0 \subset G^0$ acts transitively on the unit sphere in \mathbb{K}^2 and hence on the projective line over \mathbb{K} , it follows that $K_{i_1} = \varphi_{i_1}(K^0) \subset K$ acts chamber-transitively on $\text{lk}(\mathcal{P}_1)$, so that $C_1 = k_1 C_0$ for a suitable $k_1 \in K_{i_1} \subset K$. Let now $r \in \{2, \dots, t\}$ be arbitrary, then the induction hypothesis implies that we can find $k_{r-1} \in K$ such that $C_{r-1} = k_{r-1} C_0$. Thus, $k_{r-1}^{-1}C_r$ is s_{i_r} -adjacent to C_0 and by the same argument as before, there exists some $k'_r \in K_{i_r} = \varphi_{i_r}(K^0) \subset K$ such that $k_{r-1}^{-1}C_r = k'_r C_0$, which implies that $C_r = k_r C_0$ with $k_r := k_{r-1} k'_r \in K$, as claimed. \square

The desired decomposition on the group level now only requires a minor refinement of the product $G = KB_+$ into subgroups that intersect trivially. Observe that the group $A = \exp(\mathfrak{a})$ in the next result is the same in the real and complex case, only K and U_+ depend on the field.

Proposition 3.4.7. *The following map is bijective:*

$$\begin{aligned} K \times A \times U_+ &\rightarrow G \\ (k, a, u) &\mapsto kau \end{aligned} \tag{3.4.5}$$

Proof. Using that $G = KB_+ = K T U_+ = K M A U_+$ by Proposition 3.4.6 and $A \cap U_+ = \{1\}$, it only remains to show that $K \cap B_+ = M$. The compact involution $\Theta : G \rightarrow G$ maps B_+ to B_- , so if $b \in B_+$ was a fixed-point of Θ , then $b = \Theta(b) \in B_+ \cap B_- = T$ and hence $K \cap B_+ \subset T$. Let $\{v_1, \dots, v_d\}$ be a \mathbb{Z} -basis of Λ^\vee , then every $t \in T$ can be uniquely written as $t = r_1^{v_1} \dots r_d^{v_d}$ with $r_1, \dots, r_d \in \mathbb{K}^*$. Applying Θ yields

$$\Theta(t) = \overline{r_1}^{-v_1} \dots \overline{r_d}^{-v_d} = (\overline{r_1}^{-1})^{v_1} \dots (\overline{r_d}^{-1})^{v_d},$$

so that $t = \Theta(t)$ if and only if $r_i = \overline{r_i}^{-1}$ for all $i \in \{1, \dots, d\}$, i.e. $r_i \in \{1, -1\}$ if $\mathbb{K} = \mathbb{R}$ and $r_i \in S^1$ if $\mathbb{K} = \mathbb{C}$, which is equivalent to $t \in M$. Thus, we have shown that $K \cap B_+ = K \cap T = M$. \square

Definition 3.4.8. The decomposition $G = KAU_+$ established in Proposition 3.4.7 is called the *Iwasawa decomposition* of G . The corresponding projections onto these three subgroups are the *Iwasawa projections* of G and will be denoted by:

$$\Pi_K : G \rightarrow K \quad \Pi_A : G \rightarrow A \quad \Pi_{U_+} : G \rightarrow U_+ \quad (3.4.6)$$

For $g \in G$, the unique elements $k := \Pi_K(g) \in K$, $a := \Pi_A(g) \in A$, $u := \Pi_{U_+}(g) \in U_+$ such that $g = kau$ are the *Iwasawa components* of g .

Convention 3.4.9. To simplify the terminology, we occasionally define the Iwasawa components of an element $g \in G$ implicitly by saying that $g = kau$ is in *Iwasawa form*, where it is tacitly assumed that $k = \Pi_K(g) \in K$, $a = \Pi_A(g) \in A$ and $u = \Pi_{U_+}(g) \in U_+$.

Remark 3.4.10. The existence of the Iwasawa decomposition for certain types of Kac-Moody groups has already been established, for example, in [KP85, Proposition 5.1] or [FHHK20, Theorem 3.31]. On the one hand, our result is slightly more general since it applies to real and complex Kac-Moody groups defined by arbitrary Kac-Moody root data. For ease of reference, we have therefore chosen to provide an independent proof which is inspired by [DMGH09, Corollary 3.11]. On the other hand, Proposition 3.4.7 is weaker in the sense that it only states that the multiplication map $K \times A \times U_+ \rightarrow G$ is bijective. In fact, the authors of [FHHK20] also show that there is a natural topology on G , called the *Kac-Peterson topology*, which turns G into a Hausdorff topological group such that the Iwasawa decomposition is a homeomorphism. While their result only applies to real Kac-Moody groups defined by simply connected root data associated to 2-spherical GCM's, similar techniques were used in [KV24, Theorem 3.20] to weaken some of the assumptions and extend it to the complex case. For the purposes of our main theorems, however, it will not be necessary to view Kac-Moody groups from a topological perspective.

Recall that G is generated by the positive and negative simple root groups together with the torus which normalizes every root group. This implies that $G = TG'$, where G' is the subgroup of G generated by the simple G^0 -images $(G_i)_{i \in I}$. A nice consequence of the previous two results is that the natural analogue of this statement is true for the compact form K .

Corollary 3.4.11. *Let K' be the subgroup of K generated by $(K_i)_{i \in I}$, then $K = MK'$.*

Proof. Since $K_i = \varphi_i(K^0)$ for all $i \in I$, inspecting the proof of Proposition 3.4.6 actually shows that $G = K'B_+$ and therefore $K = (K'B_+) \cap K = K'(K \cap B_+) = K'(K \cap T) = K'M$, where the last two equalities follow from the proof of Proposition 3.4.7. By taking inverses, we conclude that $K = MK'$. \square

Example 3.4.12. Now that we have an Iwasawa decomposition $G = KAU_+$ on every real or complex Kac-Moody group, let us relate it to the Iwasawa decomposition $G^0 = K^0A^0U_+^0$ of $G^0 = \mathrm{SL}_2(\mathbb{K})$ discussed in Example 3.4.1. For every $\alpha \in \Phi_+$, the homomorphism $\varphi_\alpha : G^0 \rightarrow G$ maps K^0 to $K_\alpha \subset K$, A^0 to $A_\alpha \subset A$ and U_+^0 to $U_\alpha \subset U_+$. In particular, we obtain an Iwasawa decomposition on the G^0 -image $G_\alpha = \varphi_\alpha(G^0) = K_\alpha A_\alpha U_\alpha$, where the three subgroups are simply the respective intersections of G_α with K , A and U_+ . We will often express this by saying that φ_α is *compatible* with the Iwasawa decompositions $G^0 = K^0A^0U_+^0$ and $G = KAU_+$ and that it *intertwines* the Iwasawa projections of G^0 and G in the sense that the following three diagrams commute:

$$\begin{array}{ccc} G^0 & \xrightarrow{\varphi_\alpha} & G \\ \Pi_{K^0} \downarrow & & \downarrow \Pi_K \\ K^0 & \xrightarrow{\varphi_\alpha} & K \end{array} \quad \begin{array}{ccc} G^0 & \xrightarrow{\varphi_\alpha} & G \\ \Pi_{A^0} \downarrow & & \downarrow \Pi_A \\ A^0 & \xrightarrow{\varphi_\alpha} & A \end{array} \quad \begin{array}{ccc} G^0 & \xrightarrow{\varphi_\alpha} & G \\ \Pi_{U_+^0} \downarrow & & \downarrow \Pi_{U_+} \\ U_+^0 & \xrightarrow{\varphi_\alpha} & U_+ \end{array} \quad (3.4.7)$$

Likewise, the Lie algebra homomorphism $\varphi'_\alpha : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ maps \mathfrak{k}^0 to $\mathfrak{k}_\alpha \subset \mathfrak{k}$, \mathfrak{a}^0 to $\mathfrak{a}_\alpha \subset \mathfrak{a}$ and \mathfrak{u}_+^0 to $\mathfrak{u}_\alpha \subset \mathfrak{u}_+$, so we also obtain an Iwasawa decomposition on its image $\mathfrak{g}_\alpha = \mathfrak{k}_\alpha \oplus \mathfrak{a}_\alpha \oplus \mathfrak{u}_\alpha$, where the three subalgebras are the respective intersections of \mathfrak{g}_α with \mathfrak{k} , \mathfrak{a} and \mathfrak{u}_+ . Hence, φ'_α is compatible

with the Iwasawa decompositions $\mathfrak{g}^0 = \mathfrak{k}^0 \oplus \mathfrak{a}^0 \oplus \mathfrak{u}_+^0$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ and it intertwines the Iwasawa projections of \mathfrak{g}^0 and \mathfrak{g} by means of the following three commutative diagrams:

$$\begin{array}{ccc}
 \mathfrak{g}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{g} \\
 \pi_{\mathfrak{k}^0} \downarrow & & \downarrow \pi_{\mathfrak{k}} \\
 \mathfrak{k}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{k}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathfrak{g}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{g} \\
 \pi_{\mathfrak{a}^0} \downarrow & & \downarrow \pi_{\mathfrak{a}} \\
 \mathfrak{a}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{a}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathfrak{g}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{g} \\
 \pi_{\mathfrak{u}_+^0} \downarrow & & \downarrow \pi_{\mathfrak{u}_+} \\
 \mathfrak{u}_+^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{u}_+
 \end{array}
 \tag{3.4.8}$$

The two diagrams in the middle of (3.4.7) and (3.4.8) are connected by the respective exponential maps. In fact, it is immediate from the definitions that the diagram below commutes as well:

$$\begin{array}{ccc}
 \mathfrak{a}^0 & \xrightarrow{\varphi'_\alpha} & \mathfrak{a} \\
 \exp^0 \downarrow & & \downarrow \exp \\
 A^0 & \xrightarrow{\varphi_\alpha} & A
 \end{array}
 \tag{3.4.9}$$

Here, the map on the left is the usual exponential of (2×2) -matrices, whereas the one on the right is defined by (3.4.1). It is also clear from the definitions that the restriction of φ_α (resp. φ'_α) to A^0 (resp. \mathfrak{a}^0) is W^0 -equivariant, where the unique non-trivial element of $W^0 \cong \mathbb{Z}_2$ acts on A^0 (resp. \mathfrak{a}^0) as the inversion map (resp. $-\text{id}$) and on A (resp. \mathfrak{a}) by the natural action of $s_\alpha \in W$. Hence, the entire diagram (3.4.9) is W^0 -equivariant.

The discussion of this section can also be applied to the restrictions of \mathcal{D} which yields Iwasawa decompositions for certain subgroups of G and subalgebras of \mathfrak{g} .

Notation 3.4.13. Let $J \subset I$ be any subset and \mathcal{D}_J the J -restriction of \mathcal{D} . The real or complex Kac-Moody group $G_{\mathcal{D}_J}(\mathbb{K})$ of type \mathcal{D}_J can be identified with the subgroup $G_J \subset G$ generated by T and the root groups $(U_\alpha)_{\alpha \in \Phi_J}$. By definition, the compact involution $\Theta_J : G_J \rightarrow G_J$ coincides with the restriction of Θ to G_J , so that its fixed-point set K_J is equal to $K \cap G_J$. Let $U_{J,+}$ denote the subgroup of G_J generated by $(U_\alpha)_{\alpha \in \Phi_{J,+}}$, then the Iwasawa decomposition of G_J takes the following form:

$$G_J = K_J A U_{J,+} \tag{3.4.10}$$

Since the three groups are the respective intersections of G_J with K , A and U_+ , it follows that G_J contains the Iwasawa components of all its elements and the Iwasawa projections are simply the restrictions of Π_K , Π_A and Π_{U_+} to $G_J \subset G$.

Likewise, the real or complex Kac-Moody algebra $\mathfrak{g}_{\mathcal{D}_J}(\mathbb{K})$ of type \mathcal{D}_J can be identified with the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ generated by \mathfrak{u}_0 and the Chevalley generators $(e_j)_{j \in J}, (f_j)_{j \in J}$. The compact involution $\theta_J : \mathfrak{g}_J \rightarrow \mathfrak{g}_J$ is the restriction of θ to \mathfrak{g}_J , so its fixed-point set \mathfrak{k}_J is $\mathfrak{k} \cap \mathfrak{g}_J$ and the Iwasawa decomposition of \mathfrak{g}_J looks as follows (cf. Notation 2.3.6):

$$\mathfrak{g}_J = \mathfrak{k}_J \oplus \mathfrak{a} \oplus \mathfrak{u}_{J,+} \tag{3.4.11}$$

As on the group level, the three subalgebras are the respective intersections of \mathfrak{g}_J with \mathfrak{k} , \mathfrak{a} and \mathfrak{u}_+ , which implies that \mathfrak{g}_J also contains the Iwasawa components of all its elements and the Iwasawa projections are the restrictions of $\pi_{\mathfrak{k}}$, $\pi_{\mathfrak{a}}$ and $\pi_{\mathfrak{u}_+}$ to $\mathfrak{g}_J \subset \mathfrak{g}$.

Parallel to Remark 3.3.3, it should be noted that if $J = \{i\}$ for some $i \in I$, the subgroup $K_{\{i\}} \subset G_{\{i\}}$ defined here does usually not coincide with the K^0 -image $K_i = \varphi_i(K^0) \subset G_i$. Both groups are the respective intersections of $G_{\{i\}}$ and G_i with K , but $G_{\{i\}} = TG_i$ and $K_{\{i\}} = MK_i$ by Corollary 3.4.11. For similar reasons, $\mathfrak{k}_{\{i\}} \subset \mathfrak{g}_{\{i\}}$ is not equal to $\mathfrak{k}_i = \varphi'_i(\mathfrak{k}^0) \subset \mathfrak{g}_i$ in general as $\mathfrak{g}_{\{i\}} = \mathfrak{g}_i + \mathfrak{u}_0$ and $\mathfrak{k}_{\{i\}} = \mathfrak{k}_i + (\mathfrak{k} \cap \mathfrak{u}_0)$.

The reader who has worked through the historical introduction will have realized that we now have all the necessary ingredients to formulate natural analogues of Kostant's Convexity Theorems for real or complex Kac-Moody groups. The precise statements will be given in Section 5.1 and 5.3.

In order to prove them, however, we still have to develop a lot of tools. On the one hand, we need to study the “geometry” of convex hulls of Weyl group orbits under the \mathcal{D} -representation which is the content of Chapter 4. On the other hand, we also require some more information about Kac-Moody groups and their representations. The rest of this chapter is dedicated to establishing several auxiliary results that will be relevant to the proofs of these theorems.

Lemma 3.4.14. *If $h \in \mathfrak{a}$ and $a := \exp(h) \in A$, then $a(\mu) = e^{\langle \mu, h \rangle} \in \mathbb{R}_{>0}$ for all $\mu \in \Lambda$.*

Proof. Choose a \mathbb{Z} -basis $\{v_1, \dots, v_d\}$ of Λ^\vee and let $z_1, \dots, z_d \in \mathbb{R}$ be the unique scalars such that $h = \sum_{i=1}^d z_i v_i$ and $a = \prod_{i=1}^d (e^{z_i})^{v_i}$, then we may perform the following computation:

$$\begin{aligned} a(\mu) &= \prod_{i=1}^d (e^{z_i})^{v_i}(\mu) = \prod_{i=1}^d (e^{z_i})^{\langle \mu, v_i \rangle} = \prod_{i=1}^d e^{z_i \langle \mu, v_i \rangle} \\ &= e^{\sum_{i=1}^d z_i \langle \mu, v_i \rangle} = e^{\langle \mu, \sum_{i=1}^d z_i v_i \rangle} = e^{\langle \mu, h \rangle} \end{aligned} \quad \square$$

Lemma 3.4.15. *For $h \in \mathfrak{a}$ and $a := \exp(h) \in A$, the following statements are equivalent:*

- (i) h is central in \mathfrak{g} .
- (ii) $\langle c_i, h \rangle = 0$ for all $i \in I$.
- (iii) h is a global fixed-point of the \mathcal{D} -representation of W .
- (iv) h is a global fixed-point of the adjoint representation of G .
- (v) a is central in G .
- (vi) $a(c_i) = 1$ for all $i \in I$.
- (vii) a commutes with the extended Weyl group $\tilde{W} \subset K$.
- (viii) $a \in \ker(\text{Ad})$.

Proof. The equivalences (i) \iff (ii) \iff (iii) are both immediate from (KMA2) and (2.6.1) and (ii) \iff (iv) is proved as follows: By (3.2.4), $h \in \mathfrak{a} \subset \mathfrak{u}_0$ is fixed by $\text{Ad}(t)$ for every $t \in T$. Because G is generated by the torus and the positive and negative simple root groups, h is a global fixed-point of the adjoint representation if and only if it is fixed by $\text{Ad}(x_{\pm i}(z)) = \exp(\text{ad}(ze_{\pm i}))$ for all $i \in I$ and $z \in \mathbb{K}$. In view of (KMA2), this is equivalent to $\langle c_i, h \rangle = 0$ for all $i \in I$. The equivalence (v) \iff (vi) is a consequence of (3.1.8) since $a \in A \subset T$ is central in G if and only if it commutes with every positive or negative simple root group, (vi) \iff (viii) also follows from (3.2.4) and (ii) \iff (vi) from Lemma 3.4.14. Finally, (iii) \iff (vii) holds as conjugation by \tilde{W} induces the natural W -action on A and $\exp : \mathfrak{a} \rightarrow A$ is a W -equivariant isomorphism. \square

Lemma 3.4.16. *Let $\alpha \in \Phi_+$ be arbitrary. For every $h \in \mathfrak{a}$, there exist $h_1, h_2 \in \mathfrak{a}$ such that $h = h_1 + h_2$, $h_1 \in \mathfrak{a}_\alpha$ and h_2 centralizes \mathfrak{g}_α . Equivalently, for every $a \in A$, there exist $a_1, a_2 \in A$ such that $a = a_1 a_2$, $a_1 \in A_\alpha$ and a_2 commutes with G_α .*

Proof. Let \mathfrak{a}_2 be an arbitrary complement to $\mathfrak{a}_1 := \mathfrak{a}_\alpha$ in \mathfrak{a} , then there exist unique elements $h'_1 \in \mathfrak{a}_1$ and $h'_2 \in \mathfrak{a}_2$ such that $h = h'_1 + h'_2$. If we set $z := \langle c_\alpha, h'_2 \rangle \in \mathbb{R}$, we can rewrite h as $h = h_1 + h_2$, where $h_1 := h'_1 + \frac{z}{2} h_\alpha \in \mathfrak{a}_1$ and $h_2 := h'_2 - \frac{z}{2} h_\alpha$. Since $\langle c_\alpha, h_2 \rangle = 0$, it follows that $[h_2, e_\alpha] = [h_2, f_\alpha] = [h_2, h_\alpha] = 0$ and therefore $[h_2, \mathfrak{g}_\alpha] = \{0\}$. This proves the first statement which is equivalent to the second by the proof of Lemma 3.4.15. \square

We close this section with an important result about highest-weight representations of Kac-Moody groups. Recall from Section 2.7 that whenever $\lambda \in \mathfrak{a}^* \subset \mathfrak{h}^*$ is a dominant integral weight, there exists a symmetric bilinear or Hermitian form H on the real or complex irreducible highest-weight \mathfrak{g} -module $L(\lambda)$ which is contravariant in the sense of (2.7.1). Hence, the compact form $\mathfrak{k} \subset \mathfrak{g}$ acts on $L(\lambda)$ by skew-adjoint linear operators. If λ is even proper, it lifts to a representation of G and contravariance of H translates into the following property on the group level:

Lemma 3.4.17. *If $\lambda \in \Lambda$ is a proper dominant integral weight and H a contravariant symmetric bilinear or Hermitian form on $L(\lambda)$, then*

$$H(g \cdot v, w) = H(v, \Theta(g^{-1}) \cdot w) \quad \text{for all } g \in G \text{ and } v, w \in L(\lambda). \quad (3.4.12)$$

In particular, H is K -invariant.

Proof. Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(L(\lambda))$ and $\hat{\rho} : G \rightarrow \mathrm{GL}(L(\lambda))$ denote the respective actions of \mathfrak{g} and G . Because G is generated by the torus and the positive and negative simple root groups and $\hat{\rho}$ and Θ are homomorphisms, it suffices to verify the statement for these generators. For definiteness, let us assume that if $\mathbb{K} = \mathbb{C}$ where H is Hermitian, then H is \mathbb{C} -linear in the first variable and \mathbb{C} -antilinear in the second. If $g = x_i(z) \in U_i$ for some $i \in I$ and $z \in \mathbb{K}$, then $\theta(e_i) = f_i = e_{-i}$ and the following computation holds, where the bars can be omitted if $\mathbb{K} = \mathbb{R}$:

$$\begin{aligned} H(x_i(z) \cdot v, w) &= H(\hat{\rho}(x_i(z))v, w) = H(\exp(\rho(ze_i))v, w) = \sum_{s=0}^{\infty} \frac{z^s}{s!} H(\rho(e_i)^s v, w) \\ &= \sum_{s=0}^{\infty} (-1)^s \frac{z^s}{s!} H(v, \rho(\theta(e_i))^s w) = \sum_{s=0}^{\infty} (-1)^s \frac{z^s}{s!} H(v, \rho(e_{-i})^s w) \\ &= H(v, \exp(\rho(-\bar{z}e_{-i}))w) = H(v, \hat{\rho}(x_{-i}(-\bar{z}))w) = H(v, \hat{\rho}(x_{-i}(\bar{z})^{-1})w) \\ &= H(v, \hat{\rho}(\Theta(x_i(z)^{-1}))w) = H(v, \Theta(x_i(z)^{-1}) \cdot w) \end{aligned}$$

The same argument applies to elements of U_{-i} . Finally, if $g = t \in T$, we may assume that v and w are both contained in $L(\lambda)_\mu$ for some $\mu \in \mathrm{wt}(L(\lambda))$ because T acts diagonally on $L(\lambda)$ and different weight spaces are H -orthogonal. Further, we may suppose that $t = r^h$ for some $h \in \Lambda^\vee$ and $r \in \mathbb{K}^*$ as elements of this form generate T . Hence, the proof is concluded as follows:

$$\begin{aligned} H(r^h \cdot v, w) &= H(\hat{\rho}(r^h)v, w) = H(r^{\langle \mu, h \rangle} v, w) = r^{\langle \mu, h \rangle} H(v, w) = H(v, \bar{r}^{\langle \mu, h \rangle} w) \\ &= H(v, \hat{\rho}(\bar{r}^h)w) = H(v, \hat{\rho}(\Theta((r^h)^{-1}))w) = H(v, \Theta((r^h)^{-1}) \cdot w) \quad \square \end{aligned}$$

If the underlying GCM is symmetrizable, then H can be chosen to be positive-definite, so K acts on $L(\lambda)$ by unitary operators. Moreover, by Lemma 3.4.14, every element of A acts diagonally on $L(\lambda)$ with positive real eigenvalues and the action of U_+ can be computed iteratively using (3.2.3). Altogether, we have proved the following addendum to Proposition 2.7.8:

Proposition 3.4.18. *With the notational conventions introduced in Section 2.7, let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be a symmetrizable GCM of rank $\ell \leq n$, let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a free, cofree and cotorsion-free Kac-Moody root datum of rank $d \geq 2n - \ell$ and denote by G (resp. \mathfrak{g}) either the real or complex Kac-Moody group (resp. algebra) of type \mathcal{D} . Then the following assertions hold:*

- (i) *Each of the real or complex irreducible highest-weight \mathfrak{g} -modules $L(\omega_1), \dots, L(\omega_n)$ as well as $L(\phi_{n+1}), \dots, L(\phi_d)$ can be lifted to a representation of G that is defined by (3.2.5). Every such module can be endowed with a positive-definite, contravariant, symmetric bilinear or Hermitian form H that satisfies (3.4.12).*
- (ii) *The compact form K acts on these modules by unitary operators. Every element of A acts diagonally with respect to the weight space decomposition and on each weight space, it acts by multiplication with a positive real number. The action of U_+ is characterized by (3.2.3). In particular, U_+ acts trivially on the highest-weight space.*
- (iii) *For $i \in \{n+1, \dots, d\}$, every element of K acts on the one-dimensional module $L(\phi_i)$ by multiplication with a real or complex number of absolute value 1. An element $a \in A$ acts on $L(\phi_i)$ by multiplication with $a(\phi_i) \in \mathbb{R}_{>0}$ and the action of U_+ is trivial.* \square

3.5 SL_2 -computations

At this point, it is clear that a powerful tool to investigate the structure of a Kac-Moody group G is the existence of homomorphisms from $G^0 = \mathrm{SL}_2(\mathbb{K})$ into G . To prepare for Chapter 5 and 6, it is useful to carry out some explicit computations related to the Iwasawa decomposition of G^0 . The results of this section are rather technical, but we will see that they have important consequences for the Iwasawa decompositions of arbitrary Kac-Moody groups. In order not to distract from the main arguments later, we have chosen to collect all these computations here, but the reader may feel free to skip this part on first reading and return to it when needed.

Throughout this section, we will denote $\mathrm{SL}_2(\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ by G^0 and employ Notation 3.3.2. Recall from Example 3.4.1 that in the Iwasawa decomposition $G^0 = K^0 A^0 U_+^0$, the compact form K^0 is the special orthogonal group $\mathrm{SO}(2)$ if $\mathbb{K} = \mathbb{R}$ and the special unitary group $\mathrm{SU}(2)$ if $\mathbb{K} = \mathbb{C}$, A^0 is the group of diagonal matrices with determinant 1 and positive real entries (both in the real and complex case) and U_+^0 is the group of upper-triangular matrices with 1's on the diagonal. Moreover, the Iwasawa components of a matrix in G^0 can be computed by endowing the vector space \mathbb{K}^2 with the standard symmetric bilinear or Hermitian form and applying the Gram-Schmidt orthonormalization procedure to its column vectors. To proceed, it is useful to define

$$B := \{(z_1, z_2) \in \mathbb{K}^2 : |z_1|^2 + |z_2|^2 = 1\}. \quad (3.5.1)$$

If $\mathbb{K} = \mathbb{R}$, then B coincides with the 1-sphere $S^1 \subset \mathbb{R}^2$ and if $\mathbb{K} = \mathbb{C}$, it can be identified with the 3-sphere $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$. With this notation, the Iwasawa components can be parametrized as:

$$\begin{aligned} K^0 &= \left\{ k(z_1, z_2) := \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \mid (z_1, z_2) \in B \right\} \\ A^0 &= \left\{ a(r) := h(r, r^{-1}) = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \mid r \in \mathbb{R}_{>0} \right\} \\ U_+^0 &= \left\{ u(z) := x_+(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in \mathbb{K} \right\} \end{aligned} \quad (3.5.2)$$

In the real case, we may also write $K^0 = \mathrm{SO}(2)$ as

$$\mathrm{SO}(2) = \left\{ R_\gamma := \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \mid \gamma \in [0, 2\pi) \right\}, \quad (3.5.3)$$

so that $R_\gamma = k(\cos(\gamma), \sin(\gamma))$ for all $\gamma \in [0, 2\pi)$ in the notation of (3.5.2). In the complex case, we may view $\mathrm{SO}(2)$ as a subgroup of $K^0 = \mathrm{SU}(2)$.

The main goal of this section is to compute some explicit formulas that describe how the Iwasawa decomposition of G^0 behaves with respect to the group multiplication. In particular, we will be interested in determining how the Iwasawa components of an element $g \in G^0$ change when it is multiplied from the right by some $k \in K^0$. First, we consider the case where $g \in A^0$, i.e. we study a product of the form $a(r)k(z_1, z_2) \in G^0$ for some $r \in \mathbb{R}_{>0}$ and $(z_1, z_2) \in B$. To simplify the notation in the anticipated result, we define the following two functions:

$$\begin{aligned} \xi_A : \mathbb{R}_{>0} \times B &\rightarrow \mathbb{R}_{>0} & \eta_A : \mathbb{R}_{>0} \times B &\rightarrow \mathbb{K} \\ (r, z_1, z_2) &\mapsto \sqrt{r^2|z_1|^2 + r^{-2}|z_2|^2} & (r, z_1, z_2) &\mapsto \frac{(r^{-2} - r^2)\overline{z_1}z_2}{\xi_A(r, z_1, z_2)^2} \end{aligned} \quad (3.5.4)$$

If $r \in \mathbb{R}_{>0}$ is fixed, the image of the map $z_1 \mapsto r^2|z_1|^2 + r^{-2}(1 - |z_1|^2)$ for $|z_1| \leq 1$ is the interval $[r^{-2}, r^2] \subset \mathbb{R}_{>0}$, so ξ_A and η_A are well-defined. Furthermore, we introduce:

$$\begin{aligned} \kappa_A^1 : \mathbb{R}_{>0} \times B &\rightarrow \mathbb{K} & \kappa_A^2 : \mathbb{R}_{>0} \times B &\rightarrow \mathbb{K} \\ (r, z_1, z_2) &\mapsto \frac{rz_1}{\xi_A(r, z_1, z_2)} & (r, z_1, z_2) &\mapsto \frac{r^{-1}z_2}{\xi_A(r, z_1, z_2)} \end{aligned} \quad (3.5.5)$$

They are also well-defined and satisfy $|\kappa_A^1(r, z_1, z_2)|^2 + |\kappa_A^2(r, z_1, z_2)|^2 = 1$ for all $r \in \mathbb{R}_{>0}$ and $(z_1, z_2) \in B$. In other words, the product map $\kappa_A := \kappa_A^1 \times \kappa_A^2$ takes values in B . These four functions compute the Iwasawa components of elements in $A^0 K^0 \subset G^0$ in the desired way.

Proposition 3.5.1. *For all $r \in \mathbb{R}_{>0}$ and $(z_1, z_2) \in B$, the product $a(r)k(z_1, z_2) \in G^0$ has the following Iwasawa components:*

$$\begin{aligned} \Pi_{K^0}(a(r)k(z_1, z_2)) &= k(\kappa_A(r, z_1, z_2)) \\ \Pi_{A^0}(a(r)k(z_1, z_2)) &= a(\xi_A(r, z_1, z_2)) \\ \Pi_{U_+^0}(a(r)k(z_1, z_2)) &= u(\eta_A(r, z_1, z_2)) \end{aligned} \quad (3.5.6)$$

Proof. Throughout this proof, we denote by (\cdot, \cdot) the standard symmetric bilinear or Hermitian form on \mathbb{K}^2 and by $\|\cdot\|$ the induced norm. If $\mathbb{K} = \mathbb{C}$, we assume that (\cdot, \cdot) is \mathbb{C} -linear in the first variable and \mathbb{C} -antilinear in the second. To compute the Iwasawa components of $a(r)k(z_1, z_2)$, we need to apply the Gram-Schmidt orthonormalization procedure to the columns of the matrix

$$g := a(r)k(z_1, z_2) = \begin{pmatrix} rz_1 & -r\bar{z}_2 \\ r^{-1}z_2 & r^{-1}\bar{z}_1 \end{pmatrix} \in G^0.$$

Let v_1, v_2 be the two column vectors of g . The Gram-Schmidt procedure turns $\{v_1, v_2\}$ into an orthonormal basis $\{b_1, b_2\}$ of \mathbb{K}^2 in two steps. First, set $\tilde{b}_1 := v_1$ and define \tilde{b}_2 as follows:

$$\begin{aligned} \tilde{b}_2 &:= v_2 - \frac{(v_2, v_1)}{\|v_1\|^2} v_1 = \begin{pmatrix} -r\bar{z}_2 \\ r^{-1}\bar{z}_1 \end{pmatrix} - \frac{(r^{-2} - r^2)\bar{z}_1\bar{z}_2}{r^2|z_1|^2 + r^{-2}|z_2|^2} \begin{pmatrix} rz_1 \\ r^{-1}z_2 \end{pmatrix} \\ &= \frac{1}{r^2|z_1|^2 + r^{-2}|z_2|^2} \begin{pmatrix} -r\bar{z}_2(r^2|z_1|^2 + r^{-2}|z_2|^2) - (r^{-2} - r^2)\bar{z}_1\bar{z}_2 \cdot rz_1 \\ r^{-1}\bar{z}_1(r^2|z_1|^2 + r^{-2}|z_2|^2) - (r^{-2} - r^2)\bar{z}_1\bar{z}_2 \cdot r^{-1}z_2 \end{pmatrix} \\ &= \frac{1}{\xi_A(r, z_1, z_2)^2} \begin{pmatrix} -r^{-1}\bar{z}_2 \\ r\bar{z}_1 \end{pmatrix} \end{aligned}$$

Then \tilde{b}_1 and \tilde{b}_2 are orthogonal, so b_1 and b_2 are simply obtained by normalizing these vectors whose norms are:

$$\|\tilde{b}_1\| = \sqrt{r^2|z_1|^2 + r^{-2}|z_2|^2} = \xi_A(r, z_1, z_2) \quad \|\tilde{b}_2\| = \frac{1}{\sqrt{r^2|z_1|^2 + r^{-2}|z_2|^2}} = \frac{1}{\xi_A(r, z_1, z_2)}$$

The matrix k' with columns b_1, b_2 is now contained in K^0 . Since column operations are realized by multiplication with elementary matrices from the right, it follows that $k' = gu'^{-1}a'^{-1}$ where the factors are the following matrices:

$$\begin{aligned} k' &= \frac{1}{\xi_A(r, z_1, z_2)} \begin{pmatrix} rz_1 & -r^{-1}\bar{z}_2 \\ r^{-1}z_2 & r\bar{z}_1 \end{pmatrix} = k(\kappa_A(r, z_1, z_2)) \in K^0 \\ a'^{-1} &= \begin{pmatrix} \frac{1}{\xi_A(r, z_1, z_2)} & 0 \\ 0 & \xi_A(r, z_1, z_2) \end{pmatrix} = a(\xi_A(r, z_1, z_2)^{-1}) \in A^0 \\ u'^{-1} &= \begin{pmatrix} 1 & -\frac{(r^{-2} - r^2)\bar{z}_1\bar{z}_2}{\xi_A(r, z_1, z_2)^2} \\ 0 & 1 \end{pmatrix} = u(-\eta_A(r, z_1, z_2)) \in U_+^0 \end{aligned}$$

If we rearrange this expression as $g = k'a'u'$, the proof is finished. \square

An important consequence of this computation is the following result which can be viewed as the simplest case of Kostant's Non-Linear Convexity Theorem and will play a crucial role in the proof of our anticipated generalization to the Kac-Moody setting.

Corollary 3.5.2. *Let $r \in \mathbb{R}_{>0}$ be arbitrary, set $a := a(r) \in A^0$ and $h := \log(a) \in \mathfrak{a}^0$. The image of the map $K^0 \rightarrow A^0$, $k \mapsto \Pi_{A^0}(ak)$, is the exponential of the closed line segment in \mathfrak{a}^0 between h and $-h$ or, equivalently, the exponential of the convex hull of the orbit $W^0 \cdot h \subset \mathfrak{a}^0$.*

Proof. From the definition of ξ_A , it is clear that $(z_1, z_2) \mapsto \xi_A(r, z_1, z_2)$ restricts to a surjection $B \rightarrow [r^{-1}, r]$. Since every $k \in K^0$ can be uniquely written as $k = k(z_1, z_2)$ for some $(z_1, z_2) \in B$, Proposition 3.5.1 yields that

$$\{\Pi_{A^0}(ak) \mid k \in K^0\} = \{a(r') \mid r' \in [r^{-1}, r]\} = \exp \left\{ \begin{pmatrix} s' & 0 \\ 0 & -s' \end{pmatrix} \mid s' \in [-\log(r), \log(r)] \right\}.$$

The last set is precisely the exponential of the closed line segment in \mathfrak{a}^0 connecting

$$h = \begin{pmatrix} \log(r) & 0 \\ 0 & -\log(r) \end{pmatrix} \quad \text{and} \quad -h = \begin{pmatrix} -\log(r) & 0 \\ 0 & \log(r) \end{pmatrix},$$

so the first assertion holds. As the unique non-trivial element $s_1 \in W^0 \cong \mathbb{Z}_2$ acts on \mathfrak{a}^0 as $-\mathrm{id}$, we have $-h = s_1(h)$ and the second claim is proved as well. \square

Next, we derive an analogue of Proposition 3.5.1 for elements of U_+^0 instead of A^0 , i.e. we compute the Iwasawa components of a product $u(z)k(z_1, z_2)$ for $z \in \mathbb{K}$ and $(z_1, z_2) \in B$. We proceed in the exact same way with similar definitions:

$$\begin{aligned} \xi_U : \mathbb{K} \times B &\rightarrow \mathbb{R}_{>0} & \eta_U : \mathbb{K} \times B &\rightarrow \mathbb{K} \\ (z, z_1, z_2) &\mapsto \sqrt{1 + |zz_2|^2 + 2\operatorname{Re}(z\bar{z}_1\bar{z}_2)} & (z, z_1, z_2) &\mapsto \frac{z\bar{z}_1^2 - \bar{z}z_2^2 + |z|^2\bar{z}_1\bar{z}_2}{\xi_U(z, z_1, z_2)^2} \end{aligned} \quad (3.5.7)$$

Using that $|z_1|^2 \leq 1$ for $(z_1, z_2) \in B$, it is not difficult to check that ξ_U and η_U are well-defined. Likewise, we introduce:

$$\begin{aligned} \kappa_U^1 : \mathbb{K} \times B &\rightarrow \mathbb{K} & \kappa_U^2 : \mathbb{K} \times B &\rightarrow \mathbb{K} \\ (z, z_1, z_2) &\mapsto \frac{z_1 + zz_2}{\xi_U(z, z_1, z_2)} & (z, z_1, z_2) &\mapsto \frac{z_2}{\xi_U(z, z_1, z_2)} \end{aligned} \quad (3.5.8)$$

They are also well-defined and $|\kappa_U^1(z, z_1, z_2)|^2 + |\kappa_U^2(z, z_1, z_2)|^2 = 1$ for all $z \in \mathbb{K}$ and $(z_1, z_2) \in B$, so the product map $\kappa_U := \kappa_U^1 \times \kappa_U^2$ takes values in B . Exactly as before, these functions determine the Iwasawa components of elements in $U_+^0 K^0 \subset G^0$.

Proposition 3.5.3. *For all $z \in \mathbb{K}$ and $(z_1, z_2) \in B$, the product $u(z)k(z_1, z_2) \in G^0$ has the following Iwasawa components:*

$$\begin{aligned} \Pi_{K^0}(u(z)k(z_1, z_2)) &= k(\kappa_U(z, z_1, z_2)) \\ \Pi_{A^0}(u(z)k(z_1, z_2)) &= a(\xi_U(z, z_1, z_2)) \\ \Pi_{U_+^0}(u(z)k(z_1, z_2)) &= u(\eta_U(z, z_1, z_2)) \end{aligned} \quad (3.5.9)$$

Proof. As in the proof of Proposition 3.5.1, we need to apply the Gram-Schmidt procedure to the column vectors v_1, v_2 of the matrix

$$g := u(z)k(z_1, z_2) = \begin{pmatrix} z_1 + zz_2 & -\bar{z}_2 + z\bar{z}_1 \\ z_2 & \bar{z}_1 \end{pmatrix} \in G^0.$$

Again, we set $\tilde{b}_1 := v_1$ and $\tilde{b}_2 := v_2 - \frac{(v_2, v_1)}{\|v_1\|^2} v_1$ where:

$$\begin{aligned} \|v_1\|^2 &= (z_1 + zz_2)(\bar{z}_1 + \bar{z}\bar{z}_2) + |z_2|^2 = |z_1|^2 + z\bar{z}_1\bar{z}_2 + \bar{z}z_1\bar{z}_2 + |zz_2|^2 + |z_2|^2 \\ &= 1 + |zz_2|^2 + 2\operatorname{Re}(z\bar{z}_1\bar{z}_2) = \xi_U(z, z_1, z_2)^2 \\ (v_2, v_1) &= (-\bar{z}_2 + z\bar{z}_1)(\bar{z}_1 + \bar{z}\bar{z}_2) + \bar{z}_1\bar{z}_2 = -\bar{z}_1\bar{z}_2 + z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2 + \bar{z}_1\bar{z}_2 \\ &= z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2 = \eta_U(z, z_1, z_2)\xi_U(z, z_1, z_2)^2 \end{aligned}$$

These expressions and the fact that $|z_1|^2 + |z_2|^2 = 1$ establish the following computation:

$$\begin{aligned} \tilde{b}_2 &= \begin{pmatrix} -\bar{z}_2 + z\bar{z}_1 \\ \bar{z}_1 \end{pmatrix} - \frac{z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2}{1 + z\bar{z}_1\bar{z}_2 + \bar{z}z_1\bar{z}_2 + |zz_2|^2} \begin{pmatrix} z_1 + zz_2 \\ z_2 \end{pmatrix} \\ &= \frac{1}{\xi_U(z, z_1, z_2)^2} \left[\begin{pmatrix} (-\bar{z}_2 + z\bar{z}_1)(1 + z\bar{z}_1\bar{z}_2 + \bar{z}z_1\bar{z}_2 + |zz_2|^2) \\ \bar{z}_1(1 + z\bar{z}_1\bar{z}_2 + \bar{z}z_1\bar{z}_2 + |zz_2|^2) \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} (z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2)(z_1 + zz_2) \\ (z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2)z_2 \end{pmatrix} \right] = \frac{1}{\xi_U(z, z_1, z_2)^2} \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 + \bar{z}\bar{z}_2 \end{pmatrix} \end{aligned}$$

Furthermore, we have $\|\tilde{b}_1\| = \xi_U(z, z_1, z_2)$ and $\|\tilde{b}_2\| = \xi_U(z, z_1, z_2)^{-1}$. Thus, we conclude exactly as before that $k' = gu'^{-1}a'^{-1}$ with the factors listed below, so the proof is complete:

$$\begin{aligned} k' &= \frac{1}{\xi_U(z, z_1, z_2)} \begin{pmatrix} z_1 + zz_2 & -\bar{z}_2 \\ z_2 & \bar{z}_1 + \bar{z}\bar{z}_2 \end{pmatrix} = k(\kappa_U(z, z_1, z_2)) \in K^0 \\ a'^{-1} &= \begin{pmatrix} \frac{1}{\xi_U(z, z_1, z_2)} & 0 \\ 0 & \xi_U(z, z_1, z_2) \end{pmatrix} = a(\xi_U(z, z_1, z_2)^{-1}) \in A^0 \\ u'^{-1} &= \begin{pmatrix} 1 & -\frac{z\bar{z}_1^2 - \bar{z}\bar{z}_2^2 + |z|^2\bar{z}_1\bar{z}_2}{\xi_U(z, z_1, z_2)^2} \\ 0 & 1 \end{pmatrix} = u(-\eta_U(z, z_1, z_2)) \in U_+^0 \end{aligned} \quad \square$$

For applications in Chapter 6, we require some additional properties of the functions introduced in this section that we establish in the next two lemmas and their respective corollaries.

Lemma 3.5.4. *Let $r \in \mathbb{R}_{>0}$ and $z \in \mathbb{K}$ be arbitrary. Each of the maps $(z_1, z_2) \mapsto \kappa_A(r, z_1, z_2)$ and $(z_1, z_2) \mapsto \kappa_U(z, z_1, z_2)$ defines a bijection $B \rightarrow B$.*

Proof. We first consider the map $\kappa_r := \kappa_A(r, \cdot) : B \rightarrow B$ and show that it is injective. Suppose that $(z_1, z_2), (z'_1, z'_2) \in B$ have the same image under κ_r , then

$$\frac{rz_1}{\xi_A(r, z_1, z_2)} = \frac{rz'_1}{\xi_A(r, z'_1, z'_2)} \quad \text{and} \quad \frac{r^{-1}z_2}{\xi_A(r, z_1, z_2)} = \frac{r^{-1}z'_2}{\xi_A(r, z'_1, z'_2)}.$$

Consequently, we have $(z_1, z_2) = \lambda(z'_1, z'_2)$ with $\lambda := \frac{\xi_A(r, z_1, z_2)}{\xi_A(r, z'_1, z'_2)} \in \mathbb{R}_{>0}$. As both points lie in B , this is only possible if $\lambda = 1$, i.e. $(z_1, z_2) = (z'_1, z'_2)$.

In order to show that κ_r is also surjective, we make use of the following map:

$$f : \left[0, \frac{\pi}{2}\right] \rightarrow [0, 1] \quad \gamma \mapsto \kappa_A^1(r, \cos(\gamma), \sin(\gamma)) = \frac{r \cos(\gamma)}{\sqrt{r^2 \cos^2(\gamma) + r^{-2} \sin^2(\gamma)}}$$

It is clear that f is well-defined and continuous. In addition, it satisfies $f(0) = 1$ and $f(\frac{\pi}{2}) = 0$, so f is surjective by the intermediate value theorem. Let now $(z_1, z_2) \in B$ be arbitrary. Using polar coordinates, we find $r_1, r_2 \in [0, 1]$ and $\theta_1, \theta_2 \in [0, 2\pi)$ such that $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. If $\mathbb{K} = \mathbb{R}$, we necessarily have $\theta_1, \theta_2 \in \{0, \pi\}$. By surjectivity of f , there exists some $\gamma \in [0, \frac{\pi}{2}]$ such that $f(\gamma) = r_1$. Now $(z_1, z_2) \in B$ implies that $r_2 \in [0, 1]$ is the unique non-negative square root of $1 - r_1^2$. Likewise, since $\kappa_A(r, \cos(\gamma), \sin(\gamma)) \in B$ and $\sin(\gamma) \geq 0$, we also conclude that $\kappa_A^2(r, \cos(\gamma), \sin(\gamma)) \in [0, 1]$ is the unique non-negative square root of

$$1 - \kappa_A^1(r, \cos(\gamma), \sin(\gamma))^2 = 1 - f(\gamma)^2 = 1 - r_1^2.$$

It follows that $r_2 = \kappa_A^2(r, \cos(\gamma), \sin(\gamma))$, so if we define $q_1 := \cos(\gamma)$ and $q_2 := \sin(\gamma)$, then $(q_1, q_2) \in B$ and $\kappa_A(r, q_1, q_2) = (r_1, r_2)$. Furthermore, we also have $(y_1, y_2) := (q_1 e^{i\theta_1}, q_2 e^{i\theta_2}) \in B$ and the definition of κ_A yields that

$$\kappa_r(y_1, y_2) = \kappa_A(r, y_1, y_2) = (\kappa_A^1(r, q_1, q_2) e^{i\theta_1}, \kappa_A^2(r, q_1, q_2) e^{i\theta_2}) = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = (z_1, z_2).$$

Therefore, $\kappa_r : B \rightarrow B$ is surjective as well. Next, we investigate the map $\kappa_z := \kappa_U(z, \cdot) : B \rightarrow B$. If $(z_1, z_2), (z'_1, z'_2) \in B$ have the same image under κ_z , then

$$\frac{z_1 + zz_2}{\xi_U(z, z_1, z_2)} = \frac{z'_1 + zz'_2}{\xi_U(z, z'_1, z'_2)} \quad \text{and} \quad \frac{z_2}{\xi_U(z, z_1, z_2)} = \frac{z'_2}{\xi_U(z, z'_1, z'_2)}.$$

By the same reasoning as above, it follows that (z_1, z_2) is a positive scalar multiple of (z'_1, z'_2) which is only possible if the two points coincide, so κ_z is injective.

To prove surjectivity, we apply an abstract argument from algebraic topology for which we refer to [Hat02, Section 2.2]. Let $c : [0, 1] \rightarrow \mathbb{K}$ be a continuous curve with $c(0) = 0$ and $c(1) = z$, then $t \mapsto \kappa_{c(t)}$ defines a homotopy between the continuous maps $\kappa_0 = \text{id}$ and κ_z . If we view κ_z as a map from S^1 or S^3 to itself, it has a well-defined “degree” which is homotopy-invariant. Hence, the degree of κ_z is 1. But if κ_z was not surjective, its degree would have to be 0 since if a point $p \in B$ was not contained in the image, then κ_z would map B to the contractible space $B \setminus \{p\}$ and induce the zero map on every homology group. Thus, $\kappa_z : B \rightarrow B$ is surjective too. \square

Corollary 3.5.5. *Let $g \in G^0$ be arbitrary. The map $k \mapsto \Pi_{K^0}(gk)$ defines a bijection $K^0 \rightarrow K^0$.*

Proof. We first show that it is injective. Suppose that $k_1, k_2 \in K^0$ satisfy $gk_1 = k_0 a' u'$ and $gk_2 = k_0 a'' u''$ for some $k_0 \in K^0$ with $a', a'' \in A^0$ and $u', u'' \in U_+^0$. As A^0 normalizes U_+^0 , we have

$$k_2^{-1} k_1 = (gk_2)^{-1} (gk_1) = u''^{-1} a''^{-1} a' u' \in K^0 \cap A^0 U_+^0 = \{1\}$$

and therefore $k_1 = k_2$. To prove surjectivity, let $k' \in K^0$ be arbitrary and write $g = kau$ in Iwasawa form. In view of Proposition 3.5.1 and 3.5.3, it follows from Lemma 3.5.4 that there exist $k_1, k_2 \in K^0$ such that $\Pi_{K^0}(ak_1) = k^{-1}k'$ and $\Pi_{K^0}(uk_2) = k_1$, which implies

$$\Pi_{K^0}(gk_2) = k\Pi_{K^0}(auk_2) = k\Pi_{K^0}(ak_1) = kk^{-1}k' = k'$$

because A^0 normalizes U_+^0 . Thus, the map is surjective as well. \square

Lemma 3.5.6. *For every $r \in \mathbb{R}_{>0}$, there exists some $\gamma \in [0, 2\pi)$ such that the image of the map $\mathbb{K} \rightarrow \mathbb{R}_{>0}$, $z \mapsto \xi_U(z, \cos(\gamma), \sin(\gamma))$, contains the interval $[r, \infty) \subset \mathbb{R}_{>0}$.*

Proof. It is sufficient to consider the case $\mathbb{K} = \mathbb{R}$. For every $\gamma \in [0, 2\pi)$, we define a map:

$$g_\gamma : \mathbb{R} \rightarrow \mathbb{R}_{>0} \quad x \mapsto \xi_U(x, \cos(\gamma), \sin(\gamma))^2 = 1 + 2x \cos(\gamma) \sin(\gamma) + x^2 \sin^2(\gamma)$$

Hence, we need to show that $[r^2, \infty) \subset g_\gamma(\mathbb{R})$ for some $\gamma \in [0, 2\pi)$. If $\gamma \in \{0, \pi\}$, then g_γ is the constant function $x \mapsto 1$. Otherwise, $\sin(\gamma) \neq 0$ and the graph of g_γ is an upwards-pointing parabola whose unique global minimum $x_0(\gamma) \in \mathbb{R}$ is located at

$$x_0(\gamma) = -\frac{2 \cos(\gamma) \sin(\gamma)}{2 \sin^2(\gamma)} = -\frac{\cos(\gamma)}{\sin(\gamma)}$$

and the value of g_γ at this minimum is

$$g_\gamma(x_0(\gamma)) = 1 - 2 \frac{\cos(\gamma)}{\sin(\gamma)} \cos(\gamma) \sin(\gamma) + \frac{\cos^2(\gamma)}{\sin^2(\gamma)} \sin^2(\gamma) = 1 - \cos^2(\gamma) = \sin^2(\gamma).$$

Thus, if $\gamma \in [0, 2\pi) \setminus \{0, \pi\}$ is chosen such that $\sin^2(\gamma) \leq r^2$, then $[r^2, \infty) \subset g_\gamma(\mathbb{R})$. \square

Corollary 3.5.7. *For every $a \in A^0$, there exists some $\gamma \in [0, 2\pi)$ such that $a \in \Pi_{A^0}(U_+^0 R_\gamma)$. In particular, $\Pi_{A^0}(U_+^0 K^0) = A^0$.*

Proof. In view of Proposition 3.5.3, the assertion follows immediately from Lemma 3.5.6. \square

Parallel to the Iwasawa decomposition $G^0 = K^0 A^0 U_+^0$ on the group level, there is a similar decomposition $\mathfrak{g}^0 = \mathfrak{k}^0 \oplus \mathfrak{a}^0 \oplus \mathfrak{u}_+^0$ on the Lie algebra level, where the three components are the Lie algebras of the corresponding subgroups. More precisely, the compact form \mathfrak{k}^0 of $\mathfrak{g}^0 = \mathfrak{sl}_2(\mathbb{K})$ is the Lie algebra $\mathfrak{so}(2)$ (resp. $\mathfrak{su}(2)$) of trace-free skew-symmetric (resp. skew-Hermitian) matrices if $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), \mathfrak{a}^0 is the space of trace-free diagonal matrices with real entries (both in the real and complex case) and \mathfrak{u}_+^0 is the space of upper-triangular matrices with 0's on the diagonal. The Lie group G^0 acts on its Lie algebra \mathfrak{g}^0 via the adjoint representation $\text{Ad}^0 : G^0 \rightarrow \text{GL}(\mathfrak{g}^0)$ which is simply conjugation. We now arrive at the following analogue of Proposition 3.5.1:

Proposition 3.5.8. *Let $t \in \mathbb{R}$ be arbitrary and set $h := \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}^0$. For all $(z_1, z_2) \in B$, the Iwasawa components of $\text{Ad}^0(k(z_1, z_2))h \in \mathfrak{g}^0$ are the following:*

$$\begin{aligned} \pi_{\mathfrak{k}^0}(\text{Ad}^0(k(z_1, z_2))h) &= \begin{pmatrix} 0 & -2tz_1\bar{z}_2 \\ 2t\bar{z}_1z_2 & 0 \end{pmatrix} \\ \pi_{\mathfrak{a}^0}(\text{Ad}^0(k(z_1, z_2))h) &= \begin{pmatrix} t(|z_1|^2 - |z_2|^2) & 0 \\ 0 & -t(|z_1|^2 - |z_2|^2) \end{pmatrix} \\ \pi_{\mathfrak{u}_+^0}(\text{Ad}^0(k(z_1, z_2))h) &= \begin{pmatrix} 0 & 4tz_1\bar{z}_2 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{3.5.10}$$

Proof. This is an immediate consequence of the following calculation:

$$\text{Ad}^0(k(z_1, z_2))h = k(z_1, z_2)hk(z_1, z_2)^{-1} = t \begin{pmatrix} |z_1|^2 - |z_2|^2 & 2z_1\bar{z}_2 \\ 2\bar{z}_1z_2 & -(|z_1|^2 - |z_2|^2) \end{pmatrix} \quad \square$$

As on the group level, this computation leads to the following statement which can be seen as the simplest instance of Kostant's Linear Convexity Theorem.

Corollary 3.5.9. *Let $t \in \mathbb{R}$ be arbitrary and set $h := \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}^0$. The image of the map $K^0 \rightarrow \mathfrak{a}^0$, $k \mapsto \pi_{\mathfrak{a}^0}(\text{Ad}^0(k)h)$, is the closed line segment in \mathfrak{a}^0 between h and $-h$ or, equivalently, the convex hull of the orbit $W^0 \cdot h \subset \mathfrak{a}^0$.*

Proof. As (z_1, z_2) runs over B , the expression $|z_1|^2 - |z_2|^2 = 1 - 2|z_2|^2$ takes every value in the interval $[-1, 1]$, so Proposition 3.5.8 implies that

$$\{\pi_{\mathfrak{a}^0}(\text{Ad}^0(k(z_1, z_2))h) \mid (z_1, z_2) \in B\} = \left\{ \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} \mid s \in [-t, t] \right\}.$$

This is precisely the closed line segment in \mathfrak{a}^0 between h and $-h = s_1(h)$. \square

Similarly, we could also establish an analogue of Proposition 3.5.3 on the Lie algebra level, but this is not necessary because there, we will only be interested in generalizing Kostant's Linear Convexity Theorem to the Kac-Moody setting for which the previous result will be sufficient.

3.6 Functoriality in the Kac-Moody root datum

We conclude this chapter by explaining how the definitions of Kac-Moody groups & algebras are “functorial” with respect to the underlying Kac-Moody root datum. This property will be exploited in the proofs of our anticipated generalizations of Kostant's Convexity Theorems in Chapter 5. We follow the terminology of [Mar18, Section 7.3 and 7.4.5]. In order not to overload the text with definitions, we often employ suggestive notation that should be self-explanatory from the context. In the beginning, \mathbb{K} is again allowed to be an arbitrary field, not necessarily \mathbb{R} or \mathbb{C} .

Definition 3.6.1. Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ and $\check{\mathcal{D}} = (I, \check{\mathbb{A}}, \check{\Lambda}, (\check{c}_i)_{i \in I}, (\check{h}_i)_{i \in I})$ be two Kac-Moody root data associated to the same GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$. A *morphism* between \mathcal{D} and $\check{\mathcal{D}}$ is a \mathbb{Z} -linear map $f : \Lambda^\vee \rightarrow \check{\Lambda}^\vee$ such that $f(h_i) = \check{h}_i$ and $\check{c}_i \circ f = c_i$ for all $i \in I$.

More succinctly, we will simply say that $f : \mathcal{D} \rightarrow \check{\mathcal{D}}$ is a morphism. By definition, a morphism of Kac-Moody root data satisfies

$$\langle \check{c}_i, f(h) \rangle = \langle c_i, h \rangle \quad \text{for all } h \in \Lambda^\vee \text{ and } i \in I. \quad (3.6.1)$$

In particular, this implies that f is equivariant with respect to the natural actions of the Weyl group $W = W(\mathbb{A})$ on Λ^\vee and $\check{\Lambda}^\vee$ in the sense that

$$f(w(h)) = w(f(h)) \quad \text{for all } w \in W \text{ and } h \in \Lambda^\vee. \quad (3.6.2)$$

Let \mathbb{K} be a field and denote by \mathfrak{g} and $\check{\mathfrak{g}}$ the Kac-Moody algebras of types \mathcal{D} and $\check{\mathcal{D}}$ over \mathbb{K} with Cartan subalgebras $\mathfrak{h} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}$ and $\check{\mathfrak{h}} = \check{\Lambda}^\vee \otimes_{\mathbb{Z}} \mathbb{K}$ and Chevalley generators $(e_i)_{i \in I}, (f_i)_{i \in I}$ and $(\check{e}_i)_{i \in I}, (\check{f}_i)_{i \in I}$, respectively. The defining relations (KMA1)-(KMA4) imply that the map sending e_i to \check{e}_i and f_i to \check{f}_i for all $i \in I$ and \mathfrak{h} to $\check{\mathfrak{h}}$ via the canonical \mathbb{K} -linear extension of f uniquely extends to a Lie algebra homomorphism $\varphi' : \mathfrak{g} \rightarrow \check{\mathfrak{g}}$.

Likewise, let G and \check{G} be the Kac-Moody groups of types \mathcal{D} and $\check{\mathcal{D}}$ over \mathbb{K} with tori $T = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ and $\check{T} = \check{\Lambda}^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ and root groups $(U_\alpha)_{\alpha \in \Phi}$ and $(\check{U}_\alpha)_{\alpha \in \Phi}$, respectively. By inspecting the defining relations (KMG1)-(KMG4), it is not difficult to verify that the map taking $x_\alpha(z) \in U_\alpha$ to $\check{x}_\alpha(z) \in \check{U}_\alpha$ for all $\alpha \in \Phi$ and $z \in \mathbb{K}$ and T to \check{T} via the canonical extension of f uniquely extends to a group homomorphism $\varphi : G \rightarrow \check{G}$.

By [Mar18, Proposition 7.63 and Theorem 7.71], the kernel of the group homomorphism $\varphi : G \rightarrow \check{G}$ is contained in the center $Z = \{t \in T : t(c_i) = 1 \ \forall i \in I\}$ of G and φ is surjective if its restriction to T is surjective onto \check{T} . Likewise, the kernel of the Lie algebra homomorphism $\varphi' : \mathfrak{g} \rightarrow \check{\mathfrak{g}}$ lies in $\{h \in \mathfrak{h} : \langle c_i, h \rangle = 0 \ \forall i \in I\} \subset \mathfrak{z}$ and φ' is surjective if its restriction to \mathfrak{h} is surjective onto $\check{\mathfrak{h}}$.

Suppose now that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathfrak{a} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ and $\check{\mathfrak{a}} = \check{\Lambda}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ denote the real Cartan subalgebras of \mathfrak{g} and $\check{\mathfrak{g}}$. The Lie algebra homomorphism $\varphi' : \mathfrak{g} \rightarrow \check{\mathfrak{g}}$ restricts to a W -equivariant linear map $\mathfrak{a} \rightarrow \check{\mathfrak{a}}$, i.e. it intertwines the \mathcal{D} -representation of W on \mathfrak{a} with the $\check{\mathcal{D}}$ -representation of W on $\check{\mathfrak{a}}$. Combining this fact with (3.6.1) implies that the fundamental chambers and Tits cones of these representations are related in the natural ways:

$$\varphi'(C) \subset \check{C} \quad \varphi'(\overline{C}) \subset \overline{\check{C}} \quad \varphi'(X_\pm) \subset \check{X}_\pm \quad \varphi'(X_\pm^\circ) \subset \check{X}_\pm^\circ \quad (3.6.3)$$

It is also immediate from the definition of φ' (resp. φ) that it intertwines the compact involutions of \mathfrak{g} and $\check{\mathfrak{g}}$ (resp. G and \check{G}) in the sense that the following diagrams commute:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\theta} & \mathfrak{g} \\ \varphi' \downarrow & & \downarrow \varphi' \\ \check{\mathfrak{g}} & \xrightarrow{\check{\theta}} & \check{\mathfrak{g}} \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\Theta} & G \\ \varphi \downarrow & & \downarrow \varphi \\ \check{G} & \xrightarrow{\check{\Theta}} & \check{G} \end{array} \quad (3.6.4)$$

The first diagram implies that $\varphi'(\mathfrak{k}) \subset \check{\mathfrak{k}}$, while $\varphi'(\mathfrak{a}) \subset \check{\mathfrak{a}}$ and $\varphi'(\mathfrak{u}_+) \subset \check{\mathfrak{u}}_+$ hold by definition, so φ' is compatible with the Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ and $\check{\mathfrak{g}} = \check{\mathfrak{k}} \oplus \check{\mathfrak{a}} \oplus \check{\mathfrak{u}}_+$ and it intertwines the corresponding Iwasawa projections via three commutative diagrams:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi'} & \check{\mathfrak{g}} \\ \pi_{\mathfrak{k}} \downarrow & & \downarrow \pi_{\check{\mathfrak{k}}} \\ \mathfrak{k} & \xrightarrow{\varphi'} & \check{\mathfrak{k}} \end{array} \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi'} & \check{\mathfrak{g}} \\ \pi_{\mathfrak{a}} \downarrow & & \downarrow \pi_{\check{\mathfrak{a}}} \\ \mathfrak{a} & \xrightarrow{\varphi'} & \check{\mathfrak{a}} \end{array} \quad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi'} & \check{\mathfrak{g}} \\ \pi_{\mathfrak{u}_+} \downarrow & & \downarrow \pi_{\check{\mathfrak{u}}_+} \\ \mathfrak{u}_+ & \xrightarrow{\varphi'} & \check{\mathfrak{u}}_+ \end{array} \quad (3.6.5)$$

The same argument for the diagram on the right in (3.6.4) shows that $\varphi(K) \subset \check{K}$, $\varphi(A) \subset \check{A}$ and $\varphi(U_+) \subset \check{U}_+$, so φ is compatible with the Iwasawa decompositions $G = KAU_+$ and $\check{G} = \check{K}\check{A}\check{U}_+$ and it intertwines the associated Iwasawa projections:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & \check{G} \\ \Pi_K \downarrow & & \downarrow \Pi_{\check{K}} \\ K & \xrightarrow{\varphi} & \check{K} \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\varphi} & \check{G} \\ \Pi_A \downarrow & & \downarrow \Pi_{\check{A}} \\ A & \xrightarrow{\varphi} & \check{A} \end{array} \quad \begin{array}{ccc} G & \xrightarrow{\varphi} & \check{G} \\ \Pi_{U_+} \downarrow & & \downarrow \Pi_{\check{U}_+} \\ U_+ & \xrightarrow{\varphi} & \check{U}_+ \end{array} \quad (3.6.6)$$

Finally, if we denote the exponential maps of G and \check{G} both by \exp , the following diagram also commutes by definition:

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{\varphi'} & \check{\mathfrak{a}} \\ \exp \downarrow & & \downarrow \exp \\ A & \xrightarrow{\varphi} & \check{A} \end{array} \quad (3.6.7)$$

Example 3.6.2. Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum of rank $d \in \mathbb{N}$. There are three important morphisms of Kac-Moody root data involving \mathcal{D} .

- (i) Since the simply connected root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is cofree and coadjoint, the free \mathbb{Z} -module Λ_{sc}^\vee can be canonically identified with the coroot lattice Q^\vee by mapping h_i^{sc} to α_i^\vee for all $i \in I$. Under this identification, the cocharacter map $h : Q^\vee \rightarrow \Lambda^\vee$ can be viewed as a morphism of Kac-Moody root data. Consequently, if G^{sc} and G (resp. \mathfrak{g}^{sc} and \mathfrak{g}) denote the corresponding Kac-Moody groups (resp. algebras) of types $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ and \mathcal{D} over an arbitrary field \mathbb{K} , then h induces a group (resp. Lie algebra) homomorphism $G^{\text{sc}} \rightarrow G$ (resp. $\mathfrak{g}^{\text{sc}} \rightarrow \mathfrak{g}$).
- (ii) *Free extension:* \mathcal{D} can be “extended” to a free Kac-Moody root datum $\check{\mathcal{D}}$ of rank $d + n$ by the following construction: Let Λ_0^\vee and Λ_0 be dual free \mathbb{Z} -modules of rank n with dual bases $\{u_i \mid i \in I\} \subset \Lambda_0^\vee$ and $\{u_i^* \mid i \in I\} \subset \Lambda_0$. Define $\hat{\Lambda} := \Lambda \oplus \Lambda_0$ and $\hat{\Lambda}^\vee := \Lambda^\vee \oplus \Lambda_0^\vee$ and set

$$\hat{c}_i := c_i + u_i^* \in \hat{\Lambda} \quad \text{and} \quad \hat{h}_i := h_i \in \hat{\Lambda}^\vee \quad \text{for all } i \in I. \quad (3.6.8)$$

Then $\hat{\mathcal{D}} = (I, \mathbb{A}, \hat{\Lambda}, (\hat{c}_i)_{i \in I}, (\hat{h}_i)_{i \in I})$ is a free Kac-Moody root datum and the natural inclusion $\Lambda^\vee \rightarrow \hat{\Lambda}^\vee$ defines a morphism $\mathcal{D} \rightarrow \hat{\mathcal{D}}$. Hence, for every field \mathbb{K} , it induces homomorphisms $i : G \rightarrow \hat{G}$ and $i' : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ between the Kac-Moody groups & algebras of types \mathcal{D} and $\hat{\mathcal{D}}$ over \mathbb{K} that are both injective. In fact, it can be shown that \hat{G} (resp. $\hat{\mathfrak{g}}$) is isomorphic to a semidirect extension of G (resp. \mathfrak{g}) by $T_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ (resp. $\mathfrak{h}_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}$) whose precise structure is described in [Mar18, Section 7.3.2 and Proposition 7.66].

- (iii) *Cofree extension:* Dually, \mathcal{D} can also be extended to a cofree root datum $\tilde{\mathcal{D}}$ of rank $d + n$. As in (ii), let Λ_0^\vee and Λ_0 be dual free \mathbb{Z} -modules of rank n with dual bases $\{u_i \mid i \in I\} \subset \Lambda_0^\vee$ and $\{u_i^* \mid i \in I\} \subset \Lambda_0$ and define $\tilde{\Lambda} := \Lambda \oplus \Lambda_0$ and $\tilde{\Lambda}^\vee := \Lambda^\vee \oplus \Lambda_0^\vee$. This time, we set

$$\tilde{c}_i := c_i \in \tilde{\Lambda} \quad \text{and} \quad \tilde{h}_i := h_i + u_i \in \tilde{\Lambda}^\vee \quad \text{for all } i \in I. \quad (3.6.9)$$

Hence, $\tilde{\mathcal{D}} = (I, \mathbb{A}, \tilde{\Lambda}, (\tilde{c}_i)_{i \in I}, (\tilde{h}_i)_{i \in I})$ is a cofree Kac-Moody root datum and the canonical projection $\tilde{\Lambda}^\vee \rightarrow \Lambda^\vee$ can be viewed as a morphism $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$. Given a field \mathbb{K} , the induced homomorphisms $p : \tilde{G} \rightarrow G$ and $p' : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ between the Kac-Moody groups & algebras of types $\tilde{\mathcal{D}}$ and \mathcal{D} over \mathbb{K} are both surjective. More precisely, one can prove that \tilde{G} (resp. $\tilde{\mathfrak{g}}$) is isomorphic to a central extension of G (resp. \mathfrak{g}) or, equivalently, that G (resp. \mathfrak{g}) is a central quotient of \tilde{G} (resp. $\tilde{\mathfrak{g}}$) by $T_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$ (resp. $\mathfrak{h}_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}$). More details can be found in [Mar18, Section 7.3.2 and Proposition 7.64]. Observe that the construction of $\tilde{\mathcal{D}}$ ensures that it is even cotorsion-free.

As the final step of this chapter, we establish a simple computational result about the adjoint representation of a Kac-Moody group that will only be needed for the proof of Theorem 5.3.1. Afterwards, we have collected enough preliminary material about Kac-Moody groups & algebras to conclude Part I of this thesis and turn to the principal questions of research.

Lemma 3.6.3. *Let $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ be a Kac-Moody root datum, let \mathbb{K} be a field and denote by G (resp. \mathfrak{g}) the Kac-Moody group (resp. algebra) of type \mathcal{D} over \mathbb{K} .*

- (i) *Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be an integrable \mathfrak{g} -module with $\text{wt}(V, \rho) \subset \Lambda$ and $\hat{\rho} : G \rightarrow \text{GL}(V)$ the corresponding representation of G , then*

$$\rho(\text{Ad}(g)x) = \hat{\rho}(g)\rho(x)\hat{\rho}(g)^{-1} \quad \text{for all } g \in G \text{ and } x \in \mathfrak{g}. \quad (3.6.10)$$

- (ii) *For every $\alpha \in \Phi_+$, let $\varphi_\alpha : G^0 \rightarrow G$ and $\varphi'_\alpha : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ be the homomorphisms associated to α and denote by $\text{Ad}^0 : G^0 \rightarrow \text{GL}(\mathfrak{g}^0)$ the adjoint representation of G^0 on \mathfrak{g}^0 , then*

$$\varphi'_\alpha(\text{Ad}^0(g^0)x^0) = \text{Ad}(\varphi_\alpha(g^0))\varphi'_\alpha(x^0) \quad \text{for all } g^0 \in G^0 \text{ and } x^0 \in \mathfrak{g}^0. \quad (3.6.11)$$

- (iii) *Let $\tilde{\mathcal{D}}$ be another Kac-Moody root datum associated to \mathbb{A} and denote by \tilde{G} (resp. $\tilde{\mathfrak{g}}$) the Kac-Moody group (resp. algebra) of type $\tilde{\mathcal{D}}$ over \mathbb{K} . If $f : \mathcal{D} \rightarrow \tilde{\mathcal{D}}$ is a morphism, the canonical homomorphisms $\varphi : G \rightarrow \tilde{G}$ and $\varphi' : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ induced by f satisfy*

$$\varphi'(\text{Ad}(g)x) = \text{Ad}(\varphi(g))\varphi'(x) \quad \text{for all } g \in G \text{ and } x \in \mathfrak{g}. \quad (3.6.12)$$

- (iv) *Suppose that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\Theta : G \rightarrow G$ and $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ denote the compact involutions of G and \mathfrak{g} , then*

$$\theta(\text{Ad}(g)x) = \text{Ad}(\Theta(g))\theta(x) \quad \text{for all } g \in G \text{ and } x \in \mathfrak{g}. \quad (3.6.13)$$

In particular, in the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the ± 1 -eigenspaces of θ , both \mathfrak{k} and \mathfrak{p} are $\text{Ad}(K)$ -invariant.

Proof. (i) Since G is generated by the torus T and the root groups $(U_\alpha)_{\alpha \in \Phi}$ and Ad and $\hat{\rho}$ are homomorphisms, it suffices to verify the statement in the case where g is contained in one of these subgroups. In addition, as $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra homomorphism, it extends uniquely to an algebra homomorphism $\rho : \mathcal{U}(\mathfrak{g}) \rightarrow \mathfrak{gl}(V)$. For arbitrary $\alpha \in \Phi$ and $z \in \mathbb{K}$,

we may therefore perform the following computation in $\mathcal{U}(\mathfrak{g})$, where l_{e_α} (resp. r_{e_α}) denotes multiplication from the left (resp. right) by $e_\alpha \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$:

$$\begin{aligned}
\rho(\text{Ad}(x_\alpha(z))x) &= \rho(\exp(\text{ad}(ze_\alpha))x) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \rho(\text{ad}(e_\alpha)^s x) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \rho((l_{e_\alpha} - r_{e_\alpha})^s x) \\
&= \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{t=0}^s \binom{s}{t} (-1)^{s-t} \rho(e_\alpha^t x e_\alpha^{s-t}) \\
&= \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{t=0}^s \binom{s}{t} (-1)^{s-t} \rho(e_\alpha)^t \rho(x) \rho(e_\alpha)^{s-t} \\
&= \sum_{s=0}^{\infty} \frac{z^s}{s!} \sum_{p+q=s} \binom{p+q}{p} (-1)^q \rho(e_\alpha)^p \rho(x) \rho(e_\alpha)^q \\
&= \sum_{s=0}^{\infty} \sum_{p+q=s} \frac{z^p}{p!} \frac{(-z)^q}{q!} \rho(e_\alpha)^p \rho(x) \rho(e_\alpha)^q \\
&= \left(\sum_{p=0}^{\infty} \frac{z^p}{p!} \rho(e_\alpha)^p \right) \rho(x) \left(\sum_{q=0}^{\infty} \frac{(-z)^q}{q!} \rho(e_\alpha)^q \right) \\
&= \exp(\rho(ze_\alpha)) \rho(x) \exp(\rho(-ze_\alpha)) \\
&= \hat{\rho}(x_\alpha(z)) \rho(x) \hat{\rho}(x_\alpha(-z)) = \hat{\rho}(x_\alpha(z)) \rho(x) \hat{\rho}(x_\alpha(z))^{-1}
\end{aligned}$$

To establish the same result for an element $t \in T$ which acts diagonally on \mathfrak{g} with respect to the root space decomposition, we may w.l.o.g. assume that $x \in \mathfrak{u}_\beta$ for some $\beta \in \Phi^\infty \cup \{0\}$. On the one hand, this implies

$$\rho(\text{Ad}(t)x) = \rho(t(c_\beta)x) = t(c_\beta)\rho(x).$$

On the other hand, $\hat{\rho}(t)$ acts diagonally on V with respect to the weight space decomposition. Now for every $\mu \in \text{wt}(V, \rho)$ and $v \in V_\mu$, we have $\rho(x)v \in V_{\mu+c_\beta}$ and thus

$$\hat{\rho}(t)\rho(x)\hat{\rho}(t)^{-1}v = \hat{\rho}(t)(t^{-1}(\mu)\rho(x)v) = t(c_\beta + \mu)t(-\mu)\rho(x)v = t(c_\beta)\rho(x)v.$$

Since V is the direct sum of all weight spaces, it follows that

$$\hat{\rho}(t)\rho(x)\hat{\rho}(t)^{-1} = t(c_\beta)\rho(x) = \rho(\text{Ad}(t)x).$$

- (ii) As in the proof of (i), it suffices to verify the statement in the case where g^0 belongs to a suitable generating set of G^0 . For this, we take the groups U_+^0 and U_-^0 of upper-triangular and lower-triangular matrices with 1's on the diagonal. Let $x^0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{g}^0$ with $a, b, c \in \mathbb{K}$ be arbitrary. On the one hand, since the adjoint representation of G^0 is simply conjugation, the definition of φ'_α implies that the following holds for all $z \in \mathbb{K}$:

$$\varphi'_\alpha(\text{Ad}^0(x_+(z))x^0) = \varphi'_\alpha \begin{pmatrix} a + cz & b - 2az - cz^2 \\ c & -a - cz \end{pmatrix} = (a + cz)h_\alpha + (b - 2az - cz^2)e_\alpha - cf_\alpha$$

On the other hand, using the defining relations of \mathfrak{g} , we find:

$$\begin{aligned}
\text{Ad}(\varphi_\alpha(x_+(z)))\varphi'_\alpha(x^0) &= \text{Ad}(x_\alpha(z))(ah_\alpha + be_\alpha - cf_\alpha) = \exp(\text{ad}(ze_\alpha))(ah_\alpha + be_\alpha - cf_\alpha) \\
&= a(h_\alpha + [ze_\alpha, h_\alpha]) + be_\alpha - c \left(f_\alpha + [ze_\alpha, f_\alpha] + \frac{1}{2}[ze_\alpha, [ze_\alpha, f_\alpha]] \right) \\
&= ah_\alpha - 2aze_\alpha + be_\alpha - cf_\alpha + czh_\alpha - cz^2e_\alpha \\
&= (a + cz)h_\alpha + (b - 2az - cz^2)e_\alpha - cf_\alpha
\end{aligned}$$

In the second line, we have used that $[e_\alpha, e_\alpha] = 0$ in order to truncate the exponential series after the second, first resp. third term. Therefore, the right-hand sides of the previous two equations coincide which proves the assertion for elements in U_+^0 . A similar computation can be performed for those in U_-^0 .

- (iii) The proof follows the same strategy as in (i) but turns out to be simpler. Suppose first that $g = x_i(z) \in U_i$ for some $i \in I$ and $z \in \mathbb{K}$. Since φ' is a Lie algebra homomorphism and $\varphi'(e_i) = \check{e}_i$, the following calculation is justified:

$$\begin{aligned} \varphi'(\text{Ad}(x_i(z))x) &= \varphi'(\exp(\text{ad}(ze_i))x) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \varphi'(\text{ad}(e_i)^s x) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \text{ad}(\varphi'(e_i))^s \varphi'(x) \\ &= \sum_{s=0}^{\infty} \frac{z^s}{s!} \text{ad}(\check{e}_i)^s \varphi'(x) = \exp(\text{ad}(z\check{e}_i))\varphi'(x) = \text{Ad}(\check{x}_i(z))\varphi'(x) \\ &= \text{Ad}(\varphi(x_i(z)))\varphi'(x) \end{aligned}$$

If $g = x_{-i}(z) \in U_{-i}$ for some $i \in I$ and $z \in \mathbb{K}$, the same argument applies with e_i and \check{e}_i replaced by f_i and \check{f}_i , respectively. Finally, if $g = t \in T$, we may again w.l.o.g. assume that $x \in \mathfrak{u}_\beta$ for some $\beta \in \Phi^\infty \cup \{0\}$, so that $\varphi'(x) \in \check{\mathfrak{u}}_\beta$ and

$$\varphi'(\text{Ad}(t)x) = t(c_\beta)\varphi'(x) = \varphi(t)(\check{c}_\beta)\varphi'(x) = \text{Ad}(\varphi(t))\varphi'(x).$$

As G is generated by the torus and the positive and negative simple root groups, the proof of the assertion is complete.

- (iv) The computations in this case are almost identical to (iii) and follow the same reasoning. First, we again suppose that $g = x_i(z) \in U_i$ for some $i \in I$ and $z \in \mathbb{K}$. Using that θ is a \mathbb{C} -antilinear Lie algebra homomorphism with $\theta(e_i) = f_i$, the following holds:

$$\begin{aligned} \theta(\text{Ad}(x_i(z))x) &= \theta(\exp(\text{ad}(ze_i))x) = \sum_{s=0}^{\infty} \frac{\bar{z}^s}{s!} \theta(\text{ad}(e_i)^s x) = \sum_{s=0}^{\infty} \frac{\bar{z}^s}{s!} \text{ad}(\theta(e_i))^s \theta(x) \\ &= \sum_{s=0}^{\infty} \frac{\bar{z}^s}{s!} \text{ad}(f_i)^s \theta(x) = \exp(\text{ad}(\bar{z}f_i))\theta(x) = \text{Ad}(x_{-i}(\bar{z}))\theta(x) \\ &= \text{Ad}(\Theta(x_i(z)))\theta(x) \end{aligned}$$

Almost the same calculation works if $g = x_{-i}(z) \in U_{-i}$. Second, if $g = t \in T$, we may even assume that $t = r^h$ for some $h \in \Lambda^\vee$ and $r \in \mathbb{K}^*$ since elements of this form generate T . Given any $\beta \in \Phi^\infty \cup \{0\}$ and $x \in \mathfrak{u}_\beta$, we have $\theta(x) \in \mathfrak{u}_{-\beta}$ and therefore

$$\theta(\text{Ad}(r^h)x) = \theta(r^{(c_\beta, h)}x) = \bar{r}^{(c_\beta, h)}\theta(x) = \text{Ad}(\bar{r}^{-h})\theta(x) = \text{Ad}(\Theta(r^h))\theta(x).$$

For the same reasons as in (iii), these two computations conclude the proof of (3.6.13) and the final assertion is an immediate consequence. \square

Part II

Kostant Convexity for Kac-Moody Groups

Chapter 4

Convex Hulls of Weyl Group Orbits

The ultimate goal of the second part of this thesis is to formulate and prove analogues of Kostant's Convexity Theorems for real and complex Kac-Moody groups. This will be achieved in Chapter 5 and their applications to algebraic properties of Kac-Moody groups is the content of Chapter 6. As in the Lie group case investigated by Kostant, the statements of these theorems will involve *convex hulls of Weyl group orbits* in the real Cartan subalgebra of a real or complex Kac-Moody algebra. Hence, it is natural to first study these sets in their own right and, indeed, our proofs of the convexity theorems will require rather detailed information about their “geometric structure” in a sense that will be made more precise in the course of this chapter. Since the Weyl group orbits arising in Kac-Moody theory are typically infinite, this is a non-trivial task and it is not clear from the start what kind of properties and phenomena should be expected from their convex hulls. To the best of the author's knowledge, these sets have hitherto not been studied in this level of detail, so we will provide full proofs of all assertions in this chapter. Occasionally, similar statements have been established in different contexts that are not quite suitable for our purposes and we will mention this when appropriate. Although we have chosen to work in a setting that originates from Kac-Moody theory in order to use the results more easily in the proofs of the anticipated convexity theorems, most of the material in this chapter primarily relies on the theory of Coxeter groups and the Kac-Moody context will only play a subordinate role.

4.1 Preliminary definitions

Throughout this chapter, we denote by $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ a Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to an arbitrary GCM $\mathbb{A} = (a_{ij})_{i,j \in I} \in \mathbb{Z}^{n \times n}$ of rank $\ell \leq n$ and write $W := W(\mathbb{A})$ for the Weyl group of \mathbb{A} , $\Phi := \Phi(\mathbb{A})$ for its set of real roots and $M = (m_{ij})_{i,j \in I}$ for the Coxeter matrix of (W, S) . We let C (resp. \overline{C}) be the open (resp. closed) fundamental chamber of the \mathcal{D} -representation of W on $\mathfrak{a} := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_{\pm} \subset \mathfrak{a}$ the positive (resp. negative) Tits cone. Although much of the discussion could be simplified by assuming that \mathbb{A} is indecomposable, we prefer not to invoke this restriction and instead apply the following familiar notation:

Notation 4.1.1. We denote by $\mathbb{A}_1, \dots, \mathbb{A}_m$ the indecomposable components of \mathbb{A} with index sets $I_1, \dots, I_m \subset I$, by $\mathcal{D}_1, \dots, \mathcal{D}_m$ the corresponding indecomposable components of \mathcal{D} and by $(W_1, S_1), \dots, (W_m, S_m)$ the irreducible factors of (W, S) , which implies that $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_m$, $\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_m$ and $W \cong W_1 \times \dots \times W_m$. For every $k \in \{1, \dots, m\}$, the restriction of the \mathcal{D} -representation of W to $W_k = W(\mathbb{A}_k)$ coincides with the \mathcal{D}_k -representation of W_k and we write C_k (resp. \overline{C}_k) for its open (resp. closed) fundamental chamber and $X_{k,\pm}$ for its positive (resp. negative) Tits cone with interior $X_{k,\pm}^\circ$. We set $\Phi_k := \Phi(\mathbb{A}_k)$, so that $\Phi = \Phi_1 \sqcup \dots \sqcup \Phi_m$, and use similar notation for all other sets of (co-)roots introduced in Chapter 2. These definitions yield:

$$C = \bigcap_{k=1}^m C_k \quad \overline{C} = \bigcap_{k=1}^m \overline{C}_k \quad X_{\pm} = \bigcap_{k=1}^m X_{k,\pm} \quad X_{\pm}^\circ = \bigcap_{k=1}^m X_{k,\pm}^\circ \quad (4.1.1)$$

The goal of this chapter is to study the “geometry” of convex hulls of W -orbits in the d -dimensional real vector space \mathfrak{a} under the \mathcal{D} -representation. In other words, given an arbitrary element $h \in \mathfrak{a}$, we will be interested in the following set:

$$\Xi_h := \text{conv}(W \cdot h) \subset \mathfrak{a} \quad (4.1.2)$$

As we shall see, the “shape” of this set depends heavily on the location of h with respect to the “double Tits cone” $X_+ \cup X_- \subset \mathfrak{a}$ and the union $X_+^\circ \cup X_-^\circ$ (resp. $\overline{X_+} \cup \overline{X_-}$) of the topological interiors (resp. closures) of its two halves. We distinguish the following three possibilities, where we denote by W_h the stabilizer of h under the \mathcal{D} -representation of W :

Definition 4.1.2. An element $h \in \mathfrak{a}$ is said to be

- (i) *internal* if it is contained in $X_+ \cup X_-$. In this case, we further say that h is
 - a) *regular* if $W_h = \{1\}$.
 - b) *virtually singular* if $|W_h| < \infty$.
 - c) *almost virtually singular* if $|W_h \cap W_k| < \infty$ whenever $k \in \{1, \dots, m\}$ is such that the irreducible factor $W_k \subset W$ is not entirely contained in W_h .
 - d) *essentially singular* if there exists some $k \in \{1, \dots, m\}$ such that $W_h \cap W_k$ is infinite and a proper subgroup of W_k .
- (ii) *critical* if it is contained in $(\overline{X_+} \setminus X_+) \cup (\overline{X_-} \setminus X_-)$.
- (iii) *external* if it is not contained in $\overline{X_+} \cup \overline{X_-}$.

The reason why we consider both X_+ and $X_- = -X_+$ in this definition is that $\Xi_{-h} = -\Xi_h$ has the same properties as Ξ_h . By our choice of terminology, a regular element is virtually singular and a virtually singular element is also almost virtually singular but not essentially singular. In view of Proposition 2.6.5, these notions admit some noteworthy equivalent characterizations:

- a) h is regular if and only if h or $-h$ lies in an open Weyl chamber of X_+ .
- b) h is virtually singular if and only if h or $-h$ lies in a spherical cell of X_+ or, equivalently, in the topological interior $X_+^\circ \subset X_+$.
- c) h is almost virtually singular if and only if h or $-h$ has the following property: For every $k \in \{1, \dots, m\}$, it is contained in a spherical cell or in the intersection of all walls of $X_{k,+}$. Equivalently, it lies in the topological interior $X_{k,+}^\circ \subset X_{k,+}$ or is a global fixed-point for the W_k -action on \mathfrak{a} .

For the most part, we will investigate the set Ξ_h under the assumption that h is virtually singular since this is the setting in which it will have the most useful structure. These results also extend to almost virtually singular elements by a simple but technical modification. In the essentially singular case, however, it will turn out that some very important properties of Ξ_h no longer hold. For critical and external elements, a uniform description of Ξ_h does not seem to be available and we will content ourselves with some interesting examples. In order to simplify the discussion, we invoke Assumption 2.7.3 and the resulting notational conventions that we briefly repeat here.

Assumption 4.1.3. Throughout Chapter 4, we assume that the root datum \mathcal{D} is free and cofree, so that $d \geq 2n - \ell \geq n$. We view Q and Q^\vee as subsets of Λ and Λ^\vee , write α_i and α_i^\vee instead of c_i and h_i for all $i \in I$ and choose the dual bases

$$\mathcal{B} = \{\alpha_1^\vee, \dots, \alpha_n^\vee, h_{n+1}, \dots, h_d\} \subset \mathfrak{a} \quad \text{and} \quad \mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\} \subset \mathfrak{a}^*.$$

We denote by $\mathfrak{a}' := \bigoplus_{i \in I} \mathbb{R}\alpha_i^\vee$ the n -dimensional subspace of \mathfrak{a} spanned by the simple coroots.

Remark 4.1.4. If \mathcal{D} is not required to be free and cofree, it seems very likely that many results of this chapter can be restated and proved in a suitable way upon replacing α_i by c_i and α_i^\vee by h_i for all $i \in I$. However, we sometimes want to compute dimensions of certain sets or use arguments that rely on dimension considerations which become more involved if \mathcal{D} is not free or not cofree. Also, one very important step will require the existence of regular elements or, equivalently, that \mathcal{D} is admissible, which is automatically satisfied if \mathcal{D} is free. We emphasize once again that \mathcal{D} is necessarily free and cofree if \mathbb{A} is invertible.

For the convexity theorems in Chapter 5, this assumption will not be a restriction. We will first prove them for free and cofree (and cotorsion-free) Kac-Moody root data using the results of this chapter and afterwards generalize them to arbitrary root data via the functoriality of Section 3.6.

Convention 4.1.5. When we need to specify the Kac-Moody root datum under consideration, we add a prefix “ \mathcal{D} –” to all notions derived from the \mathcal{D} –representation. For instance, we call a regular element “ \mathcal{D} –regular” in this case and speak of “ \mathcal{D} –chambers” of the “ \mathcal{D} –Tits cone”.

Remark 4.1.6. If the Weyl group $W = W(\mathbb{A})$ is finite, then $X_+ = X_+^\circ = \mathfrak{a}$ by Proposition 2.6.5 (vi). In this situation, Ξ_h is a convex polytope for every $h \in \mathfrak{a}$ (cf. Definition 4.2.5) and most results of Section 4.3 can also be deduced from the general theory of these sets, which is a classical topic in convex geometry. However, W is finite if and only if \mathbb{A} is spherical or, equivalently, \mathbb{A} is the Cartan matrix of a finite-dimensional, complex, semisimple Lie algebra. Since this is a very small subclass of GCM’s and we are interested in the “generic” Kac-Moody setting, the reader should typically think of W as an infinite group. More precisely, the lemma below shows that Weyl group orbits of points in \mathfrak{a} should generally be viewed as infinite sets. In this case, the theory of convex polytopes cannot be applied and it is not a priori clear whether Ξ_h has any desirable properties. For example, one major issue that we are going to encounter is the question whether Ξ_h is a closed subset of \mathfrak{a} .

Lemma 4.1.7. *If $h \in \mathfrak{a}$ is such that $W \cdot h \subset \mathfrak{a}$ is finite, then h is almost virtually singular and contained in $X_+ \cap X_-$.*

Proof. First, we claim that $h \in X_+$. To prove this, we define the functional $\eta := \sum_{i \in I} \omega_i \in \mathfrak{a}^*$ which satisfies $\langle \eta, \alpha_i^\vee \rangle = 1$ for all $i \in I$. As $W \cdot h$ is finite, there exists some $w_0 \in W$ such that $\langle \eta, w_0(h) \rangle = \max_{w \in W} \langle \eta, w(h) \rangle$. Then $w_0(h) \in \overline{C}$ because otherwise, there would be some $i \in I$ with $\langle \alpha_i, w_0(h) \rangle < 0$ and hence

$$\langle \eta, s_i w_0(h) \rangle = \langle \eta, w_0(h) \rangle - \langle \eta, \alpha_i^\vee \rangle \langle \alpha_i, w_0(h) \rangle = \langle \eta, w_0(h) \rangle - \langle \alpha_i, w_0(h) \rangle > \langle \eta, w_0(h) \rangle,$$

contradicting the choice of w_0 , so we have $w_0(h) \in \overline{C} \subset X_+$ and $h \in X_+$, as claimed. The same argument yields that $-h \in X_+$ and thus $h \in X_+ \cap X_-$, so it remains to show that h is almost virtually singular. Upon replacing h by $w_0(h)$ if necessary, we may assume that $h \in \overline{C}$. Let $J \subset I$ be the unique set such that h lies in the cell $D_J \subset \overline{C}$, then the stabilizer of h is the standard subgroup W_J . Since $W \cdot h$ is finite and the W –translates of h are in bijective correspondence with the left cosets of $W_h = W_J$ in W , it follows that W_J is a finite-index standard subgroup of W . In the notation of Theorem 1.1.6, this implies that $W_{I_\infty} \subset W_J = W_h$. For $k \in \{1, \dots, m\}$ with $|W_k| = \infty$, we therefore have $W_k \subset W_h$, so h is almost virtually singular. \square

We conclude this introductory section with a simple result that will be used extensively in many proofs in this chapter. From this point onwards, we often apply Notation 1.2.8.

Lemma 4.1.8. *Given any $i \in I$, the stabilizer of ω_i under the W –action on \mathfrak{a}^* is the standard subgroup $W^{(i)}$. More precisely, for every $w \in W$, there exists $\alpha \in Q_+$ such that $w(\omega_i) = \omega_i - \alpha$, where $\alpha = 0$ if and only if $w \in W^{(i)}$.*

Proof. The dual of the W –action on \mathfrak{a} is precisely the \mathcal{D}^* –representation of W , where \mathcal{D}^* denotes the dual root datum of \mathcal{D} . Let $\overline{C}^* \subset \mathfrak{a}^*$ be the closed fundamental \mathcal{D}^* –chamber and $\{D_J^* \mid J \subset I\}$ the \mathcal{D}^* –cells of \overline{C}^* . Now $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all $j \in I$ implies that ω_i is contained in the cell $D_{I^{(i)}}^*$ whose stabilizer is the standard subgroup $W_{I^{(i)}} = W^{(i)}$.

It follows that $\alpha := \omega_i - w(\omega_i) = 0$ if and only if $w \in W^{(i)}$. To prove the second statement, it only remains to show that $\alpha \in Q_+$ for all $w \in W$. We argue by induction on $l(w)$. If $l(w) = 0$, then $w = 1$ and $\alpha = 0 \in Q_+$. If $l(w) > 0$, there exists some $j \in I$ such that $w' := ws_j$ satisfies $l(w') < l(w)$. By the induction hypothesis, we have $\alpha' := \omega_i - w'(\omega_i) \in Q_+$ and

$$\alpha = \omega_i - w(\omega_i) = \omega_i - w'(\omega_i - \delta_{ij}\alpha_j) = \alpha' + \delta_{ij}w'(\alpha_j).$$

Since $w'(\alpha_j) \in \Phi_+ \subset Q_+$ by (2.5.4), we conclude that $\alpha \in Q_+$, as required. \square

4.2 Some facts from convex geometry

To prepare for the study of Ξ_h in Section 4.3, we begin by collecting some notions and results about convex sets in finite-dimensional real vector spaces. Our terminology mostly follows [Roc70]. More details and proofs of all assertions in this section are contained in [Roc70, §1, 2, 6, 11, 18] or [Gru07, Chapters 3-5]. A thorough discussion of convex polytopes that goes way beyond what is needed for the purposes of this thesis can be found in [Gru07, Chapters 14-20].

Throughout this section, we denote by V a finite-dimensional real vector space of dimension $d \in \mathbb{N}$ equipped with its standard Euclidean topology. By an *affine hyperplane* in V , we always mean an affine subspace of codimension 1. If $E \subset V$ is an affine hyperplane and $P \subset V$ any subset, we say that P *lies on one side of E* if P is entirely contained in one of the two closed half-spaces defined by E . For two points $x, y \in V$, we write $[x, y] := \{(1-t)x + ty \mid t \in [0, 1]\}$ for the closed line segment in V connecting x to y and (x, y) , $[x, y)$ and $(x, y]$ for the corresponding open and half-open line segments. A subset $P \subset V$ is said to be *convex* if $[x, y] \subset P$ for all $x, y \in P$ and in this case, the *affine span* of P is defined as the smallest affine subspace of V containing P and denoted by $\text{aff}(P)$. It is naturally endowed with the trace topology of V . The topological interior of P , viewed as a subset of the topological space $\text{aff}(P)$, is called the *relative interior* of P and written as $\text{relint}(P)$. A point $x \in P$ lies in $\text{relint}(P)$ if and only if it satisfies any of the following two equivalent conditions:

- a) There exists an open neighbourhood $U \subset \text{aff}(P)$ of x (w.r.t. the trace topology on $\text{aff}(P)$) that is contained in P .
- b) For every $y \in P \setminus \{x\}$, there exists some $z \in P \setminus \{x\}$ such that $x \in (y, z)$. In other words, every closed line segment starting at a point in $P \setminus \{x\}$ and ending at x can be extended beyond x to another point in $P \setminus \{x\}$.

Since $\text{aff}(P)$ is closed, the topological closure \overline{P} of P does not depend on whether P is viewed as a subset of $\text{aff}(P)$ or V . The sets $\text{relint}(P)$, P and \overline{P} are all convex and they have the same affine span and topological closure, namely $\text{aff}(P)$ and \overline{P} . The *relative boundary* of P is defined as $\partial_{\text{rel}}(P) := \overline{P} \setminus \text{relint}(P)$. If $\text{aff}(P) = V$, then $\text{relint}(P)$ coincides with the topological interior P° of P in V and $\partial_{\text{rel}}(P)$ with the usual boundary $\partial P = \overline{P} \setminus P^\circ$.

Definition 4.2.1. Let $P \subset V$ be a convex set. A non-empty convex subset $F \subset P$ is called a *face* of P if it has the property that whenever $p_1, p_2 \in P$ satisfy $(p_1, p_2) \cap F \neq \emptyset$, then $[p_1, p_2] \subset F$. The *dimension* $\dim(F)$ of a face F is defined as the dimension of its affine span. The set of all faces of P , partially-ordered by inclusion, is called the *face poset* of P and denoted by $\mathcal{F}(P)$.

Given a convex set $P \subset V$, the entire set P is a face of itself, but we have excluded the empty set from being a face. This is merely a convention that will simplify the terminology in Section 4.3. Hence, $\mathcal{F}(P)$ has a maximal element but usually not a minimal one. We call P the *trivial face* and every other face is said to be *proper*. It follows easily from the definition that the intersection of any two faces of P is either empty or another face of P and for every $F \in \mathcal{F}(P)$, the faces of F are precisely the faces of P contained in F . Moreover, if $A \subset P$ is any convex subset and F a face of P which contains a point from the relative interior of A , then $A \subset F$. In particular, every proper face $F \in \mathcal{F}(P) \setminus \{P\}$ is contained in the relative boundary of P and the relative interiors of different faces are disjoint. For two points $p_1, p_2 \in P$, it therefore follows that $[p_1, p_2] \subset \partial_{\text{rel}}(P)$ if and only if p_1 and p_2 lie on a common proper face. Note that non-trivial faces do not exist if $P \cap \partial_{\text{rel}}(P) = \emptyset$ or, equivalently, if P is relatively open in $\text{aff}(P)$. We will mostly be interested in a slightly less general notion of faces.

Definition 4.2.2. Let $P \subset V$ be a convex set. A non-empty convex subset $F \subset P$ is called an *exposed face* of P if $F = P$ or there exists an affine hyperplane $E \subset V$ such that P lies on one side of E and $F = P \cap E$. In the latter case, we say that E is a *supporting hyperplane* of P and that the corresponding closed half-space containing P is a *supporting half-space*. Exposed faces of dimension 0 and 1 are called *vertices* and *edges*, respectively. Those of dimension $\dim(P) - 1$ are referred to as *facets* and their supporting hyperplanes (resp. supporting half-spaces) as *defining hyperplanes* (resp. *defining half-spaces*).

The definition easily implies that an exposed face is also a face in the sense of Definition 4.2.1, but the converse is not true in general.

Example 4.2.3. Consider the convex set $P \subset \mathbb{R}^2$ depicted as the green region in Figure 4.1. It is enclosed by two parallel line segments of equal length and two half-circles of equal radius connecting their endpoints.

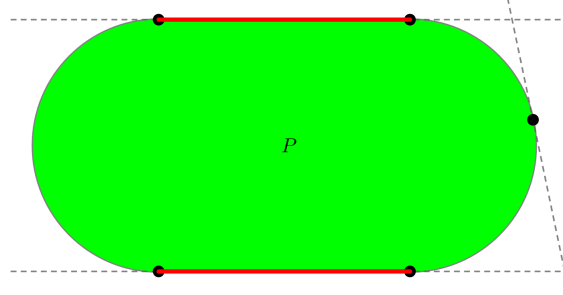


Figure 4.1: A convex set with non-exposed faces

Its proper faces are the two thick red line segments as well as every point on the two half-circles, including the four highlighted points where the half-circles and the straight line segments meet. Apart from those four points, all faces of P are exposed and some supporting hyperplanes are drawn by dashed lines.

When a convex set $P \subset V$ satisfies $\dim(P) = d$, its defining hyperplanes are uniquely determined and more generally, two defining hyperplanes give rise to the same face of P if and only if their intersections with the affine span of P coincide. By definition, we consider the trivial face P to be exposed and we call an affine hyperplane $E \subset V$ a *trivial supporting hyperplane* if $P = P \cap E$ or, equivalently, if $P \subset E$.¹ While it is possible that there exist non-exposed proper faces in the relative boundary of P , it follows from a separation argument that every point of $P \cap \partial_{\text{rel}}(P)$ lies on a non-trivial supporting hyperplane of P . Consequently, every proper face of P is contained in a proper exposed face. This phenomenon is easy to observe in Example 4.2.3. Moreover, if $P = \overline{P}$ is closed, it implies that for every point in $V \setminus P$, there exists a supporting half-space of P not containing it, which shows that P can be reconstructed as the intersection of all its supporting half-spaces (cf. [Roc70, Theorem 11.6] and [Gru07, Corollary 4.1]). In algebraic terms, this means that a point in V lies in P if and only if it satisfies a family of (weak) linear (in-)equalities. If $\dim(P) = d$, then every supporting hyperplane of P is non-trivial and none of these inequalities are equalities. By replacing V with the affine span of P , it is usually no restriction to assume that $\dim(P) = d$, but for our purposes, it will be more convenient to avoid this assumption.

Given any $F \in \mathcal{F}(P)$, we have already mentioned that the faces of F are exactly the faces of P contained in F and the ones that are exposed in P are also exposed in F . More generally, if we intersect P with any affine hyperplane $E \subset V$, not necessarily supporting, then every face of $P \cap E$ has a canonical “pre-image” in P .

Lemma 4.2.4. *Let $P \subset V$ be a convex set and $E \subset V$ an affine hyperplane with $P' := P \cap E \neq \emptyset$. For every face F' of P' , there exists a unique minimal face F of P such that $F' = F \cap E$ and it satisfies the following:*

$$\dim(F) = \begin{cases} \dim(F') & \text{if } F' \text{ is a face of } P \text{ contained in } E \\ \dim(F') + 1 & \text{otherwise} \end{cases} \quad (4.2.1)$$

¹Allowing that $F = P$ in Definition 4.2.2 is only necessary if $\dim(P) = d$. Otherwise, P is contained in an affine hyperplane of V , so P is exposed even when this condition is removed from the definition. If $\dim(P) = d$, there does not exist a trivial supporting hyperplane, but in this case, one could simply extend V to a vector space of dimension $d + 1$ in order to reduce this situation to the previous one. Hence, considering P to be an exposed face does not cause any contradictions and will be convenient to simplify the terminology.

Proof. First, if G is any face of P , then $G' := G \cap E$ is either empty or a face of P' and in the latter case, there are three possibilities:

- (i) If $G \subset E$, then $G = G'$. This is necessarily the case if $\dim(G) = 0$.
- (ii) If $G \not\subset E$ but there is a point $x \in \text{relint}(G) \cap E$, then $\dim(G') = \dim(G) - 1$. To see this, set $k := \dim(G) \in \mathbb{N}$ and choose an arbitrary norm on V , which are all equivalent since V is finite-dimensional. As x lies in the relative interior of G , there exists a k -dimensional ball $B_R(x)$ of some sufficiently small radius $R \in \mathbb{R}_{>0}$ centered at x that is contained in G . Now $x \in E$ but $B_R(x) \not\subset E$, so $B_R(x) \cap E$ is a $(k - 1)$ -dimensional ball centered at x that is contained in G' . Hence, the dimension of G' is at least $k - 1 \geq 0$, but it cannot be larger as G is not a subset of E .
- (iii) If neither (i) nor (ii) is satisfied, then $G' \subset \partial_{\text{rel}}(G)$. Since the relative interiors of the proper faces of G form a partition of $\partial_{\text{rel}}(G)$, there exists a proper face of G that contains a point from the relative interior of G' , which implies that G' is entirely contained in that face. Let H be the intersection of all faces of G containing G' , then H is also a proper face of G (and hence of P) such that $G' = H \cap E$ and $G' \cap \text{relint}(H) \neq \emptyset$. If $H \subset E$, then $G' = H$ is a face of P contained in E and $\dim(G') = \dim(H)$. Otherwise, the hyperplane E contains an interior point of H , so that $\dim(G') = \dim(H) - 1$ by (ii).

Conversely, let F' be any face of P' , choose a point $x \in \text{relint}(F')$ and let F be the unique face of P such that $x \in \text{relint}(F)$, then $F' = F \cap E$ since both sets are faces of P' containing x in their relative interiors. If $F \subset E$, then $F' = F$ is a face of P contained in E . Otherwise, we have $F \not\subset E$ and $\text{relint}(F) \cap E \neq \emptyset$, so that $\dim(F') = \dim(F) - 1$ by (ii). If $\tilde{F} \neq F$ is another face of P with $F' = \tilde{F} \cap E$, the point $x \in F' \subset F$ has to lie in $\partial_{\text{rel}}(\tilde{F})$, which implies that F is a proper face of \tilde{F} and $\dim(F) < \dim(\tilde{F})$. Hence, F is uniquely determined by F' and the minimality condition. \square

For an arbitrary set $Y \subset V$, the *convex hull* of Y is defined as the smallest convex subset of V containing Y and denoted by $\text{conv}(Y)$. It consists of all finite *convex combinations* of elements of Y , i.e. all vectors of the form $\sum_{y \in Y} \lambda_y y \in V$, where $\lambda_y \in [0, 1]$ for all $y \in Y$, $\lambda_y > 0$ only for finitely many $y \in Y$ and $\sum_{y \in Y} \lambda_y = 1$. Trivially, every convex set is equal to its own convex hull. An important special case occurs when it can be reduced to a finite set in this way.

Definition 4.2.5. A convex set $P \subset V$ is called a *convex polytope* if there exists a finite set $Y \subset V$ such that $P = \text{conv}(Y)$.

Convex polytopes enjoy many special properties that are not true for convex hulls of infinite sets. For example, if Y is finite, then $P := \text{conv}(Y)$ is compact and therefore closed. Moreover, it has only finitely many faces and all of them are exposed. In fact, every face $F \in \mathcal{F}(P)$ is of the form $F = \text{conv}(Y')$ for some subset $Y' \subset Y$ and thus again a convex polytope. Furthermore, for every $k \in \{0, \dots, \dim(P)\}$, there exists a k -dimensional face of P and all maximal chains in the face poset $\mathcal{F}(P)$ have the same length. In particular, every non-trivial face of P is contained in a facet, i.e. a maximal proper (exposed) face.

If Y is infinite, it is not necessarily true that its convex hull is closed, not even if Y appears to have “nice” properties, e.g. being uniformly discrete. It might even be relatively open in which case it cannot have any non-trivial faces. Hence, there is no hope of transferring the theory of convex polytopes to this situation and it is not guaranteed that the face poset of $\text{conv}(Y)$ has any interesting structure. Nevertheless, the following result at least states that if faces exist at all, they can be constructed in a simple way.

Lemma 4.2.6. Let $Y \subset V$ be any subset and $P := \text{conv}(Y)$.

- (i) If F is a face of P , then $F = \text{conv}(Y \cap F)$.
- (ii) If $E \subset V$ is a supporting hyperplane for P , then $P \cap E = \text{conv}(Y \cap E)$.

Proof. The first assertion is proved in [Roc70, Theorem 18.3] and the second follows immediately by applying (i) to the exposed face $F := P \cap E$. Alternatively, a simpler and self-contained proof of (ii) goes as follows: It is clear that $\text{conv}(Y \cap E) \subset P \cap E$. Conversely, assume that $x \in P \cap E$ and write this point as a convex combination $x = \sum_{y \in Y} \lambda_y y$, where $\lambda_y \in [0, 1]$ for all $y \in Y$, $\lambda_y > 0$ only for finitely many $y \in Y$ and $\sum_{y \in Y} \lambda_y = 1$. Choose a linear functional $f \in V^*$ and a

constant $c \in \mathbb{R}$ such that $E = \{v \in V : \langle f, v \rangle = c\}$. Since P lies on one side of E , we may assume that $\langle f, p \rangle \leq c$ for all $p \in P$. Now $x \in P \cap E$ implies that

$$c = \langle f, x \rangle = \sum_{y \in Y} \lambda_y \langle f, y \rangle \leq \sum_{y \in Y} \lambda_y c = c.$$

This inequality would be strict if we had $\lambda_y > 0$ for some $y \in Y \setminus E$, so it follows that $\lambda_y = 0$ for all $y \in Y \setminus E$ and $x \in \text{conv}(Y \cap E)$. \square

4.3 Orbits of internal elements

With the basic terminology from convex geometry at hand, we now return to the situation of Section 4.1 and begin to study the set $\Xi_h = \text{conv}(W \cdot h)$ for a fixed element $h \in \mathfrak{a}$. First, we consider the case where $h \in X_+ \cup X_-$ is internal. As $\Xi_{-h} = -\Xi_h$ and $\Xi_{w(h)} = \Xi_h$ for all $w \in W$, we may w.l.o.g. assume that $h \in \overline{C} \subset X_+$. This also implies that $\Xi_h \subset X_+$ because the Tits cone is W -invariant and convex. We define

$$I_h := \{i \in I : \langle \alpha_i, h \rangle = 0\}, \quad (4.3.1)$$

so that D_{I_h} is the unique cell of \overline{C} containing h and its stabilizer W_h coincides with the standard subgroup W_{I_h} . Throughout this section, we suppose that the Coxeter diagram $\Delta := \Delta(W, S)$ is coloured by I_h (i.e. the vertices in I_h are black and those in $I \setminus I_h$ are white) and we adapt the terminology of Section 1.2 (e.g. visible and invisible vertices) to the case $I^\bullet = I_h$. For every subset $J \subset I$, this induces a colouring on the subgraph $\Delta_J := \Delta(W_J, S_J)$ of Δ which is simply the colouring on Δ_J defined by $J_h := J \cap I_h$. We have $I = I_{\text{vis}} \sqcup I_{\text{inv}}$ and $W \cong W_{\text{vis}} \times W_{\text{inv}}$, where the action of $W_{\text{inv}} \subset W_h$ on \mathfrak{a} fixes Ξ_h point-wise. This group may well be infinite, but it does not have any influence on the structure of $\Xi_h = \text{conv}(W_{\text{vis}} \cdot h)$ and Definition 4.1.2 can be simplified: An element $h \in \overline{C}$ is almost virtually singular if and only if $W_h \cap W_{\text{vis}}$ is finite and essentially singular if and only if $W_h \cap W_{\text{vis}}$ is infinite.

One of our main objectives is to give a combinatorial description of the face poset of Ξ_h in terms of the Coxeter system (W, S) in the case where $h \in \mathfrak{a}$ is almost virtually singular. The result will be particularly appealing if h is regular, where we shall see that $\mathcal{F}(\Xi_h)$ is isomorphic to the reversed Coxeter complex $\Sigma^*(W, S)$. Our argument consists of two steps: First, we explicitly construct exposed faces of Ξ_h using the approach of Lemma 4.2.6. This can be carried out for all internal elements simultaneously. Afterwards, we show that every face of Ξ_h arises from this construction and for this step, it will be necessary to first treat the regular case separately. In order to extend it to virtually singular elements, our assumption that the Kac-Moody root datum \mathcal{D} under consideration is, in particular, admissible will be crucial. Once this is achieved, the almost virtually singular case will be a simple but technical generalization. In the second step, we further establish that Ξ_h is closed if h is almost virtually singular and find defining inequalities that decide whether or not a point in \mathfrak{a} is contained in Ξ_h . This characterization of Ξ_h will be a fundamental ingredient for the proofs of the convexity theorems in Chapter 5. For essentially singular elements, however, a similar reduction argument to the regular case will not be possible and, indeed, we are going to see that Ξ_h is not closed in this situation. Two simple but important observations that will be used frequently in this section are the following:

- a) Since the W -action on \mathfrak{a} is linear, the set Ξ_h is W -invariant, i.e. $w(\Xi_h) = \Xi_h$ for all $w \in W$, and it follows immediately from Definition 4.2.1 that if F is a face of Ξ_h and $w \in W$ is arbitrary, then $w(F)$ is also a face of Ξ_h . Therefore, W acts on the face poset $\mathcal{F}(\Xi_h)$ and this preserves the partial order (i.e. inclusion) among the faces as well as their dimensions. Moreover, if F is exposed and E a supporting hyperplane of Ξ_h with $F = \Xi_h \cap E$, then $w(\Xi_h) = \Xi_h$ lies on one side of $w(E)$ and $w(F) = w(\Xi_h \cap E) = \Xi_h \cap w(E)$, so $w(F)$ is an exposed face of Ξ_h and $w(E)$ a supporting hyperplane. Hence, the subposet of $\mathcal{F}(\Xi_h)$ consisting of all exposed faces is W -invariant. Note that the properties in this paragraph do not require h to be internal.

b) The assumption that $h \in \overline{C}$ implies the following for every real root $\alpha \in \Phi$:

$$\alpha \in \Phi_+ \implies \langle \alpha, h \rangle \geq 0 \quad \text{and} \quad \alpha \in \Phi_- \implies \langle \alpha, h \rangle \leq 0 \quad (4.3.2)$$

Equality holds if and only if $I(\alpha) \subset I_h$.

Remark 4.3.1. In order to study the face poset of Ξ_h , we are going to establish some properties of the Coxeter system (W, S) that are purely algebraic and not related to the Kac-Moody setting we work in. The notion of a “visible subset” of I plays an important role in the following discussion. As mentioned in Remark 1.2.2, this concept had previously been introduced in [Max89] and some results of the next section were proved there in the context of general Coxeter groups. Maxwell’s ideas are also connected to the theory of “shadows” developed in [Tit74, Appendix 1].

Loosely speaking, in the full generality of Tits’ approach, one is interested in describing certain partially-ordered sets constructed from coloured Coxeter diagrams. Here, the poset is viewed from a completely combinatorial perspective and does not contain any information other than its partial order. It is not important if and how it arises from a more “concrete” geometric object, e.g. as the face poset of a convex polytope. Any “geometric realization” that may be associated to a shadow is merely a particular kind of incidence geometry and not endowed with any “metric” structure. This is intended to generalize the way in which the simplicial complex underlying a spherical building can often be interpreted as the geometric realization of the flag complex of an appropriate incidence geometry, such as a projective or polar geometry.

The scope of Maxwell’s work, on the other hand, is rather different. Although his arguments often rely on purely combinatorial properties of Coxeter groups that are similar in style to Tits’ theory, his ultimate interest lies in interpreting his “shadow lattices” as the face posets or tessellations of “rigid” geometric objects, namely certain convex cones or polytopes in finite-dimensional real vector spaces that are constructed using a particular action of a Coxeter group. Thus, concrete metric considerations play an important role but even in his approach, these objects are not uniquely determined by the underlying poset. Hence, the poset should still be viewed as the more “fundamental” concept in Maxwell’s setting and all associated geometric objects as being “derived” from the poset. To some extent, Scharlau’s work on geometric realizations of shadow geometries (cf. [Sch90]) provides a link between the ideas of Maxwell and Tits, but it also tends to favour the spirit of incidence geometry rather than the metric viewpoint.

For our purposes, it will be crucial that we do not only consider $\mathcal{F}(\Xi_h)$ as an abstract poset that we want to understand from a combinatorial perspective, but instead that we view Ξ_h as a metric object that is embedded in the real vector space \mathfrak{a} in a specified way. For the applications we have in mind, it is not sufficient to know how the faces of Ξ_h are arranged as a poset, but we need to know precisely which subsets of Ξ_h are faces and we require explicit equations for the supporting hyperplanes of exposed faces. This metric aspect is completely ignored in Tits’ approach and plays a subordinate role in the works of Maxwell and Scharlau. For the reader’s convenience, the author has therefore chosen to provide self-contained proofs for all assertions that follow the terminology of this thesis, but the connection to these references will be made as explicit as possible.

4.3.1 Construction of faces

We begin with the first step of our plan and try to construct faces of Ξ_h explicitly. It follows from Lemma 4.2.6 that every face can be written as $\text{conv}(W' \cdot h)$ for a suitable subset $W' \subset W$. In order to find subsets of W that do indeed define a face in this way, it is natural to first consider standard subgroups. For every $J \subset I$, we therefore write

$$F_J := \text{conv}(W_J \cdot h) \subset \Xi_h \quad \text{and} \quad \mathfrak{a}'_J := \bigoplus_{j \in J} \mathbb{R}\alpha_j^\vee \subset \mathfrak{a}'. \quad (4.3.3)$$

Hence, we have $F_J \subset h + \mathfrak{a}'_J$ by (2.6.1) and $\dim(\mathfrak{a}'_J) = |J|$. The Coxeter diagram Δ_J of (W_J, S_J) is the union of its connected components $\Delta_{J_1}, \dots, \Delta_{J_s}$. As in Chapter 1, we slightly abuse the terminology by calling the corresponding index sets J_1, \dots, J_s the connected components of J .

Proposition 4.3.2. *Let $J \subset I$ be arbitrary and J_1, \dots, J_s the connected components of J .*

- (i) *For every $r \in \{1, \dots, s\}$, the convex set F_{J_r} is either a single point or its dimension is $|J_r|$. The former is the case if and only if $J_r \subset I_h$.*
- (ii) *Let J_{r_1}, \dots, J_{r_t} be the connected components of J that are not contained in I_h , then*

$$\dim(F_J) = \sum_{p=1}^t |J_{r_p}| = |J_{\text{vis}}|. \quad (4.3.4)$$

- (iii) *If (W_J, S_J) is irreducible, then h is either a global fixed-point of W_J or $\dim(F_J) = |J|$.*

Proof. (i) Upon replacing J by J_r , we may w.l.o.g. assume that (W_J, S_J) is irreducible and Δ_J is connected. If $J \subset I_h$, i.e. if all vertices of Δ_J are black, then $W_J \subset W_h$ and $F_J = \text{conv}(W_J \cdot h) = \text{conv}(\{h\}) = \{h\}$. Hence, it remains to show that if $J \not\subset I_h$, i.e. if at least one vertex of Δ_J is white, then $\dim(F_J) = |J|$. As F_J is contained in the affine subspace $h + \mathfrak{a}'_J \subset \mathfrak{a}$, its dimension cannot be larger than $|J|$, so it suffices to find $|J| + 1$ points in F_J that are affinely independent.

Since Δ_J is connected and contains a white vertex, the set J is visible, so by Lemma 1.2.5, there exists an exposing forest Δ_J° for Δ_J and we denote by $\{w_j \mid j \in J\}$ the corresponding exposing set for W_J . We claim that $\{h, w_j(h) \mid j \in J\} \subset F_J$ is an affinely independent set. To prove this, we put $\delta := \max_{j \in J} \delta_J(j) < \infty$. If $\delta = 0$, then $J_h = \emptyset$, so that $w_j = s_j$ and $s_j(h) = h - \langle \alpha_j, h \rangle \alpha_j^\vee$ with $\langle \alpha_j, h \rangle > 0$ for all $j \in J$. As the simple coroots are linearly independent, the claim follows in this case, so we may assume that $\delta > 0$. Suppose that $\{\lambda_j \mid j \in J\} \subset \mathbb{R}$ is a set of real numbers such that

$$\sum_{j \in J} \lambda_j (h - w_j(h)) = 0. \quad (4.3.5)$$

We need to show that $\lambda_j = 0$ for all $j \in J$. To this end, choose any $j' \in J$ with $\delta_J(j') = \delta$ and consider the unique path $\gamma = (j_0, \dots, j_\delta = j')$ in Δ_J° which connects j' with a white vertex $j_0 \in J \setminus I_h$. By definition, s_{j_0} does not stabilize h , $m_{j_{k-1}j_k} > 2$ for all $k \in \{1, \dots, \delta\}$ and $w_{j'} = s_{j'} s_{j_{\delta-1}} \dots s_{j_0}$. As $m_{j_{k-1}j_k} > 2$ is equivalent to $a_{j_{k-1}j_k} < 0$ by (2.5.1), it follows by induction on δ that

$$h - w_{j'}(h) = \sum_{k=0}^{\delta} x_k \alpha_{j_k}^\vee \quad \text{for some } x_0, \dots, x_\delta \in \mathbb{R}_{>0}.$$

Moreover, since $\delta_J(j) \leq \delta = \delta_J(j')$ for all $j \in J$, the only element of the exposing set for which $s_{j'}$ is a simple factor is $w_{j'}$ itself, so $h - w_{j'}(h)$ is the only summand in (4.3.5) in which $\alpha_{j'}^\vee$ occurs with a positive coefficient. As the simple coroots are linearly independent, the equation can only be satisfied if $\lambda_{j'} = 0$. The same argument can be applied to every other vertex whose distance to $J \setminus I_h$ is δ .

Let now $j'' \in J$ be such that $\delta_J(j'') = \delta - 1$, then the only elements of the exposing set containing $s_{j''}$ as a simple factor are $w_{j''}$ and those $w_{j'}$ for which $j' \in J$ satisfies $\delta_J(j') = \delta$ and the unique path in Δ_J° connecting j' with a vertex in $J \setminus I_h$ runs through j'' . As $\lambda_{j'} = 0$ for all indices of the latter type, we must also have $\lambda_{j''} = 0$. Iterating this process shows that $\lambda_j = 0$ for all $j \in J$, which proves the claim.

- (ii) As $\bigcup_{p=1}^t J_{r_p} = J_{\text{vis}}$ and $W_J \cong W_{J_{\text{vis}}} \times W_{J_{\text{inv}}}$ with $W_{J_{\text{inv}}} \subset W_h$ and $W_J \cdot h = W_{J_{\text{vis}}} \cdot h$, we may w.l.o.g. assume that $J_{\text{inv}} = \emptyset$ or, equivalently, that $t = s$. We argue by induction on s . If $s = 1$, the statement follows directly from (i). If $s > 1$, we set $J' := \bigcup_{r=1}^{s-1} J_r$ and $J'' := J_s$, so that $J = J_{\text{vis}} = J' \sqcup J''$ and $W_J \cong W_{J'} \times W_{J''}$. Hence, we have $\dim(F_J) = \sum_{r=1}^{s-1} |J_r|$ by the induction hypothesis and $\dim(F_{J''}) = |J''| = |J_s|$ by (i). More precisely, the affine span of $F_{J'}$ (resp. $F_{J''}$) is $h + \mathfrak{a}'_{J'}$ (resp. $h + \mathfrak{a}'_{J''}$), so $\text{aff}(F_J) = h + (\mathfrak{a}'_{J'} \oplus \mathfrak{a}'_{J''}) = h + \mathfrak{a}'_J$, which implies that $\dim(F_J) = \dim(\mathfrak{a}'_J) = |J| = |J'| + |J''| = \sum_{r=1}^s |J_r|$.
- (iii) This is simply a restatement of (i) and (ii). \square

For every $J \subset I$, it follows from Proposition 4.3.2 that the affine span of $F_J = F_{J_{\text{vis}}}$ is $h + \mathfrak{a}'_{J_{\text{vis}}}$. In particular, if $J = I$, we conclude that $\Xi_h = F_I = F_{I_{\text{vis}}}$ has dimension $|I_{\text{vis}}|$ and its affine span is the W -invariant affine subspace

$$E_h := h + \mathfrak{a}'_{\text{vis}} \subset \mathfrak{a}, \quad \text{where} \quad \mathfrak{a}'_{\text{vis}} := \mathfrak{a}'_{I_{\text{vis}}}. \quad (4.3.6)$$

Since every face of Ξ_h can be written as $\text{conv}(W' \cdot h)$ for some $W' \subset W$ and $\mathcal{F}(\Xi_h)$ is W -invariant, every face is W -equivalent to one that contains the point $h \in \overline{C}$. Recall from Proposition 2.6.5 (v) that for every $w \in W$, the vector $h - w(h) \in \mathfrak{a}'_{\text{vis}} \subset \mathfrak{a}'$ is a linear combination of simple coroots with non-negative coefficients. To construct faces of Ξ_h containing h , it is important to understand precisely which coefficients are positive. Note that $w(h) = w'(h)$ for some $w' \in W$ if and only if $wW_h = w'W_h$, so the W -translates of h are in bijective correspondence with the left cosets of W_h in W . Hence, in order to investigate which coroots appear in $h - w(h)$ with a positive coefficient, we may assume that w is right I_h -reduced, i.e. $l(w) < l(ws_i)$ for all $i \in I_h$. The following simple properties will be useful in the subsequent discussion:

Lemma 4.3.3. *If $h \in \overline{C}$, $w \in W$ is right I_h -reduced and $i \in I$ is arbitrary, then exactly one of the following three conditions holds:*

- (i) *If $l(s_i w) < l(w)$, then $s_i w$ is right I_h -reduced and $\langle \alpha_i, w(h) \rangle < 0$.*
- (ii) *If $l(s_i w) > l(w)$ and $s_i w$ is right I_h -reduced, then $\langle \alpha_i, w(h) \rangle > 0$.*
- (iii) *If $l(s_i w) > l(w)$ and $s_i w$ is not right I_h -reduced, then $\langle \alpha_i, w(h) \rangle = 0$.*

Proof. That exactly one of these situations occurs is the statement of the folding condition of a Coxeter group which has been discussed in Section 1.1. Thus, in each of the three cases, it suffices to prove the assertion about the value of $\langle \alpha_i, w(h) \rangle \in \mathbb{R}$.

If $l(s_i w) < l(w)$, then $w^{-1}(\alpha_i) \in \Phi_-$ by (2.5.4) and $\langle \alpha_i, w(h) \rangle = \langle w^{-1}(\alpha_i), h \rangle \leq 0$ by (4.3.2). Suppose for a contradiction that equality holds, then $I(w^{-1}(\alpha_i)) \subset I_h$, which is equivalent to $w^{-1}(\alpha_i) \in \Phi_{I_h} = \Phi(\mathbb{A}_{I_h})$. Hence, there exists an element $u \in W(\mathbb{A}_{I_h}) = W_{I_h} = W_h$ and an index $j \in I_h$ such that $uw^{-1}(\alpha_i) = \alpha_j$, which implies that $(uw^{-1})s_i(uw^{-1})^{-1} = s_j$ or, equivalently, $w^{-1}s_i w = u^{-1}s_j u \in W_h$. It follows that w and $s_i w$ lie in the same left coset of W_h , but this is not possible as $l(s_i w) < l(w)$ and w is assumed to be of minimal length in wW_h .

Similarly, if $l(s_i w) > l(w)$, then $w^{-1}(\alpha_i) \in \Phi_+$ and $\langle \alpha_i, w(h) \rangle = \langle w^{-1}(\alpha_i), h \rangle \geq 0$. If $s_i w$ is also right I_h -reduced, the same argument as above shows that $\langle \alpha_i, w(h) \rangle > 0$. Otherwise, the folding condition ensures that $s_i w = ws_j$ for some $j \in I_h$ and therefore

$$\langle \alpha_i, w(h) \rangle = \langle s_i(\alpha_i), s_i w(h) \rangle = -\langle \alpha_i, ws_j(h) \rangle = -\langle \alpha_i, w(h) \rangle.$$

This proves that $\langle \alpha_i, w(h) \rangle = 0$, as claimed. \square

Corollary 4.3.4. *If $h \in \overline{C}$, then Ξ_h is bounded if and only if $W \cdot h \subset \mathfrak{a}$ is finite.*

Proof. In this proof, we identify \mathfrak{a} with \mathbb{R}^d by mapping the basis $\mathcal{B} \subset \mathfrak{a}$ to the standard basis of \mathbb{R}^d in the natural order and we endow \mathfrak{a} with the Euclidean norm $\|\cdot\|$ of \mathbb{R}^d via this isomorphism. In particular, this implies that $\|\alpha_i^\vee\| = 1$ for all $i \in I$.

If $W \cdot h$ is finite, then Ξ_h is a convex polytope and hence bounded, so we may assume that $W \cdot h$ is infinite. Equivalently, the quotient space W/W_h is infinite, so W contains right I_h -reduced elements of arbitrarily large length. To conclude the proof, it suffices to show that if $w \in W$ is right I_h -reduced and $l := l(w)$, then

$$\|h - w(h)\| \geq l \min_{i \in I \setminus I_h} \langle \alpha_i, h \rangle.$$

Note that the minimum on the right-hand side is positive because $h \in \overline{C}$ and I_h has to be a proper subset of I if $W \cdot h$ is infinite.

We argue by induction on l . If $l = 0$, there is nothing to show. Suppose that $l > 0$ and let $i \in I$ be such that $l(s_i w) < l(w)$. The proof of Lemma 4.3.3 shows that $w' := s_i w$ is right I_h -reduced and $I(w'^{-1}(\alpha_i)) \not\subset I_h$, so that $\langle \alpha_i, w'(h) \rangle = \langle w'^{-1}(\alpha_i), h \rangle \geq \min_{i \in I \setminus I_h} \langle \alpha_i, h \rangle$. Now

$$h - w(h) = h - s_i w'(h) = h - w'(h) + \langle \alpha_i, w'(h) \rangle \alpha_i^\vee$$

and the induction hypothesis yields that

$$\|h - w'(h)\| \geq (l - 1) \min_{i \in I \setminus I_h} \langle \alpha_i, h \rangle.$$

As $h - w'(h) \in \mathfrak{a}'$ is a linear combination of the simple coroots with non-negative coefficients, adding the vector $\langle \alpha_i, w'(h) \rangle \alpha_i^\vee$ increases its norm by the required amount. \square

Lemma 4.3.5. *If $h \in \overline{C}$ and $w \in W$ is right I_h -reduced, there exist unique $x_1, \dots, x_n \in \mathbb{R}$ such that $h - w(h) = \sum_{i \in I} x_i \alpha_i^\vee$, where $x_i > 0$ for $i \in I(w)$ and $x_i = 0$ for $i \notin I(w)$.*

Except for a slightly different setup, this statement is the content of [Max89, Proposition 5.4], but its proof is not made explicit there.

Proof. Existence is clear and uniqueness holds since the simple coroots are linearly independent. In view of (2.6.7), it suffices to show that $x_i > 0$ for all $i \in I(w)$. Suppose that $x_i = 0$ and thus $\langle \omega_i, w(h) \rangle = \langle \omega_i, h \rangle$ for some $i \in I(w)$. By Lemma 4.1.8, the stabilizer of ω_i under the W -action on \mathfrak{a}^* is the standard subgroup $W^{(i)}$, so every element in the double coset $W^{(i)}wW_h$ also satisfies this equation and we may assume that w is $(I^{(i)}, I_h)$ -reduced. In particular, it is left $I^{(i)}$ -reduced and $s_i \in S(w)$, which implies that $w = s_i w'$ for some $w' \in W$ with $l(w') < l(w)$. By Lemma 4.3.3, w' and $s_i w' = w$ are both right I_h -reduced and $\langle \alpha_i, w'(h) \rangle > 0$, but this leads to the following contradiction, where the last inequality holds by (2.6.7):

$$\langle \omega_i, h \rangle = \langle \omega_i, w(h) \rangle = \langle s_i(\omega_i), w'(h) \rangle = \langle \omega_i, w'(h) \rangle - \langle \alpha_i, w'(h) \rangle < \langle \omega_i, w'(h) \rangle \leq \langle \omega_i, h \rangle \quad \square$$

Lemma 4.3.5 is the crucial ingredient to construct faces of Ξ_h because it will enable us to find supporting hyperplanes, so the resulting faces will even be exposed. Since the affine span of Ξ_h is $E_h = h + \mathfrak{a}'_{\text{vis}} \subset \mathfrak{a}$, it suffices to study affine hyperplanes in that subspace, i.e. intersections of E_h with affine hyperplanes in \mathfrak{a} , which may be empty or coincide with E_h . For every $i \in I$, we set

$$E^{(i)} := \{h' \in E_h : \langle \omega_i, h' \rangle = \langle \omega_i, h \rangle\}. \quad (4.3.7)$$

By Lemma 4.1.8, $E^{(i)}$ is invariant under the standard subgroup $W^{(i)}$. If $i \in I_{\text{inv}}$, then ω_i is constant on E_h and $E^{(i)} = E_h$ is a trivial supporting hyperplane as Ξ_h is entirely contained in it. If $i \in I_{\text{vis}}$, then ω_i is not constant on E_h and $E^{(i)}$ separates it into two half-spaces. In this case, it gives rise to a non-trivial face of Ξ_h .

Proposition 4.3.6. *If $h \in \overline{C}$, then $E^{(i)}$ is a supporting hyperplane of Ξ_h for every $i \in I$ and the corresponding exposed face is given by $\Xi_h \cap E^{(i)} = F^{(i)}$. In addition, $E^{(i)}$ is a defining hyperplane and $F^{(i)}$ a facet of Ξ_h if and only if $i \in I$ is a non-degenerate vertex of Δ .*

Proof. First, we show that Ξ_h lies on one side of $E^{(i)}$ and $F^{(i)} \subset \Xi_h \cap E^{(i)}$. Every $h' \in \Xi_h$ can be written as a convex combination $h' = \sum_{w \in W} \lambda_w w(h)$ with $\lambda_w \in [0, 1]$ for all $w \in W$, $\lambda_w > 0$ only for finitely many $w \in W$ and $\sum_{w \in W} \lambda_w = 1$. Using (2.6.7), we obtain that

$$\langle \omega_i, h' \rangle = \sum_{w \in W} \lambda_w \langle \omega_i, w(h) \rangle \leq \sum_{w \in W} \lambda_w \langle \omega_i, h \rangle = \langle \omega_i, h \rangle.$$

Therefore, Ξ_h lies on one side of $E^{(i)}$. Moreover, $E^{(i)}$ contains $w(h)$ for every $w \in W^{(i)}$ since it is $W^{(i)}$ -invariant and $h \in E^{(i)}$, which implies that $F^{(i)} = \text{conv}(W^{(i)} \cdot h) \subset \Xi_h \cap E^{(i)}$ by convexity. In particular, it follows that $\Xi_h \cap E^{(i)}$ is non-empty and thus an exposed face.

Conversely, since $\Xi_h \cap E^{(i)} = \text{conv}((W \cdot h) \cap E^{(i)})$ by Lemma 4.2.6, we only need to show that whenever $w \in W$ satisfies $w(h) \in E^{(i)}$, then $w(h) \in W^{(i)} \cdot h$. Suppose that $w(h) \notin W^{(i)} \cdot h$, then $\pi_{I_h}^r(w) \notin W^{(i)}$ because $w(h) = \pi_{I_h}^r(w)(h)$, so we may assume that $w \in W$ is right I_h -reduced and not contained in $W^{(i)}$ or, equivalently, that $i \in I(w)$. Hence, Lemma 4.3.5 states that α_i^\vee occurs in $h - w(h)$ with a positive coefficient, which implies that $\langle \omega_i, w(h) \rangle < \langle \omega_i, h \rangle$ and $w(h) \notin E^{(i)}$. Thus, we also conclude that $\Xi_h \cap E^{(i)} \subset F^{(i)}$.

To prove the final statement, we compute the dimension of $F^{(i)}$ using Proposition 4.3.2. Observe that the connected components of $I^{(i)}$ are the same as that of I , except for the unique component $I_k \subset I$ containing i which splits into a finite number of connected components after removing i . Hence, the proposition implies that

$$\dim(\Xi_h) - \dim(F^{(i)}) = \dim(F_{I_k}) - \dim(F_{I_k \setminus \{i\}}).$$

If i is invisible, then $I_k \subset I_h$ and both dimensions on the right-hand side are zero. In this case, we have $I_{\text{vis}} \subset I^{(i)}$, which shows that $\Xi_h = F_{I_{\text{vis}}} = F^{(i)}$ and $E^{(i)} = E_h$ is a trivial supporting hyperplane. Thus, we may suppose that i is visible, so that $\dim(F_{I_k}) = |I_k|$. If i is non-degenerate, then no connected component of $I_k \setminus \{i\}$ is contained in I_h and the dimension of $F_{I_k \setminus \{i\}}$ is the sum of the cardinalities of all these components which is equal to $|I_k| - 1$. In this situation, it follows that $\dim(\Xi_h) - \dim(F^{(i)}) = 1$, so $E^{(i)}$ is a defining hyperplane and $F^{(i)}$ a facet of Ξ_h . By the same argument, if i is degenerate, then at least one component of $I_k \setminus \{i\}$ is a subset of I_h and its cardinality is not counted in the dimension of $F_{I_k \setminus \{i\}}$, so that $\dim(F_{I_k \setminus \{i\}}) < |I_k| - 1$ and $\dim(F^{(i)}) < \dim(\Xi_h) - 1$, showing that $F^{(i)}$ is not a facet. \square

Using the faces $F^{(1)}, \dots, F^{(n)} \in \mathcal{F}(\Xi_h)$ and W -invariance of the face poset, we can immediately construct many other faces of Ξ_h by taking intersections and translating them by the W -action. The next result is therefore a direct consequence of Proposition 4.3.2 and 4.3.6 and the fact that intersections and W -translates of exposed faces remain exposed.

Corollary 4.3.7. *If $h \in \overline{C}$, then for every $w \in W$ and $J \subset I$, the set*

$$w(F_J) = \text{conv}(wW_J \cdot h) \tag{4.3.8}$$

is an exposed face of Ξ_h of dimension $|J_{\text{vis}}|$. In particular, every W -translate of h is a vertex. \square

Hence, every standard left coset of (W, S) defines a face of Ξ_h and if one such coset is contained in another, the corresponding faces have the same property. As a result, we obtain a W -equivariant order-preserving map:

$$\begin{aligned} \psi : \Sigma^*(W, S) &\rightarrow \mathcal{F}(\Xi_h) \\ wW_J &\mapsto w(F_J) = \text{conv}(wW_J \cdot h) \end{aligned} \tag{4.3.9}$$

Remark 4.3.8. Since $\psi(wW_J) = \psi(wW_{J_{\text{vis}}})$ holds for all $w \in W$ and $J \subset I$, it would suffice to restrict ψ to the subposet of $\Sigma^*(W, S)$ of those cosets whose type is a visible subset of I . We do not invoke this restriction on purpose because it would lead to inconveniences in describing the stabilizer of a face under the W -action in Corollary 4.3.14.

If $h \in \overline{C}$ has non-trivial stabilizer, then ψ is not injective. For example, given any $w \in W_h \setminus \{1\}$, the cosets $\{w\} = wW_\emptyset$ and $\{1\} = W_\emptyset$ are distinct, but their image under ψ is the singleton $\{h\}$. More generally, we now study the question under which conditions two standard cosets of (W, S) give rise to the same face of Ξ_h . For every $J \subset I$, we define

$$J^\perp := \{i \in I : a_{ij} = 0 \ \forall j \in J\} \quad \text{and} \quad J^* := (J^\perp)_h = J^\perp \cap I_h. \tag{4.3.10}$$

As $a_{ij} = 0$ is equivalent to $m_{ij} = 2$, the set J^\perp consists precisely of those $i \in I \setminus J$ that are not connected to any $j \in J$ by an edge of Δ . Observe that $(J^\perp)_h \neq (J_h)^\perp$ in general, so this notation will be useful to avoid confusion. Since Δ_{J^*} only contains black vertices, we have $(J^*)_{\text{vis}} = \emptyset$ and it is natural to write $J_{\text{vis}}^* := (J_{\text{vis}})^*$. Moreover, we denote by $C(W_J)$ the centralizer of W_J in W and set $C_h(W_J) := C(W_J) \cap W_h$. As $m_{ij} = 2$ holds if and only if $i \neq j$ and s_i commutes with s_j , W_{J^\perp} is the largest standard subgroup of W that intersects W_J trivially and commutes with it. In particular, it is contained in $C(W_J)$, but it need not coincide with the full centralizer.²

²That W_{J^\perp} is usually a proper subgroup of $C(W_J)$ can be seen in finite Coxeter groups with non-trivial center (e.g. the even dihedral groups) or from the possibility that Δ_J may have isolated nodes. Alternatively, it follows from a much more precise description of the centralizer of a simple reflection that we briefly discuss in Remark 4.3.12 (ii).

- Remark 4.3.9.* (i) There is a slight subtlety in (4.3.10) that is worth mentioning: Since $a_{ii} = 2$ for all $i \in I$, the set J^\perp is disjoint from J . However, if $j \in J$ is an isolated node of Δ_J , then $s_j \in W_J$ centralizes W_J but our definition forces $s_j \notin W_{J^\perp}$. Thus, in order to relate W_{J^\perp} to $C(W_J)$, it might appear more natural to define J^\perp as the set of all $i \in I$ that satisfy $a_{ij} \in \{0, 2\}$ for all $j \in J$ or, equivalently, $a_{ij} = 0$ for all $j \in J \setminus \{i\}$, but this alternative definition is only different from the original one if Δ_J contains isolated nodes. Likewise, it provides another possible definition for J^* , but this only affects the resulting set if Δ_J has isolated black nodes. In particular, if J is visible, then J^* is the same for both choices of J^\perp . The interested reader may verify that Lemma 4.3.10 does not depend on which definition of J^\perp is chosen and all subsequent results only involve J_{vis}^* or explicitly require J to be visible, so this issue will not be relevant any further.
- (ii) At this point, our discussion overlaps significantly with [Max89]. In fact, for every $J \subset I$, the set J_{vis} is the largest “ I_h -minimal” subset of J in Maxwell’s terminology, J_{vis}^* is the “ I_h -complement” of J and $J_{\text{vis}}^* \sqcup J_{\text{vis}}$ is its “ I_h -closure”. In addition, except for a different setup and notation, the argument in the proof of Lemma 4.3.10 is precisely the statement of [Max89, Proposition 1.1] which is not proved there. The assertion of Proposition 4.3.13 is identical to [Max89, Proposition 2.2] and the proof is similar to [Max89, Lemma 1.4]. Several other of Maxwell’s results also have analogues in our setting, but they will not be needed for the purposes of this thesis.

In order to find necessary and sufficient conditions under which two standard cosets of (W, S) define the same face of Ξ_h , we require a few purely combinatorial results that make sense for arbitrary Coxeter systems, not just those arising from Kac-Moody theory, but we prove them in our setting in order not to distract from the main line of reasoning.

Lemma 4.3.10. *If $w \in W$ and $J \subset I$ satisfy $w \in C(W_J)$ and $I(w) \cap J = \emptyset$, then $w \in W_{J^\perp}$.*

Proof. Since $W_{J^\perp} = \bigcap_{j \in J} W_{\{j\}^\perp}$ and an element of W centralizes W_J if and only if it commutes with every simple reflection in S_J , it suffices to show the following: Whenever $w \in W$ and $j \in J$ satisfy $ws_j = s_j w$ and $j \notin I(w)$, then $I(w) \subset \{j\}^\perp$.

Suppose to the contrary that there exist $w \in W$ and $j \in J$ for which the conclusion does not hold and choose a reduced decomposition $w = s_{i_1} \dots s_{i_t}$. Since we assume that $I(w) \not\subset \{j\}^\perp$, we have $t > 0$ and there exists a maximal index $r \in \{1, \dots, t\}$ such that $i_r \notin \{j\}^\perp$. We set $i := i_r$ and write $w = us_i v$, where $u := s_{i_1} \dots s_{i_{r-1}}$ and $v := s_{i_{r+1}} \dots s_{i_t}$. On the one hand, we now have $I(v) \subset \{j\}^\perp$ and hence $v(\alpha_j) = \alpha_j$. On the other hand, we also have $ws_j w^{-1} = s_j$ and thus $w(\alpha_j) \in \{\alpha_j, -\alpha_j\}$, where the sign is positive by (2.5.4) as $j \notin I(w)$ implies that $l(w) < l(ws_j)$. It follows that

$$\alpha_j = w(\alpha_j) = us_i(\alpha_j) = u(\alpha_j) - a_{ij}u(\alpha_i)$$

with $a_{ij} < 0$ since $i \notin \{j\}^\perp$ and $i \neq j$. But $l(u) < l(us_i)$ as u and us_i are initial subwords of a reduced decomposition of w and $l(u) < l(us_j)$ because $I(u) \subset I(w)$ does not contain $j \notin I(w)$, so (2.5.4) yields that $u(\alpha_i), u(\alpha_j) \in \Phi_+$. Thus, the equation above can only hold if $a_{ij} = 0$ and $u(\alpha_j) = \alpha_j$, a contradiction. \square

Proposition 4.3.11. *If $J \subset I$ is visible, then $W_{J^*} = C_h(W_J)$.*

Proof. It is clear that $W_{J^*} \subset C_h(W_J)$, so it remains to prove the converse inclusion. If $J = \emptyset$, then $W_J = \{1\}$, $J^* = I_h$ and $C_h(\{1\}) = W_h$, so the claim is trivially true. Suppose that $J \neq \emptyset$ and $w \in W_h$ commutes with W_J . By Lemma 1.2.5, there exists an exposing forest Δ_J° for Δ_J . For every $j \in J$, we are going to show by induction on $\delta := \delta_J(j) < \infty$ that $I(w) \subset \{j\}^\perp$. Once this is established, it will follow that $w \in W_h \cap \bigcap_{j \in J} W_{\{j\}^\perp} = W_h \cap W_{J^\perp} = W_{J^*}$.

If $\delta = 0$, then $j \in J \setminus I_h$ is not contained in $I(w) \subset I_h$ but w centralizes $W_{\{j\}}$, so Lemma 4.3.10 shows that $I(w) \subset \{j\}^\perp$. Now assume that $\delta > 0$ and let $\gamma = (j_0, \dots, j_\delta = j)$ be the unique path in Δ_J° connecting j with a white vertex $j_0 \in J \setminus I_h$, then $I(w) \subset \{j_{\delta-1}\}^\perp$ by the induction hypothesis. In particular, this implies that $j \notin I(w)$ because $j = j_\delta$ and $j_{\delta-1}$ are adjacent in Δ_J . As w centralizes $W_{\{j\}}$, it follows again from Lemma 4.3.10 that $I(w) \subset \{j\}^\perp$, as claimed. \square

Proposition 4.3.11 is a straightforward inductive consequence of Lemma 4.3.10, but it should be observed that when $J \subset I$ is arbitrary, it only applies to the visible part J_{vis} and does not extend to the full set if the invisible part J_{inv} is non-empty. In general, it holds that

$$W_{J^*} = W_{(J_{\text{vis}})^*} \cap W_{(J_{\text{inv}})^*} \quad \text{and} \quad C_h(W_J) = C_h(W_{J_{\text{vis}}}) \cap C_h(W_{J_{\text{inv}}}).$$

The groups on the left in the respective intersections coincide, but the ones on the right may be different as $W_{J_{\text{inv}}} \subset W_h$. For example, it is possible that $W_{J_{\text{inv}}}$ has non-trivial center. In this case, every non-trivial central element would be contained in $C_h(W_{J_{\text{inv}}})$ but not in $W_{(J_{\text{inv}})^*}$.

Remark 4.3.12. From the general perspective of Coxeter groups, the previous results and their proofs are interesting for two reasons:

- (i) The statements of Lemma 4.3.10 and Proposition 4.3.11 make sense for any Coxeter system (W, S) , but in the special case where $W = W(\mathbb{A})$ is the Weyl group of a GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$, the proof of the lemma relied on properties of the W -action on the set of real roots $\Phi = \Phi(\mathbb{A})$. In particular, it used the positivity criterion (2.5.4). As $\Phi = W \cdot \Phi_0 \subset Q$ is defined via the W -action on the root lattice and the simple reflections act on Q by

$$s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \quad \text{for all } i, j \in I, \quad (4.3.11)$$

the definition of Φ involves the matrix entries of \mathbb{A} . However, the same arguments apply in the general case via an analogous construction.

Let (W, S) be any Coxeter system of rank $n \in \mathbb{N}_0$ with Coxeter matrix $M = (m_{ij})_{i,j \in I}$ and consider the *canonical linear representation* of W (cf. [AB08, Section 2.5]). By definition, this is a representation of W on an n -dimensional real vector space $V \cong \mathbb{R}^n$ with basis $\{e_1, \dots, e_n\}$ which is uniquely determined by the following equation:

$$s_i(e_j) = e_j + 2 \cos\left(\frac{\pi}{m_{ij}}\right) e_i \quad \text{for all } i, j \in I \quad (4.3.12)$$

It can be shown that this gives rise to a faithful representation of W . Let $B : V \times V \rightarrow \mathbb{R}$ denote the symmetric bilinear form defined by $B(e_i, e_j) := -\cos(\pi/m_{ij})$ for all $i, j \in I$, then B is W -invariant and for every $i \in I$, the simple reflection $s_i \in S$ acts on V as the linear reflection fixing the hyperplane $H_i := \{v \in V : B(e_i, v) = 0\}$ and mapping e_i to $-e_i$. The roots of (W, S) are defined as the vectors in $\Phi(W, S) := \{w(e_i) \mid w \in W, i \in I\}$ and $\{e_1, \dots, e_n\} \subset \Phi(W, S)$ is the set of simple roots. If $\alpha \in \Phi(W, S)$ is written as $\alpha = w(e_i)$ for some $w \in W$ and $i \in I$, the reflection $s_\alpha := ws_iw^{-1} \in R \subset W$ is independent of the chosen expression and acts on V as the linear reflection which fixes the plane $H_\alpha := w(H_i)$ and maps α to $-\alpha$. By duality, W also acts on V^* and there, the reflection $s_\alpha \in W$ fixes the hyperplane $H_\alpha^* := w(H_i^*)$, where $H_i^* := \{f \in V^* : \langle f, e_i \rangle = 0\}$. The hyperplanes $\{H_\alpha^* \mid \alpha \in \Phi(W, S)\}$ are called the walls of V^* . They partition V^* into chambers, their intersections define cells and the W -action on V^* preserves this hyperplane arrangement. The existence of this representation is the reason why Coxeter groups are natural generalizations of finite reflection groups. Indeed, one can show that W is finite if and only if the symmetric bilinear form B is an inner product, i.e. positive-definite, in which case B induces a W -equivariant isomorphism between V and V^* . Furthermore, every root is a linear combination of the simple roots with coefficients of the same sign, so there is an evident notion of positive and negative roots. The map $\alpha \mapsto s_\alpha$ defines a bijection between the positive roots in $\Phi(W, S)$ and the set of reflections $R \subset W$ and the obvious analogue of (2.5.4) holds in this setting. With these modifications, the proof of Lemma 4.3.10 applies verbatim to (W, S) .

If $W = W(\mathbb{A})$ is the Weyl group of a GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$, the difference between the W -action on the root lattice Q and the canonical linear representation of the Coxeter system (W, S) is that the coefficients appearing in (4.3.11) and (4.3.12) do not coincide in general. The latter definition is symmetric in i and j , but for the former, this is the case if and only if \mathbb{A} is a symmetric matrix. More precisely, it follows from (2.5.1) that $2 \cos(\pi/m_{ij}) = -a_{ij}$ if and only if $i = j$ or $i \neq j$ and $a_{ij} = a_{ji} \in \{0, -1\}$. Thus, the real representations of W

on $Q \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and V are not equivalent, but their properties are essentially the same. Indeed, there is some degree of freedom in choosing the coefficients of the canonical linear representation and the W -action on $Q \otimes_{\mathbb{Z}} \mathbb{R}$ also fits into this general picture. This follows from the fact that the matrix entries of $-\mathbb{A}$ fulfil the conditions in [BB05, (4.1),(4.2),(4.3)], so [BB05, Proposition 4.2.5] implies that (2.5.4) holds for both $\Phi(\mathbb{A})$ and $\Phi(W, S)$. Not all properties of $\Phi(W, S)$ can be transferred to $\Phi(\mathbb{A})$ in this way because some statements of [BB05, Section 4.4] require the coefficients to be symmetric but in such cases, the results are usually still valid for $\Phi(\mathbb{A})$ by an alternative argument. For example, given any $\alpha \in \Phi(W, S)$, the only scalar multiples of α that are contained in $\Phi(W, S)$ are α and $-\alpha$. The same is true for $\alpha \in \Phi(\mathbb{A})$, but the proofs of these assertions are very different (cf. [BB05, (4.27)] and [Mar18, Lemma 3.22 (1)]). On the one hand, this connection suggests that the real roots of a GCM “fit into” the theory of Coxeter groups. On the other hand, the imaginary roots have no counterpart in the canonical linear representation, so their appearance is a genuinely new phenomenon of Kac-Moody theory.

- (ii) The proof of Lemma 4.3.10 shows that if $w \in W$ commutes with a simple reflection $s_j \in S$ for some $j \in I$, then either $j \in I(w)$ or $I(w) \subset \{j\}^\perp$. This statement is a weak special case of a much more detailed result of Brink who found an explicit description of the centralizer of a simple reflection in a Coxeter group. Let (W, S) be a Coxeter system of rank $n \in \mathbb{N}_0$ with Coxeter matrix $M = (m_{ij})_{i,j \in I}$ and consider its canonical linear representation on the vector space $V \cong \mathbb{R}^n$ as defined in (4.3.12). The main theorem of [Bri96] asserts that for every $j \in I$, the centralizer $C(s_j) := C(W_{\{j\}}) \subset W$ of the simple reflection $s_j \in S$ has the structure of a semidirect product:

$$C(s_j) \cong W_{\hat{\Gamma}_j} \rtimes Y_j \quad (4.3.13)$$

Here, the first factor $W_{\hat{\Gamma}_j}$ is the subgroup of W generated by all reflections $s_\gamma \in R \subset W$ with $\gamma \in \hat{\Gamma}_j := \Gamma_j \cup \{e_j, -e_j\}$, where $\Gamma_j := \{\gamma \in \Phi(W, S) : B(\gamma, e_j) = 0\}$, and the second is $Y_j := \{w \in W : w(e_j) = e_j \text{ and } \Phi_w \cap \Gamma_j = \emptyset\}$, where $\Phi_w \subset \Phi(W, S)$ is defined as in (2.5.5) but for the roots of the canonical linear representation.

It was shown independently by Deodhar and Dyer that every subgroup of a Coxeter group that is generated by reflections is a Coxeter group with respect to a canonical generating set (cf. [Deo89] and [Dye90]). Hence, the first factor in (4.3.13) is again a Coxeter group, but it need not be a standard (or parabolic) subgroup of (W, S) , so its Coxeter diagram is not necessarily a subgraph of $\Delta(W, S)$. The second factor is a free group of rank $e - v + 1$, where e (resp. v) denotes the number of edges (resp. vertices) of the connected component containing j of the “odd Coxeter diagram” of (W, S) , i.e. the subgraph of $\Delta(W, S)$ where all edges with even labels have been removed. In particular, if that component is a tree, then Y_j is trivial. The discussion in [Bri96] also shows that $j \in I(w)$ if $w \in Y_j$ and that $W_{\hat{\Gamma}_j}$ contains $W_{\{j\}^\perp}$ and s_j but may be larger than $W_{\{j\} \cup \{j\}^\perp}$.

In the case where $W = W(\mathbb{A})$ is the Weyl group of a GCM \mathbb{A} and $\Phi(W, S)$ can be identified with the set of real roots $\Phi = \Phi(\mathbb{A})$, the condition $B(\gamma, e_j) = 0$ is equivalent to $\langle \gamma, \alpha_j^\vee \rangle = 0$. Also, the only possible odd label for the edges of $\Delta(W, S)$ is 3, so Y_j is trivial unless the Dynkin diagram $\Gamma(\mathbb{A})$ contains a subdiagram of untwisted affine type $A_l^{(1)}$ for some $l \geq 2$.

Let us return to our study of the faces of Ξ_h for $h \in \overline{C}$. We now use the preceding results to examine under which conditions two standard left cosets of (W, S) define the same face of Ξ_h under the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$. More generally, suppose that $J, K \subset I$ and $w_1, w_2 \in W$ are such that $w_2(F_K) \subset w_1(F_J)$, then $w := w_1^{-1}w_2$ maps F_K into F_J and every element in the double coset $W_J w W_K \subset W$ has the same property. Upon replacing w_1 and w_2 by some elements in $w_1 W_J$ and $w_2 W_K$, we may therefore assume that w is (J, K) -reduced.

Proposition 4.3.13. *Let $h \in \overline{C}$ and $J, K \subset I$ be arbitrary. If $w \in W$ is (J, K) -reduced, then the following conditions are equivalent:*

- (i) $w(F_K) \subset F_J$
- (ii) $K_{\text{vis}} \subset J_{\text{vis}}$ and $w \in W_{K_{\text{vis}}}^*$

In order to prove the proposition, we need to distinguish the closed fundamental chambers and Tits cones of various restrictions of \mathcal{D} , so we apply Notation 2.6.8.

Proof. (i) \implies (ii): As the dimension of $w(F_K)$ cannot exceed that of F_J , Corollary 4.3.7 yields that $|K_{\text{vis}}| \leq |J_{\text{vis}}|$. If $J_{\text{vis}} = K_{\text{vis}} = \emptyset$, then $F_J = F_K = \{h\}$. In this case, (i) reduces to $w(h) = h$, i.e. $w \in W_h$, but we also have $K_{\text{vis}}^\perp = I$ and $K_{\text{vis}}^* = I_h$, so (ii) is satisfied. Thus, we may assume that $|J_{\text{vis}}| > 0$, so that F_J and F_K both contain h , but F_J does not only consist of that vertex.

Claim 1. $w(h) = h$

Proof of Claim. As $h \in F_K$ and $w(h) \in w(F_K) \subset F_J = F_{J_{\text{vis}}} \subset h + \mathfrak{a}'_{J_{\text{vis}}}$, we have $h - w(h) \in \mathfrak{a}'_{J_{\text{vis}}}$. Since $h \in \overline{C}$, it follows that $h - w(h) = \sum_{j \in J_{\text{vis}}} x_j \alpha_j^\vee$, where $x_j \in \mathbb{R}_{\geq 0}$ for all $j \in J_{\text{vis}}$, and

$$\langle \alpha_i, w(h) \rangle = \langle \alpha_i, h \rangle - \langle \alpha_i, h - w(h) \rangle \geq - \sum_{j \in J_{\text{vis}}} x_j a_{ji} \quad \text{for all } i \in I.$$

If $i \notin J_{\text{vis}}$, then $a_{ji} \leq 0$ for all $j \in J_{\text{vis}}$, so that $\langle \alpha_i, w(h) \rangle \geq 0$. If $i \in J_{\text{vis}}$, then $l(w) < l(s_i w)$ because w is left J -reduced, so (2.5.4) implies that $w^{-1}(\alpha_i) \in \Phi_+$. As $h \in \overline{C}$, we also obtain $\langle \alpha_i, w(h) \rangle = \langle w^{-1}(\alpha_i), h \rangle \geq 0$ in this case. We conclude that $w(h) \in \overline{C}$, but this is only possible if $w(h) = h$ because \overline{C} is a strict fundamental domain for the W -action on $X_+ \subset \mathfrak{a}$. The proof of Claim 1 is therefore finished.

Let now $\Delta_{K_{\text{vis}}}^\circ$ be an exposing forest for $\Delta_{K_{\text{vis}}}$ and $\{w_k \mid k \in K_{\text{vis}}\} \subset W_{K_{\text{vis}}}$ the corresponding exposing set for $W_{K_{\text{vis}}}$. For $k \in K_{\text{vis}}$, we denote by $\gamma(k) = (k_0, \dots, k_{\delta_K(k)} = k)$ the unique path in $\Delta_{K_{\text{vis}}}^\circ$ connecting k with a white vertex $k_0 \in K \setminus I_h$ and set $K_{\gamma(k)} := \{k_0, \dots, k_{\delta_K(k)}\} \subset K_{\text{vis}}$. By Definition 1.2.6, it holds that $I(w_k) = K_{\gamma(k)}$.

Claim 2. For every $k \in K_{\text{vis}}$, we have $K_{\gamma(k)} \subset J_{\text{vis}}$ and $w \in W_{K_{\gamma(k)}^*}$. In particular, $w_k \in W_{J_{\text{vis}}}$ and $ww_k = w_k w$ is $(J \setminus K_{\gamma(k)}, K \setminus K_{\gamma(k)})$ -reduced.

Once Claim 2 has been shown, it will follow that $K_{\text{vis}} \subset J_{\text{vis}}$ and $w \in W_{K_{\text{vis}}^*}$, so the first part of the proof will be complete.

Proof of Claim. We argue by induction on $\delta := \delta_K(k)$. Suppose first that $\delta = 0$, i.e. $k \in K \setminus I_h$. In this situation, the claim reduces to the assertion that $k \in J_{\text{vis}}$ and $w \in W_{\{k\}^*}$.

Consider the point $ws_k(h) \in w(F_K) \subset F_J$. As w is left J -reduced, the folding condition ensures that ws_k is either also left J -reduced or $ws_k = s_j w$ for some $j \in J$. If the first case occurred, the same argument as in the proof of Claim 1 would imply that $ws_k(h) = h$. This is impossible since the action of w on \mathfrak{a} is bijective and $w(h) = h$, but $s_k(h) \neq h$ because $k \in K \setminus I_h$. Hence, we have $ws_k = s_j w$ for some $j \in J$ and $h \neq ws_k(h) = s_j w(h) = s_j(h)$, which shows that $j \in J \setminus I_h \subset J_{\text{vis}}$. Moreover, the simple reflections s_k and $s_j = ws_k w^{-1}$ are both contained in the double coset $W_{I(w)s_k} W_{I(w)} \subset W$. Since $I(w) \subset I_h$ does not contain $k \in K \setminus I_h$, it follows that s_k is the unique element of minimal length in that double coset, which implies that $s_j = s_k$ and $k = j \in J \setminus I_h \subset J_{\text{vis}}$. In addition, we have $ws_k = s_k w$ with $k \notin I(w)$, so Lemma 4.3.10 yields that $I(w) \subset \{k\}^\perp$ and $w \in W_{\{k\}^\perp} \cap W_h = W_{\{k\}^*}$. This proves the first assertion and the second follows immediately as $w_k = s_k$ in this case.

Now assume that $\delta > 0$. Let $\gamma := \gamma(k) = (k_0, \dots, k_\delta = k)$ be the unique path in $\Delta_{K_{\text{vis}}}^\circ$ from k to a white vertex $k_0 \in K \setminus I_h$ and set $k' := k_{\delta-1}$ and $\gamma' := \gamma(k') = (k_0, \dots, k_{\delta-1} = k')$. The induction hypothesis yields that $K_{\gamma'} \subset J_{\text{vis}}$, $w \in W_{K_{\gamma'}^*}$, and $ww_{k'} = w_{k'} w$ is $(J \setminus K_{\gamma'}, K \setminus K_{\gamma'})$ -reduced. Furthermore, we have

$$ww_k = ws_k w_{k'} = ww_{k'} t,$$

where $t := w_{k'}^{-1} s_k w_{k'} \in R \subset W$ satisfies $l(ww_{k'} t) = l(ww_k) = l(ww_{k'}) + 1$ because w is right K -reduced and $l(w_k) = l(w_{k'}) + 1$. It follows that $ww_{k'}$ is covered by ww_k in the Bruhat order of (W, S) (cf. [BB05, Chapter 2]). As $ww_{k'}$ is left $(J \setminus K_{\gamma'})$ -reduced, a generalized version of the folding condition implies that ww_k is either left $(J \setminus K_{\gamma'})$ -reduced as well or there exists some $j \in J \setminus K_{\gamma'}$ such that $ww_k = s_j ww_{k'}$ (cf. the analogue of [BB05, Corollary 2.5.2] for right cosets). We are going to show that the first case is impossible.

Suppose to the contrary that ww_k is left $(J \setminus K_{\gamma'})$ -reduced. In this case, a simple modification of the proof of Claim 1 shows that $ww_k(h) \in w(F_K) \subset F_J$ is contained in $\overline{C_{I \setminus K_{\gamma'}}$. In addition, we have $I(w_{k'}) = K_{\gamma'}$, so $w_{k'}$ is left $(I \setminus K_{\gamma'})$ -reduced and another variation of this argument proves that $w_{k'}(h) \in \overline{C_{I \setminus K_{\gamma'}}$. Now ws_k maps $w_{k'}(h)$ to $ww_k(h)$ and it is an element of $W_{I \setminus K_{\gamma'}}$, since $I(w) \subset K_{\gamma'}^* \subset I \setminus K_{\gamma'}$ and $k \notin K_{\gamma'}$. Because $\overline{C_{I \setminus K_{\gamma'}}$ is a strict fundamental domain for the action of $W_{I \setminus K_{\gamma'}}$ on $X_{I \setminus K_{\gamma'}, +} \subset \mathfrak{a}$, we conclude that $ww_k(h) = w_{k'}(h) = w_{k'}w(h) = ww_{k'}(h)$. This contradicts the fact that the action of w is bijective and $w_k(h) \neq w_{k'}(h)$, which is clear from the definition of an exposing set (cf. the proof of Proposition 4.3.2). As a result, $ww_k = s_j ww_{k'} = s_j w_{k'}w$ for some $j \in J \setminus K_{\gamma'}$ and $ww_{k'}(h) \neq ww_k(h) = s_j ww_{k'}(h)$. Thus, s_j acts non-trivially on a point in $w(F_K) \subset F_J$, so that $j \in J_{\text{vis}}$. Furthermore, we obtain

$$s_j w_{k'} = ww_k w^{-1} = ws_k w_{k'} w^{-1} \in W_{I(w)} s_k w_{k'} W_{I(w)}.$$

As $I(w) \subset K_{\gamma'}^*$ is disjoint from $I(w_k) = K_{\gamma}$, it follows that $w_k = s_k w_{k'}$ is the unique element of minimal length in the double coset $W_{I(w)} s_k w_{k'} W_{I(w)}$. But $s_j w_{k'}$ lies in that coset as well and has the same length, so that $s_j w_{k'} = s_k w_{k'}$, which shows that $k = j \in J_{\text{vis}}$ and hence $K_{\gamma} \subset J_{\text{vis}}$. In addition, we find that $ww_k = s_j w_{k'}w = s_k w_{k'}w = w_k w$, so w centralizes $W_{K_{\gamma'}}$ and commutes with w_k , which implies that w centralizes $W_{K_{\gamma}}$. Since $I(w)$ is disjoint from K_{γ} , Lemma 4.3.10 finally yields that $I(w) \subset K_{\gamma}^{\perp}$ and thus $w \in W_{K_{\gamma}^{\perp}} \cap W_h = W_{K_{\gamma}^*}$. This completes the induction step and the proof of Claim 2.

(ii) \implies (i): Because $F_K = F_{K_{\text{vis}}}$ and $w \in W_{K_{\text{vis}}^*} = W_{K_{\text{vis}}^{\perp}} \cap W_h$ fixes h and commutes with $W_{K_{\text{vis}}}$, it fixes F_K point-wise, so $K_{\text{vis}} \subset J_{\text{vis}}$ implies that $w(F_K) = F_K = F_{K_{\text{vis}}} \subset F_{J_{\text{vis}}} = F_J$. \square

Corollary 4.3.14. *Suppose that $h \in \overline{C}$.*

- (i) *For every two standard cosets $w_1 W_J, w_2 W_K \subset W$, we have $w_1(F_J) = w_2(F_K)$ if and only if $J_{\text{vis}} = K_{\text{vis}}$ and the unique element of minimal length in the double coset $W_J w_1^{-1} w_2 W_K \subset W$ is contained in $W_{J_{\text{vis}}^*}$. The latter condition holds if and only if $W_J w_1^{-1} w_2 W_K \cap W_{J_{\text{vis}}^*} \neq \emptyset$.*
- (ii) *For every standard coset $w W_J \subset W$, the stabilizer of the face $w(F_J)$ of Ξ_h under the action of W is the parabolic subgroup $w(W_{J_{\text{vis}}^*} \times W_{J_{\text{vis}}})w^{-1} \subset W$.*

Proof. The first statement of (i) is a direct consequence of Proposition 4.3.13. The second holds because if w' is of minimal length in $W_J w_1^{-1} w_2 W_K$, then $I(w') \subset I(w'')$ for all $w'' \in W_J w_1^{-1} w_2 W_K$. It remains to prove (ii). Since the stabilizer of $w(F_J)$ is the w -conjugate of that of F_J , we may assume that $w = 1$. If we have $u = u_1 u_2 \in W_{J_{\text{vis}}^*} \times W_{J_{\text{vis}}}$, where $u_2 \in W_{J_{\text{vis}}}$ stabilizes $F_J = F_{J_{\text{vis}}}$ and $u_1 \in W_{J_{\text{vis}}^*}$ fixes it point-wise, then $u(F_J) = u_1 u_2(F_J) = u_1(F_J) = F_J$. Conversely, suppose that $u \in W$ satisfies $u(F_J) = F_J$ and let $u_1, u_2 \in W_J$ be such that $u' := u_1 u u_2$ is (J, J) -reduced. As $u'(F_J) = F_J$, Proposition 4.3.13 implies that $u' \in W_{J_{\text{vis}}^*}$ and $u = u_1^{-1} u' u_2^{-1} \in W_J W_{J_{\text{vis}}^*} W_J$. Using that $W_J \cong W_{J_{\text{vis}}} \times W_{J_{\text{inv}}}$ and $J_{\text{inv}} \subset J_{\text{vis}}^*$, it follows that $u \in W_{J_{\text{vis}}^*} \times W_{J_{\text{vis}}}$. \square

If a face $F \in \mathcal{F}(\Xi_h)$ is of the form $F = w(F_J) = \psi(w W_J)$ for some standard left coset $w W_J \subset W$, we define the *type* of F as

$$\tau(F) = \tau(w(F_J)) := J_{\text{vis}}. \quad (4.3.14)$$

In view of Corollary 4.3.14, this notion is well-defined and the action of W preserves the type of such a face. The corresponding type function τ is defined on the subset $\psi(\Sigma^*(W, S)) \subset \mathcal{F}(\Xi_h)$ and surjective onto the set $\mathcal{P}_{\text{vis}}(I)$ of all visible subsets of I .

Remark 4.3.15. The proof of Proposition 4.3.13 did not fully make use of the condition that w is (J, K) -reduced. Indeed, the argument in Claim 1 actually shows that if $w(F_K) \subset F_J$, it is sufficient to require that w is left J_{vis} -reduced to conclude that $w \in W_h$. Under this assumption, the proof of the first statement in Claim 2 works in almost the same way and yields that $K_{\text{vis}} \subset J_{\text{vis}}$ and $w \in W_{K_{\text{vis}}^*}$. In particular, it follows that w commutes with $W_{K_{\text{vis}}} \subset W_{J_{\text{vis}}}$, which implies that w is right K_{vis} -reduced as well. What we cannot ensure in this way is that w is also left J_{inv} -reduced and right K_{inv} -reduced. Geometrically, this mirrors the fact that $W_{J_{\text{inv}}}$ (resp. $W_{K_{\text{inv}}}$) acts trivially on F_J (resp. F_K), so replacing w by an element of the double coset $W_{J_{\text{inv}}} w W_{K_{\text{inv}}}$ does not have any geometric impact. In fact, condition (i) in Proposition 4.3.13 is equivalent to $w(F_{K_{\text{vis}}}) \subset F_{J_{\text{vis}}}$.

Thus, it would have been possible to assume from the start that J and K were visible subsets of I . However, for the computation of the stabilizers in the proof of Corollary 4.3.14, it was convenient that we avoided this restriction. More generally, the arguments just described also imply that

$$\{w \in W : w(F_J) \subset F_J\} = W_{J_{\text{vis}}} W_{K_{\text{vis}}}^* \quad \text{if } J, K \subset I \text{ satisfy } K_{\text{vis}} \subset J_{\text{vis}}. \quad (4.3.15)$$

The product on the right-hand side need not be a subgroup of W , but if $J_{\text{vis}} = K_{\text{vis}}$, then the two factors commute and their product is the standard subgroup $W_{J_{\text{vis}}}^* \times W_{J_{\text{vis}}}$, so we recover the assertion of Corollary 4.3.14 (ii).

Given any $J \subset I$, the subgroup $W_{J_{\text{vis}}}^* \subset W$ acts trivially on $F_J = F_{J_{\text{vis}}}$. To conclude this section, we show that it is precisely the subgroup of W that fixes F_J point-wise. To this end, we first establish a slightly technical lemma that allows us to perturb h in a controlled way in order to reduce its stabilizer. This result has already been used in the proof of Proposition 2.6.11 (iii) and will have another very important application in the next section.

Lemma 4.3.16. *Suppose that $h \in \overline{C}$. Let $J \subset I$ be a visible subset, let $\Delta_J^\circ = \bigcup_{s=1}^t \gamma_s$ be an exposing forest for Δ_J and $\{w_j \mid j \in J\}$ the corresponding exposing set for W_J . For every open neighbourhood $U \subset \mathfrak{a}$ of h , there exists a point $p \in U \cap \text{conv}(\{h, w_j(h) \mid j \in J\}) \subset \Xi_h$ such that $\langle \alpha_j, p \rangle > 0$ for all $j \in J$.*

Proof. After possibly replacing U by an open ball around h , we may assume that U is convex. Set $A := \text{conv}(\{h, w_j(h) \mid j \in J\}) \subset \Xi_h$. We are going to show that for every $j \in J$, there exists a point $x_j \in U \cap A$ with the following properties, where $\delta := \delta_J(j)$ and $\gamma = (j_0, \dots, j_\delta = j)$ is the unique path in Δ_J° connecting j with a white vertex $j_0 \in J \setminus I_h$:

- a) $\langle \alpha_{j_r}, x_j \rangle > 0$ for all $r \in \{0, \dots, \delta\}$.
- b) $\langle \alpha_k, x_j \rangle \geq 0$ for all $k \in J$ with $\delta_J(k) \leq \delta$.
- c) $\langle \alpha_k, x_j \rangle = 0$ for all $k \in J$ with $\delta_J(k) > \delta$.
- d) $x_j \in \text{conv}(\{h, w_{j_r}(h) \mid 0 \leq r \leq \delta - 1\}) \subset A$.

Once this is established, we may form a convex combination of the points $x_{j_{\delta_1}}, \dots, x_{j_{\delta_t}} \in U \cap A$, where $j_{\delta_1}, \dots, j_{\delta_t} \in J$ are the endpoints of the maximal paths $\gamma_1, \dots, \gamma_t$, to find a point $p \in U \cap A$ that satisfies $\langle \alpha_j, p \rangle > 0$ for all $j \in J$.

We argue by induction on $\delta = \delta_J(j)$. If $\delta = 0$, then $\langle \alpha_j, h \rangle > 0$ and $x_j := h$ satisfies a)-d), so we may assume that $\delta > 0$. Set $i := j_{\delta-1}$, then by the induction hypothesis, there exists a point $x_i \in U \cap A$ having the properties a)-d) with δ replaced by $\delta - 1$. Since $\langle \alpha_i, x_i \rangle > 0$ by a), it holds that $s_i(x_i) \neq x_i$. In addition, we have $\langle \alpha_j, x_i \rangle = 0$ by c) and $a_{ij} < 0$ because i and j are joined by an edge of Δ_J . Hence, every point x_j on the non-empty open line segment $(x_i, s_i(x_i))$ satisfies $\langle \alpha_j, x_j \rangle > 0$ and by choosing x_j sufficiently close to x_i , we may ensure that $x_j \in U$ and $\langle \alpha_i, x_j \rangle > 0$. For any other vertex $k \in J \setminus \{i, j\}$, we have $\langle \alpha_k, x_j \rangle \geq \langle \alpha_k, x_i \rangle$, where equality holds if and only if $a_{ik} = 0$. If $\delta_J(k) > \delta = \delta_J(i) + 1$, then i and k are not connected in Δ_J , so that $a_{ik} = 0$ and $\langle \alpha_k, x_j \rangle = \langle \alpha_k, x_i \rangle = 0$ by c). If $\delta_J(k) \leq \delta$, then $\langle \alpha_k, x_j \rangle \geq \langle \alpha_k, x_i \rangle \geq 0$ by b) and c), where the second inequality is strict if $k = j_r$ for some $r \in \{0, \dots, \delta - 2\}$, so x_j satisfies a)-c). In view of d), x_i is a convex combination of $\{h, w_{j_r}(h) \mid 0 \leq r \leq \delta - 2\}$. If $\delta \geq 2$, then $s_i(h) = h$ and the definition of an exposing set yields $s_i w_{j_{\delta-2}} = w_{j_{\delta-1}}$ and $s_i w_{j_r}(h) = w_{j_r} s_i(h) = w_{j_r}(h)$ for all $r \in \{0, \dots, \delta - 3\}$, so $s_i(x_i)$ is a convex combination of $\{h, w_{j_r}(h) \mid 0 \leq r \leq \delta - 1\}$, which implies that x_j also has property d). If $\delta = 1$, then $x_i = h$ and x_j is a convex combination of h and $s_i(h) \neq h$, but in this case, we have $w_i = s_i$ and the condition holds as well. \square

Corollary 4.3.17. *For every standard coset $wW_J \subset W$, the subgroup of W that fixes the face $w(F_J)$ of Ξ_h point-wise is the parabolic subgroup $wW_{J_{\text{vis}}}^* w^{-1}$.*

Proof. The subgroup of W that fixes $w(F_J)$ point-wise is simply the w -conjugate of that of F_J , so we may assume that $w = 1$. Since $W_{J_{\text{vis}}}^*$ commutes with $W_{J_{\text{vis}}}$ and fixes h , it acts trivially on $F_J = F_{J_{\text{vis}}}$. Conversely, the stabilizer of F_J is $W_{J_{\text{vis}}}^* \times W_{J_{\text{vis}}}$ by Corollary 4.3.14 (ii), so it suffices to show that every $u \in W_{J_{\text{vis}}} \setminus \{1\}$ acts non-trivially on F_J . By applying Lemma 4.3.16 to J_{vis} , it follows that the face $F_J = F_{J_{\text{vis}}}$ intersects the open fundamental $\mathcal{D}_{J_{\text{vis}}}$ -chamber $C_{J_{\text{vis}}} \subset \mathfrak{a}$. For every point $p \in F_J \cap C_{J_{\text{vis}}}$, the stabilizer of p under the action of $W_{J_{\text{vis}}}$ is trivial, so that $u(p) \neq p$, i.e. u does not fix F_J point-wise. \square

4.3.2 The face poset of Ξ_h

Although the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$ is not injective if the stabilizer of $h \in \overline{C}$ is non-trivial, we now want to prove that it is surjective if h is almost virtually singular. Hence, our goal is to show that every face $F \in \mathcal{F}(\Xi_h)$ is of the form $F = w(F_J)$ for some standard coset $wW_J \subset W$. We proceed in two steps: First, we establish the result under the assumption that h is regular. Afterwards, we reduce the (almost) virtually singular case to this situation. In both cases, we also show that Ξ_h is a closed subset of \mathfrak{a} and this topological property will be crucial for the proofs of the convexity theorems in Chapter 5 as it will allow us to reconstruct Ξ_h from its proper faces. The essentially singular case, however, is fundamentally different and will be discussed at the end of this section.

The regular case

Throughout this subsection, we suppose that $h \in \overline{C}$ is a regular element of \mathfrak{a} . This means that h has trivial stabilizer under the W -action, i.e. $W_h = \{1\}$, or equivalently, that h is already contained in the open fundamental chamber $C \subset \overline{C}$. Hence, we specialize the previous discussion to the case $I^\bullet = I_h = \emptyset$, i.e. all vertices of the Coxeter diagram $\Delta = \Delta(W, S)$ are coloured white. In particular, this implies that $J_{\text{vis}} = J$ and $J^* = \emptyset$ for every subset $J \subset I$. The following result is therefore an immediate consequence of Corollary 4.3.7 and 4.3.14:

Corollary 4.3.18. *If $h \in C$, then the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$, $wW_J \mapsto w(F_J)$, is injective. For every standard coset $wW_J \subset W$, the face $\psi(wW_J) = w(F_J)$ of Ξ_h has dimension $|J|$ and its stabilizer under the action of W is the parabolic subgroup $wW_Jw^{-1} \subset W$. \square*

In order to show that ψ is surjective as well, we need a simple topological property that will also be useful for proving that Ξ_h is closed. Note that $\text{aff}(\Xi_h) = E_h = h + \mathfrak{a}'_{\text{vis}} = h + \mathfrak{a}'$.

Lemma 4.3.19. *Let $h \in C$ and $h' \in \Xi_h$ be such that the vector $v := h - h' \in \mathfrak{a}'$ is of the form $v = \sum_{i \in I} x_i \alpha_i^\vee$ for some $x_1, \dots, x_n \in \mathbb{R}_{>0}$. Then the open line segment (h, h') is contained in the relative interior of $\Xi_h \subset E_h$.*

Proof. Let $\lambda \in (0, 1)$ be arbitrary and set $p := h - \lambda v \in \Xi_h$. First, we show that for every $i \in I$, there exists some $y_i \in \mathbb{R}_{>0}$ with $p - y_i \alpha_i^\vee \in \Xi_h$ by writing this point as a convex combination of $\{h, h', s_i(h)\}$. Since $\langle \alpha_i, h \rangle > 0$ by regularity of h and $\lambda \in (0, 1)$ by assumption, we may choose some $y_i \in \mathbb{R}_{>0}$ such that $c := \frac{y_i}{\langle \alpha_i, h \rangle} < 1 - \lambda$. Upon setting $b := \lambda$ and $a := 1 - b - c$, it holds that $a, b, c \in (0, 1)$ and $a + b + c = 1$, so the following computation proves that the corresponding convex combination is the desired point:

$$\begin{aligned} ah + bh' + cs_i(h) &= ah + bh + b(h' - h) + c(h - \langle \alpha_i, h \rangle \alpha_i^\vee) \\ &= (a + b + c)h - bv - c\langle \alpha_i, h \rangle \alpha_i^\vee \\ &= h - \lambda v - y_i \alpha_i^\vee = p - y_i \alpha_i^\vee \in \Xi_h \end{aligned}$$

Second, the point $h = p + \lambda v = p + \sum_{i \in I} \lambda x_i \alpha_i^\vee$ also lies in Ξ_h and all coefficients in this sum are positive. Given any $i \in I$, it therefore follows by forming a suitable convex combination of the points $\{h, p - y_j \alpha_j^\vee \mid j \in I^{(i)}\}$ that there exists some sufficiently small $y'_i \in \mathbb{R}_{>0}$ such that Ξ_h also contains $p + y'_i \alpha_i^\vee$. Altogether, we conclude that Ξ_h contains the cube

$$p + \sum_{i \in I} [-y_i, y'_i] \alpha_i^\vee \subset h + \mathfrak{a}' = E_h.$$

Since the simple coroots form a basis of \mathfrak{a}' , this cube has non-empty relative interior in $E_h = h + \mathfrak{a}'$, so p lies in the relative interior of $\Xi_h \subset E_h$. \square

Proposition 4.3.20. *If $h \in C$ and $F \in \mathcal{F}(\Xi_h)$, there exists a unique standard coset $wW_J \subset W$ such that $F = w(F_J)$. In particular, every face of Ξ_h is exposed.*

Proof. The uniqueness of wW_J has already been established in Corollary 4.3.18, so it remains to prove its existence. If $F = \Xi_h$ is the trivial face, the assertion is satisfied by taking the trivial standard coset $W_I = W$, so we may assume that F is a proper face of Ξ_h . We argue by induction on $n = |I| = \dim(\Xi_h)$. If $n = 1$, then Ξ_h is the closed line segment connecting the two distinct points h and $s_1(h)$, so the only proper faces are these two points which correspond to the two non-trivial standard cosets $W_\emptyset = \{1\}$ and $s_1W_\emptyset = \{s_1\}$ of $W \cong \mathbb{Z}_2$.³

Suppose now that $n > 1$. By Lemma 4.2.6 (i), there exists a proper subset $W' \subsetneq W$ such that $F = \text{conv}(W' \cdot h)$. After possibly replacing F by $w^{-1}(F)$ for an arbitrary $w \in W'$, we may assume that $1 \in W'$ and $h \in F$. In this situation, we claim that $W' \subset W^{(i)}$ for some $i \in I$.

Assume to the contrary that for every $i \in I$, there exists some $w_i \in W'$ with $i \in I(w_i)$. As $I_h = \emptyset$, every element of W is trivially right I_h -reduced, so Lemma 4.3.5 implies that $h - w_i(h) \in \mathfrak{a}'$ is a linear combination of the simple coroots, where all coefficients are non-negative and that of α_i^\vee is positive. By forming a suitable convex combination of the points $\{w_i(h) \mid i \in I\}$, we can construct a point $h' \in F$ such that $h - h' = \sum_{i \in I} x_i \alpha_i^\vee$ for some $x_1, \dots, x_n \in \mathbb{R}_{>0}$. Now Lemma 4.3.19 states that the open line segment $(h, h') \subset F \subset \partial_{\text{rel}}(\Xi_h)$ runs through the relative interior of Ξ_h , a contradiction. Thus, the claim holds and $W' \subset W^{(i)}$ for some $i \in I$, which implies that F is contained in the facet $F^{(i)}$ and is therefore a face of $F^{(i)}$. The induction hypothesis now yields that there is a unique (not necessarily non-trivial) standard coset $wW_J \subset W^{(i)}$ with $F = w(F_J)$. As we have already shown that every face of this form is exposed, the proof is complete. \square

Consequently, the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$ is surjective as well. The following theorem is now an immediate consequence of Corollary 4.3.18 and Proposition 4.3.20 and provides a completely combinatorial description of the face poset of Ξ_h .

Theorem 4.3.21. *If $h \in C$, the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$, $wW_J \mapsto w(F_J)$, is a well-defined W -equivariant poset isomorphism. More precisely, for every standard left coset $wW_J \subset W$, the set $w(F_J) = \text{conv}(wW_J \cdot h)$ is an exposed face of Ξ_h of dimension $|J|$ and, conversely, every face is of this form for a unique standard left coset of (W, S) .* \square

In particular, it follows that the number of faces of Ξ_h is finite if and only if the group W is finite. More precisely, for every subset $J \subset I$, the faces of Ξ_h of type J are in bijective correspondence with the left cosets of W_J in W , so their number is equal to the index $[W : W_J]$ which is finite if and only if J contains the set I_∞ of Theorem 1.1.6. However, as one standard left coset of (W, S) is contained in another if and only if the corresponding faces have the same property, only finitely many faces of Ξ_h can intersect non-trivially.

Corollary 4.3.22. *If $h \in C$, then every face of Ξ_h only lies in finitely many other faces. More precisely, let $k, l \in \{0, \dots, n\}$ with $k \leq l$ be arbitrary. For every k -dimensional face $F \in \mathcal{F}(\Xi_h)$, the number of l -dimensional faces containing F is equal to $\binom{n-k}{l-k}$ and the number of all faces containing F is 2^{n-k} . In particular, every vertex of Ξ_h is contained in precisely n edges, n facets and 2^n faces and every intersection of more than 2^n distinct faces of Ξ_h is empty.*

Proof. Let $wW_J \subset W$ with $|J| = k$ be the unique standard coset such that $F = w(F_J)$, then the l -dimensional faces of Ξ_h containing F correspond bijectively to the standard cosets $w'W_{J'} \subset W$ with $|J'| = l$ that contain wW_J . This is satisfied if and only if $w'W_{J'} = wW_{J'}$, so these cosets are in bijective correspondence with the subsets of I of cardinality l containing J or, equivalently, the subsets of $I \setminus J$ of cardinality $l - k$. Their number is equal to $\binom{n-k}{l-k}$ and the remaining claims are direct consequences of this result. \square

We conclude from Corollary 4.3.22 that the underlying graph of Ξ_h , i.e. the set of all its vertices and edges, is n -regular and thus locally finite. Moreover, Theorem 4.3.21 provides two alternative descriptions of this graph. In the formulation of the next result, we use the fact that the set of chambers of the Coxeter complex $\Sigma(W, S)$, together with the n equivalence relations generated by s_i -adjacency, can be viewed as a “chamber system” in the sense of [AB08, Definition 5.21].

³In principle, we allow the vacuous case $n = 0$, but $\Xi_h = \{h\}$ does not have any non-trivial faces in this situation, so the assumption that F is proper forces $n \geq 1$.

Corollary 4.3.23. *If $h \in C$, the underlying graph of Ξ_h is isomorphic to the chamber system of $\Sigma(W, S)$. In other words, given any $w_1, w_2 \in W$ and $i \in I$, the vertices $w_1(h)$ and $w_2(h)$ of Ξ_h are joined by an edge of type $\{i\}$ if and only if the chambers $\{w_1\}$ and $\{w_2\}$ of $\Sigma(W, S)$ are s_i -adjacent. Under this isomorphism, the graph distance in Ξ_h coincides with the chamber distance in $\Sigma(W, S)$ and minimal paths between vertices of Ξ_h are in bijective correspondence with minimal galleries between the corresponding chambers. Equivalently, the underlying graph of Ξ_h is also isomorphic to the (right) Cayley graph of (W, S) . \square*

Although we have now arrived at a satisfying combinatorial description of the face poset of Ξ_h , the result will only be useful once we know that $\mathcal{F}(\Xi_h)$ contains “sufficient information” about Ξ_h . More precisely, our final goal in this subsection is to show that Ξ_h can be reconstructed from its proper faces as the intersection of all supporting half-spaces. For such a result to hold, it is clearly necessary that Ξ_h is a closed subset of \mathfrak{a} which does not appear to be obvious if the orbit $W \cdot h$ is not finite, so we provide an explicit proof.

Proposition 4.3.24. *If $h \in C$, then Ξ_h is closed.*

Proof. If W is finite, then Ξ_h is a convex polytope and thus compact, so for the rest of the proof, we may assume that W is infinite. For every $k \in \mathbb{N}_0$, we set

$$U_k := \{w \in W : l(w) \leq k\} \quad \text{and} \quad P_k := \text{conv}(U_k \cdot h).$$

Finiteness of U_k implies that P_k is a convex polytope for every $k \in \mathbb{N}_0$. In particular, it is a closed subset of \mathfrak{a} . Moreover, these polytopes provide a “filtration” of Ξ_h in the sense that $P_k \subset P_{k+1}$ for all $k \in \mathbb{N}_0$ and $\Xi_h = \bigcup_{k=0}^{\infty} P_k$.

Let $(y_l)_{l \in \mathbb{N}}$ be a sequence in Ξ_h converging to some $y \in \mathfrak{a}$. We are going to show that there exists some $k' \in \mathbb{N}_0$ such that $y_l \in P_{k'}$ for all $l \in \mathbb{N}$. Once this is established, it will follow that the sequence $(y_l)_{l \in \mathbb{N}}$ is contained in the closed set $P_{k'}$, so that $y \in P_{k'} \subset \Xi_h$. The proof of this assertion will be reduced to a number of claims. Before we formulate them, we need to introduce some notation. Throughout this proof, we identify $E_h = h + \mathfrak{a}'$ with \mathbb{R}^n by moving h to 0 and mapping Φ_0^\vee to the standard basis of \mathbb{R}^n in the natural order. More precisely, for every $p \in E_h$, there exist unique scalars $p_1, \dots, p_n \in \mathbb{R}$ such that $p = h + \sum_{i \in I} p_i \alpha_i^\vee$ and we identify p with the vector $(p_1, \dots, p_n) \in \mathbb{R}^n$. For every $i \in I$, the affine map $p \mapsto \langle \omega_i, p - h \rangle$ corresponds to the i -th coordinate functional under this identification, so if we write $\eta := \sum_{i \in I} \omega_i \in \mathfrak{a}^*$, then

$$|\langle \eta, p \rangle - \langle \eta, q \rangle| = \left| \sum_{i=1}^n (p_i - q_i) \right| \leq \sum_{i=1}^n |p_i - q_i| = \|p - q\|_1 \quad \text{for all } p, q \in E_h \cong \mathbb{R}^n, \quad (4.3.16)$$

where $\|\cdot\|_1$ denotes the ℓ^1 -norm on \mathbb{R}^n . Since all norms on a finite-dimensional real vector space are equivalent, the ℓ^1 -topology on \mathbb{R}^n coincides with the standard Euclidean topology. Next, we set $U_{-1} := \emptyset$ and define the following objects for every $k \in \mathbb{N}_0$:

$$\begin{aligned} V_k &:= U_k \setminus U_{k-1} = \{w \in W : l(w) = k\} \\ m_+(k) &:= \max_{w \in V_k} \langle \eta, w(h) \rangle \\ m_-(k) &:= \min_{w \in V_k} \langle \eta, w(h) \rangle \\ m(k) &:= m_+(k) - m_-(k) \end{aligned}$$

As W is assumed to be infinite, the set V_k is non-empty and finite, so $m_+(k)$, $m_-(k)$ and $m(k)$ are well-defined real numbers for all $k \in \mathbb{N}_0$. We further write $c := \min_{i \in I} \langle \alpha_i, h \rangle \in \mathbb{R}_{>0}$.

Claim 1. If $w \in W$ and $i \in I$ satisfy $l(w) < l(s_i w)$, then $\langle \eta, s_i w(h) \rangle \leq \langle \eta, w(h) \rangle - c$.

Proof of Claim. Since $w^{-1}(\alpha_i) \in \Phi_+$ by (2.5.4), we have $\langle \alpha_i, w(h) \rangle = \langle w^{-1}(\alpha_i), h \rangle \geq c$ and so

$$\langle \eta, s_i w(h) \rangle = \langle \eta, w(h) \rangle - \langle \alpha_i, w(h) \rangle \langle \eta, \alpha_i^\vee \rangle = \langle \eta, w(h) \rangle - \langle \alpha_i, w(h) \rangle \leq \langle \eta, w(h) \rangle - c.$$

Claim 2. For every $k \in \mathbb{N}_0$ and $R \in \mathbb{R}_{\geq 0}$, there exists some $k' \in \mathbb{N}$ with $m_+(k') < m_-(k) - R$.

Proof of Claim. Let $k' \geq k$ and $w \in V_{k'}$ be arbitrary and write $w = w_1 w_2$ with $l(w_1) = k' - k$ and $l(w_2) = k$. Iterating the computation in Claim 1 shows that

$$\langle \eta, w(h) \rangle \leq \langle \eta, w_2(h) \rangle - (k' - k)c \leq m_+(k) - (k' - k)c = m_-(k) + m(k) - (k' - k)c.$$

The right-hand side does not depend on w , so we conclude that

$$m_+(k') \leq m_-(k) + m(k) - (k' - k)c.$$

If we choose $k' \in \mathbb{N}$ so large that $(k' - k)c > m(k) + R$, the claim is satisfied.

Claim 3. Given any $k \in \mathbb{N}_0$, every proper face F of P_k either is the intersection of P_k with a proper face of Ξ_h or there exists a subset $U \subset V_k$ such that $F = \text{conv}(U \cdot h)$.

Proof of Claim. The statement is vacuous if $k = 0$ since $P_0 = \{h\}$ is a vertex of Ξ_h , so we may assume that $k \geq 1$. By Lemma 4.2.6, every proper face F of P_k is of the form $F = \text{conv}(U \cdot h)$ for some proper subset $U \subsetneq U_k$.⁴ If $U \subset wW_J$ for some $w \in W$ and $J \subsetneq I$, then F is contained in the proper face $w(F_J) = \text{conv}(wW_J \cdot h)$ of Ξ_h . In this case, we assert that if $wW_J \subsetneq W$ is the smallest standard coset with this property, then $F = w(F_J) \cap P_k$.

This is proved as follows: On the one hand, it is clear that $w(F_J) \cap P_k \neq \emptyset$ defines a face of P_k . On the other hand, F has to intersect the relative interior of $w(F_J)$ since otherwise, it would be contained in a proper face of $w(F_J)$ which is of the form $ww'(F_{J'})$ for some $w' \in W_J$ and $J' \subsetneq J$ by Theorem 4.3.21, but this would only be possible if $U \subset ww'W_{J'} \subsetneq wW_J$ was contained in a smaller standard coset. Hence, the relative interiors of F and $w(F_J) \cap P_k$ meet, so the two faces must coincide and the assertion holds.

Suppose now that U is not contained in any non-trivial standard left coset of (W, S) and fix an arbitrary element $u \in U$. For every $i \in I$, there exists some $u_i \in U$ such that $u_i \notin uW^{(i)}$ or, equivalently, $u^{-1}u_i \notin W^{(i)}$ and thus $i \in I(u^{-1}u_i)$. Since $I_h = \emptyset$, it follows from Lemma 4.3.5 that $h - u^{-1}u_i(h) \in \mathfrak{a}'$ is a linear combination of the simple coroots in which all coefficients are non-negative and that of α_i^\vee is positive. Now consider the convex combination

$$h' := \sum_{i=1}^n \frac{1}{n} u^{-1}u_i(h) = u^{-1} \left(\sum_{i=1}^n \frac{1}{n} u_i(h) \right) \in u^{-1}(F) \subset u^{-1}(P_k).$$

Then we have

$$h - h' = \sum_{i=1}^n \frac{1}{n} (h - u^{-1}u_i(h)),$$

so $h - h' \in \mathfrak{a}'$ is a linear combination of the simple coroots in which all coefficients are positive. By Lemma 4.3.19, the open line segment (h, h') is contained in the relative interior of Ξ_h . More precisely, if $l(u) < k$, the proof of Lemma 4.3.19 actually shows that this line segment runs through the relative interior of $u^{-1}(P_k)$. Indeed, in the notation of that proof, we have only used convex combinations of $\{h, h', s_i(h) \mid i \in I\}$ to create an open neighbourhood of an arbitrary point $p \in (h, h')$ that is contained in Ξ_h . If $l(u) < k$, then $l(us_i) \leq l(u) + 1 \leq k$ and therefore $s_i(h) = u^{-1}us_i(h) \in u^{-1}(P_k)$ for every $i \in I$. Thus, the line segment $(u(h), u(h')) \subset F$ lies in the relative interior of P_k , contradicting the assumption that $F \subset \partial_{\text{rel}}(P_k)$ is a proper face. As $u \in U$ was arbitrary, F can only be a proper face of P_k if $l(u) = k$ for all $u \in U$, i.e. $U \subset V_k$.

Claim 4. If $k \in \mathbb{N}_0$ and $q \in \Xi_h \setminus P_k$, then $\langle \eta, q \rangle < m_+(k)$.

Proof of Claim. The claim clearly holds if $k = 0$ because $P_0 = \{h\}$ and every point $q \in \Xi_h \setminus \{h\}$ satisfies $\langle \eta, q \rangle < \langle \eta, h \rangle = m_+(0)$ by Lemma 4.3.5, so we may assume that $k \geq 1$. Consider the closed line segment $L := [h, q] \subset \Xi_h$ connecting h and q . Since $h \in P_0 \subset P_k$ and $q \notin P_k$, there exists a supporting hyperplane E of P_k whose supporting half-space contains h but not q , which

⁴As every face of a convex polytope is exposed, the second assertion of Lemma 4.2.6 is actually sufficient.

implies that L meets the proper face $F := P_k \cap E$ in a point $x \in [h, q]$. If F was the intersection of P_k with a face of Ξ_h , we could choose E to be a supporting hyperplane of Ξ_h , but then $q \in \Xi_h$ would require that q and h were both contained in its supporting half-space. Thus, F is of the form $F = \text{conv}(U \cdot h)$ for some $U \subset V_k$ by Claim 3. Since $\langle \eta, u(h) \rangle \leq m_+(k)$ for all $u \in U$ and x is a convex combination of these vertices, we have $\langle \eta, x \rangle \leq m_+(k)$. In addition, as $k \geq 1$, Claim 1 yields that $m_+(k) < \langle \eta, h \rangle$, so $x \neq h$ and η is strictly monotonically decreasing in the direction from h to x . Because x lies strictly between h and q , it follows that $\langle \eta, q \rangle < \langle \eta, x \rangle \leq m_+(k)$.

Claim 5. Let $k, k' \in \mathbb{N}_0$ be such that $k' > k$ and $R' := m_-(k) - m_+(k') > 0$, then $\|p - q\|_1 > R'$ for all $p \in P_k$ and $q \in \Xi_h \setminus P_{k'}$.

Proof of Claim. On the one hand, $p \in P_k$ is a convex combination of W -translates of h of length at most k . As m_- is monotonically decreasing by Claim 1, the value of η at each of these points is at least $m_-(k)$, so that $\langle \eta, p \rangle \geq m_-(k)$. On the other hand, we have $\langle \eta, q \rangle < m_+(k')$ by Claim 4, so (4.3.16) implies that

$$\|p - q\|_1 \geq |\langle \eta, p \rangle - \langle \eta, q \rangle| > m_-(k) - m_+(k') = R'.$$

Conclusion of the proof. A convergent sequence is bounded, so we find $R \in \mathbb{R}_{>0}$ such that $(y_l)_{l \in \mathbb{N}}$ is contained in a closed ball of radius R (w.r.t. the ℓ^1 -norm on $E_h \cong \mathbb{R}^n$) around y_1 . In addition, for every $l \in \mathbb{N}$, the point $y_l \in \Xi_h$ is a finite convex combination of W -translates of h , so that $y_l \in P_{k(l)}$ for some $k(l) \in \mathbb{N}_0$. We set $k := k(1)$. By Claim 2, there exists some $k' > k$ with $R' := m_-(k) - m_+(k') \geq R$. Since $\|y_1 - y_l\|_1 \leq R \leq R'$ for all $l \in \mathbb{N}$, Claim 5 implies that the sequence $(y_l)_{l \in \mathbb{N}}$ must be contained in $P_{k'}$, as required. \square

Combining this topological property with the combinatorial description of the face poset of Ξ_h leads to an elegant characterization which implies that Ξ_h is completely determined by its affine span and its proper faces.

Corollary 4.3.25. *If $h \in C$, then a point $h' \in \mathfrak{a}$ is contained in Ξ_h if and only if it satisfies the following (in-)equalities:*

$$\begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W \text{ and } i \in I \\ \langle \phi_i, h' \rangle = \langle \phi_i, h \rangle & \text{for all } i \in \{n+1, \dots, d\} \end{cases} \quad (4.3.17)$$

Proof. Since $\Xi_h \subset \mathfrak{a}$ is a closed convex set, it follows from the discussion in Section 4.2 that it coincides with the intersection of all its supporting half-spaces. On the one hand, its affine span E_h is a trivial supporting hyperplane and the condition $h' \in E_h$ is equivalent to $h - h' \in \mathfrak{a}'$ and hence to $\langle \phi_i, h' \rangle = \langle \phi_i, h \rangle$ for all $i \in \{n+1, \dots, d\}$. On the other hand, every proper face of Ξ_h is contained in a facet, so inside E_h , it even suffices to intersect its defining half-spaces. Now every facet is of the form $w^{-1}(F^{(i)}) = w^{-1}(E^{(i)}) \cap \Xi_h$ for some $w \in W$ and $i \in I$, where $w^{-1}(E^{(i)}) = \{h' \in E_h : \langle \omega_i, w(h') \rangle = \langle \omega_i, h \rangle\}$, and the corresponding defining half-space is the “ \leq -side” of this plane. Thus, another point $h' \in \mathfrak{a}$ is contained in Ξ_h if and only if it lies in E_h and satisfies the inequalities $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ for all $w \in W$ and $i \in I$. \square

Remark 4.3.26. There is a lot of redundancy in the conditions of Corollary 4.3.25. For instance, given any $w \in W$ and $i \in I$, the inequality $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ implies $\langle \omega_i, w'w(h') \rangle \leq \langle \omega_i, h \rangle$ for all $w' \in W^{(i)}$ because this standard subgroup is the stabilizer of ω_i . Thus, it would suffice to require the first inequality in (4.3.17) to hold whenever $w \in W$ is left $I^{(i)}$ -reduced. For our purposes, however, it will not be necessary to find a minimal set of defining inequalities for Ξ_h .

Given any subset $J \subset I$, the entire discussion of this subsection also applies to the J -restriction \mathcal{D}_J of \mathcal{D} and every \mathcal{D}_J -regular element $\tilde{h} \in C_J$. For ease of reference, we state the following more general variant of Corollary 4.3.25 explicitly:

Corollary 4.3.27. *Let $J \subset I$ and $\tilde{h} \in C_J$ be arbitrary, then another point $\tilde{h}' \in \mathfrak{a}$ is contained in the set $\text{conv}(W_J \cdot \tilde{h}) \subset \mathfrak{a}$ if and only if it lies in $\tilde{h} + \mathfrak{a}'_J \subset \mathfrak{a}$ and satisfies $\langle \omega_j, w(\tilde{h}') \rangle \leq \langle \omega_j, \tilde{h} \rangle$ for all $w \in W_J$ and $j \in J$. \square*

The virtually singular case

Having successfully dealt with regular elements, we now want to extend the previous results to the more general virtually singular case. Thus, throughout this subsection, we assume that $h \in \overline{C}$ satisfies $|W_h| < \infty$ or, equivalently, that h is contained in a spherical cell of \overline{C} and hence in the topological interior X_+° of the Tits cone $X_+ \subset \mathfrak{a}$. This property will allow us to perturb h to a regular element in a controlled way and to show directly that Ξ_h is an intersection of closed half-spaces. The following result is a generalization of Corollary 4.3.25:

Proposition 4.3.28. *If $h \in \overline{C}$ with $|W_h| < \infty$, then a point $h' \in \mathfrak{a}$ is contained in Ξ_h if and only if it satisfies the following (in-)equalities:*

$$\begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W_{\text{vis}} \text{ and } i \in I_{\text{vis}} \\ \langle \omega_i, w(h') \rangle = \langle \omega_i, h \rangle & \text{for all } w \in W_{\text{inv}} \text{ and } i \in I_{\text{inv}} \\ \langle \phi_i, h' \rangle = \langle \phi_i, h \rangle & \text{for all } i \in \{n+1, \dots, d\} \end{cases} \quad (4.3.18)$$

Proof. It follows from the proof of Proposition 4.3.6 and W -invariance of Ξ_h that (4.3.18) holds for every point in Ξ_h , so it suffices to show the converse. If $I'_h := (I_{\text{vis}})_h = I_{\text{vis}} \cap I_h = \emptyset$, then I_h is a union of connected components of I and every visible vertex of Δ is white. Equivalently, h is contained in the open fundamental $\mathcal{D}_{I_{\text{vis}}}$ -chamber $C_{I_{\text{vis}}} \subset \mathfrak{a}$ and the equalities in (4.3.18) imply that h' lies in the affine subspace $E_h = h + \mathfrak{a}'_{\text{vis}} \subset \mathfrak{a}$. Thus, we conclude from the inequalities and Corollary 4.3.27 that $h' \in \text{conv}(W_{\text{vis}} \cdot h) = F_{I_{\text{vis}}} = \Xi_h$.

Thus, we may suppose that $k := |I'_h| > 0$. First, we establish that (4.3.18) implies that $h' \in X_+$. Let $\mathbb{B} := \mathbb{A}_{I'_h} \in \mathbb{Z}^{k \times k}$ denote the submatrix of \mathbb{A} with rows and columns indexed by I'_h . Since $W(\mathbb{B}) = W_{I'_h} \subset W_h$ is finite, every indecomposable component of \mathbb{B} (and \mathbb{B}^T) is of finite type, so there exists a vector $x = (x_1, \dots, x_k)^T \in \mathbb{R}^k$ with $x > 0$ and $\mathbb{B}^T x > 0$. Equivalently, if we write $I'_h = \{i_1, \dots, i_k\}$ with $1 \leq i_1 < \dots < i_k \leq n$, the vector $v := \sum_{r=1}^k x_r \alpha_{i_r}^\vee \in \mathfrak{a}'_{I'_h}$ satisfies $\langle \alpha_i, v \rangle > 0$ for all $i \in I'_h$. More generally, for every $i \in I$, one of the following three conditions holds:

- If $i \in I \setminus I_h$, then $\langle \alpha_i, h \rangle > 0$ and $\langle \alpha_i, v \rangle \leq 0$.
- If $i \in I'_h = I_h \cap I_{\text{vis}}$, then $\langle \alpha_i, h \rangle = 0$ and $\langle \alpha_i, v \rangle > 0$.
- If $i \in I_h \setminus I'_h = I_{\text{inv}}$, then $\langle \alpha_i, h \rangle = 0$ and $\langle \alpha_i, v \rangle = 0$.

Hence, if $\varepsilon \in \mathbb{R}_{>0}$ is chosen sufficiently small, then $\tilde{h} := h + \varepsilon v \in E_h$ satisfies $\langle \alpha_i, \tilde{h} \rangle > 0$ for all $i \in I_{\text{vis}} = (I \setminus I_h) \sqcup I'_h$ and is thus contained in the open fundamental $\mathcal{D}_{I_{\text{vis}}}$ -chamber $C_{I_{\text{vis}}} \subset \mathfrak{a}$. As $\langle \alpha_i, \tilde{h} \rangle = 0$ for all $i \in I_{\text{inv}}$, this point still lies in the closed fundamental \mathcal{D} -chamber $\overline{C} \subset \mathfrak{a}$. In particular, the convex hull of its W -orbit is a subset of X_+ . Moreover, it follows from (4.3.18) that h' is contained in $E_h = h + \mathfrak{a}'_{\text{vis}} = \tilde{h} + \mathfrak{a}'_{\text{vis}} \subset \mathfrak{a}$ and fulfils

$$\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle \leq \langle \omega_i, \tilde{h} \rangle \quad \text{for all } w \in W_{\text{vis}} \text{ and } i \in I_{\text{vis}}.$$

We now conclude that $h' \in \text{conv}(W_{\text{vis}} \cdot \tilde{h}) \subset X_+$ by applying Corollary 4.3.27 to the set I_{vis} and the point $\tilde{h} \in C_{I_{\text{vis}}}$. Because the (in-)equalities in (4.3.18) also hold for every W -translate of h' , we may from now on assume that $h' \in \overline{C}$ and it remains to show that $h' \in \Xi_h$.

To this end, we claim that the set $J := \{i \in I : \langle \omega_i, h' \rangle < \langle \omega_i, h \rangle\} \subset I$ is visible. Suppose to the contrary that there exists a connected component J' of J that is contained in I_h and write

$$h - h' = \sum_{j \in J \setminus J'} y_j \alpha_j^\vee + \sum_{j \in J'} y_j \alpha_j^\vee \quad \text{with } y_j \in \mathbb{R}_{>0} \text{ for all } j \in J.$$

Let w_+ be the longest word of the finite group $W_{J'} \subset W_h$, then $w_+ s_j w_+ = s_{\sigma(j)}$ for all $j \in J'$, where the map $j \mapsto \sigma(j)$ defines a diagram automorphism of $\Delta_{J'}$. Together with $l(w_+ s_j) < l(w_+)$, this implies that $w_+(\alpha_j^\vee) = -\alpha_{\sigma(j)}^\vee$ for all $j \in J'$. In addition, since $W_{J'}$ commutes with $W_{J \setminus J'}$, we also have $w_+(\alpha_j^\vee) = \alpha_j^\vee$ for all $j \in J \setminus J'$. Combining these observations leads to

$$\langle \omega_{\sigma(j)}, h - w_+(h') \rangle = \langle \omega_{\sigma(j)}, w_+(h - h') \rangle = -y_j < 0 \quad \text{for all } j \in J'.$$

This is a contradiction to (4.3.18), so the claim is proved. Hence, $J = J_{\text{vis}}$ and $\langle \omega_j, h' \rangle < \langle \omega_j, h \rangle$ for all $j \in J$, so Lemma 4.3.16 states that there is a point $p \in F_J \subset \Xi_h$ with $\langle \omega_j, h' \rangle < \langle \omega_j, p \rangle$ and $\langle \alpha_j, p \rangle > 0$ for all $j \in J$. It follows that p is contained in the open fundamental \mathcal{D}_J -chamber $C_J \subset \mathfrak{a}$ while h' lies in the affine subspace $h + \mathfrak{a}'_J = p + \mathfrak{a}'_J \subset \mathfrak{a}$ and satisfies

$$\langle \omega_j, w(h') \rangle \leq \langle \omega_j, h' \rangle < \langle \omega_j, p \rangle \quad \text{for all } w \in W_J \text{ and } j \in J,$$

where the first inequality holds by (2.6.7) and the assumption that $h' \in \overline{C}$. Therefore, applying Corollary 4.3.27 to the set J and the point $p \in C_J$ shows that $h' \in \text{conv}(W_J \cdot p) \subset F_J \subset \Xi_h$. \square

As in the regular case, the equalities in the third line of (4.3.18) are equivalent to the assertion $h' \in h + \mathfrak{a}'$ and those in the second and third line together to $h' \in h + \mathfrak{a}'_{\text{vis}} = E_h$, which implies that $w(h') = h'$ for all $w \in W_{\text{inv}}$. In order to apply this characterization of Ξ_h more easily later, we establish that the equalities in the second line are consequences of the other two.

Lemma 4.3.29. *Let $h \in \overline{C}$ with $|W_h| < \infty$ and $h' \in h + \mathfrak{a}' \subset \mathfrak{a}$ be arbitrary, then the following conditions are equivalent:*

- (i) $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ for all $w \in W_{\text{vis}}$ and $i \in I_{\text{vis}}$ and $\langle \omega_i, w(h') \rangle = \langle \omega_i, h \rangle$ for all $w \in W_{\text{inv}}$ and $i \in I_{\text{inv}}$.
- (ii) $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ for all $w \in W$ and $i \in I$.

Proof.

- (i) \implies (ii): Choosing $w = 1$ in the equalities of (i) shows that $h' \in h + \mathfrak{a}'_{\text{vis}} = E_h$. Let $w \in W$ be arbitrary, then there exist unique elements $w_1 \in W_{\text{vis}}$ and $w_2 \in W_{\text{inv}}$ such that $w = w_1 w_2$. The latter acts trivially on E_h , so given any $i \in I$, we find that $\langle \omega_i, w(h') \rangle = \langle \omega_i, w_1(h') \rangle$. If $i \in I_{\text{vis}}$, then $\langle \omega_i, w_1(h') \rangle \leq \langle \omega_i, h \rangle$ by assumption and if $i \in I_{\text{inv}}$, then $w_1^{-1}(\omega_i) = \omega_i$ and thus $\langle \omega_i, w_1(h') \rangle = \langle \omega_i, h \rangle$.
- (ii) \implies (i): The inequalities in (i) are a special case of (ii), so it suffices to prove the equalities. Let $w \in W_{\text{inv}}$ and $i \in I_{\text{inv}}$ be arbitrary. As we already have $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$, it remains to show the reverse inequality. Let w_+ be the longest word of the finite group $W_{\text{inv}} \subset W_h$. By the same argument as in the proof of Proposition 4.3.28, it satisfies $w_+(\alpha_j^\vee) = \alpha_j^\vee$ if $j \in I_{\text{vis}}$ and $w_+(\alpha_j^\vee) = -\alpha_{\sigma(j)}^\vee$ if $j \in I_{\text{inv}}$, where $j \mapsto \sigma(j)$ defines an automorphism of $\Delta_{I_{\text{inv}}}$ of order two. Thus, the functionals $w_+(\omega_i)$ and $-\omega_{\sigma(i)}$ coincide on the linear subspace $\mathfrak{a}' \subset \mathfrak{a}$, so if we set $c := \langle w_+(\omega_i) + \omega_{\sigma(i)}, h \rangle \in \mathbb{R}$, then $w_+(\omega_i)$ is equal to $-\omega_{\sigma(i)} + c$ on the affine subspace $h + \mathfrak{a}' \subset \mathfrak{a}$. Since $w_+(h) = h$, we may conclude the proof as follows:

$$\begin{aligned} \langle \omega_i, w(h') \rangle &= \langle w_+(\omega_i), w_+ w(h') \rangle = -\langle \omega_{\sigma(i)}, w_+ w(h') \rangle + c \\ &\geq -\langle \omega_{\sigma(i)}, h \rangle + c = -\langle \omega_{\sigma(i)}, w_+(h) \rangle + c \\ &= \langle w_+(\omega_i), w_+(h) \rangle = \langle \omega_i, h \rangle \end{aligned} \quad \square$$

The following result is now an immediate consequence of Proposition 4.3.28 and Lemma 4.3.29. It shows that Corollary 4.3.25 is indeed true for all virtually singular elements.

Corollary 4.3.30. *If $h \in \overline{C}$ with $|W_h| < \infty$, then a point $h' \in \mathfrak{a}$ is contained in Ξ_h if and only if it satisfies the following (in-)equalities:*

$$\begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W \text{ and } i \in I \\ \langle \phi_i, h' \rangle = \langle \phi_i, h \rangle & \text{for all } i \in \{n+1, \dots, d\} \end{cases} \quad \square$$

In addition, it allows us to extend Proposition 4.3.20 and 4.3.24 to the virtually singular case.

Corollary 4.3.31. *If $h \in \overline{C}$ with $|W_h| < \infty$, then Ξ_h is closed and for every $F \in \mathcal{F}(\Xi_h)$, there exists a standard coset $wW_J \subset W$ such that $F = w(F_J)$. In particular, every face of Ξ_h is exposed.*

Proof. Corollary 4.3.30 shows that Ξ_h is the intersection of closed half-spaces in \mathfrak{a} and therefore closed. In order to prove the claim about its faces, we argue by induction on $k := |I_{\text{vis}}| = \dim(\Xi_h)$.

If $k = 0$, then $\Xi_h = \{h\} = F_\emptyset$ does not have any proper faces, so we may assume that $k > 0$.⁵ Let F be a face of Ξ_h , choose a point $x \in \text{relint}(F) \subset X_+$ and $w_0 \in W$ such that $x' := w_0(x) \in \overline{C}$, then there are two possibilities:

- If $\langle \omega_i, x' \rangle < \langle \omega_i, h \rangle$ for all $i \in I_{\text{vis}}$, then the assumption that $x' \in \overline{C}$ and (2.6.7) show that $\langle \omega_i, w(x') \rangle \leq \langle \omega_i, x' \rangle < \langle \omega_i, h \rangle$ for all $w \in W_{\text{vis}}$ and $i \in I_{\text{vis}}$. Consequently, Proposition 4.3.28 implies that x' does not lie on any non-trivial supporting hyperplane of Ξ_h and is therefore contained in its relative interior. But this is only possible if $w_0(F) = \Xi_h$ is the trivial face, so that $F = \Xi_h = F_I = F_{I_{\text{vis}}}$ holds as well.
- Otherwise, there exists some $i \in I_{\text{vis}}$ with $\langle \omega_i, x' \rangle = \langle \omega_i, h \rangle$ and hence $x' \in \Xi_h \cap E^{(i)} = F^{(i)}$ by Proposition 4.3.6. As $F^{(i)}$ meets the relative interior of $w_0(F)$, it contains the entire set, so that $w_0(F)$ is a face of $F^{(i)}$. Since $i \in I_{\text{vis}}$, we have $\dim(F^{(i)}) < k$ by Proposition 4.3.2 (ii). Thus, $F^{(i)}$ is a proper face of Ξ_h and the induction hypothesis yields that $w_0(F) = w'(F_J)$ for some standard coset $w'W_J \subset W^{(i)}$ and $F = w(F_J)$ for $w := w_0^{-1}w' \in W$. \square

Corollary 4.3.31 implies that the order-preserving map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$ is surjective and we have determined in Corollary 4.3.14 under which conditions two standard left cosets of (W, S) give rise to the same face of Ξ_h . This leads to a completely combinatorial description of $\mathcal{F}(\Xi_h)$. We define an equivalence relation \sim on $\Sigma^*(W, S)$ by

$$w_1W_J \sim w_2W_K : \Longleftrightarrow J_{\text{vis}} = K_{\text{vis}} \text{ and } W_Jw_1^{-1}w_2W_K \cap W_{J_{\text{vis}}}^* \neq \emptyset \quad (4.3.19)$$

and we set $\tilde{\Sigma}^*(W, S) := \Sigma^*(W, S)/\sim$. Equivalently, it holds that $w_1W_J \sim w_2W_K$ if and only if $\psi(w_1W_J) = \psi(w_2W_K)$, so ψ descends to a poset isomorphism $\tilde{\psi} : \tilde{\Sigma}^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$. Moreover, the second condition in (4.3.19) is satisfied if and only if the unique element of minimal length in $W_Jw_1^{-1}w_2W_K$ is contained in $W_{J_{\text{vis}}}^*$. Alternatively, it is also equivalent to $w_1^{-1}w_2 \in W_{J_{\text{vis}}}^* \times W_{J_{\text{vis}}}$. The following result is a generalization of Theorem 4.3.21 and summarizes the discussion:

Theorem 4.3.32. *If $h \in \overline{C}$ with $|W_h| < \infty$, the map $\tilde{\psi} : \tilde{\Sigma}^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$, $[wW_J] \mapsto w(F_J)$, is a well-defined W -equivariant poset isomorphism. More precisely, for every standard left coset $wW_J \subset W$, the set $w(F_J) = \text{conv}(wW_J \cdot h)$ is an exposed face of Ξ_h of dimension $|J_{\text{vis}}|$. Every face is of this form for some standard left coset of (W, S) and*

$$w_1(F_J) = w_2(F_K) \Longleftrightarrow w_1W_J \sim w_2W_K \text{ for all } w_1W_J, w_2W_K \subset W. \quad \square$$

Example 4.3.33. We illustrate the statement of Theorem 4.3.32 in the concrete example of the GCM of finite type C_3 :

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \Longleftrightarrow \Gamma(\mathbb{A}) = \bullet \text{---} \bullet \text{---} \bullet \LongRightarrow \Delta = \bullet \text{---} \bullet \text{---}^4 \bullet$$

Its Weyl group $W = W(\mathbb{A})$ is isomorphic to the signed permutation group \tilde{S}_3 on three letters. Since \mathbb{A} is invertible, the simply connected Kac-Moody root datum $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is free and cofree. Let $\{e_1, e_2, e_3\}$ denote the standard basis of \mathbb{R}^3 and $\{e_1^*, e_2^*, e_3^*\}$ the corresponding coordinate functionals. If we identify $\mathfrak{a} = \Lambda_{\text{sc}}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with \mathbb{R}^3 by mapping $\mathcal{B} = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\} \subset \mathfrak{a}$ to the basis $\{e_1 - e_2, e_2 - e_3, e_3\} \subset \mathbb{R}^3$, the simple roots $\alpha_1, \alpha_2, \alpha_3 \in \Phi_0 \subset \mathfrak{a}^*$ turn into the linear functionals $e_1^* - e_2^*$, $e_2^* - e_3^*$ and $2e_3^*$. Under this identification, the \mathcal{D} -representation of W on \mathfrak{a} coincides with the natural action of \tilde{S}_3 on \mathbb{R}^3 and the simple reflections $s_1, s_2, s_3 \in S$ are the maps $(x, y, z) \mapsto (y, x, z)$, $(x, y, z) \mapsto (x, z, y)$ and $(x, y, z) \mapsto (x, y, -z)$. Moreover, the closed fundamental chamber is given by

$$\overline{C} = \{(x, y, z) \in \mathbb{R}^3 : x \geq y \geq z \geq 0\}.$$

⁵In contrast to Proposition 4.3.20, the situation $k = 0$ is also possible if $n > 0$ and \mathbb{A} is a “non-empty” GCM because h might be a global fixed-point for the W -action on \mathfrak{a} . As h is assumed to be virtually singular, this is only possible if W is finite or, equivalently, \mathbb{A} is spherical, but we have never excluded this case from our discussion.

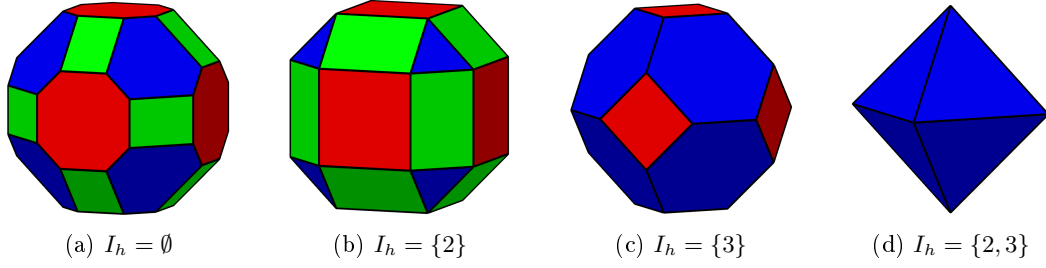
Figure 4.2: Four possible shapes of Ξ_h

Figure 4.2 depicts four possible configurations of the set Ξ_h . In 4.2a, we have chosen the regular element $h = (4, 2, 1) \in C \subset \overline{C}$ for which $I^\bullet = I_h = \emptyset$ and all vertices of $\Delta = \circ \text{---} \circ \text{---} \circ^4 \text{---} \circ$ are white. In this case, Ξ_h has 48 vertices that are in bijective correspondence with the elements of $W \cong \tilde{S}_3$. The two-dimensional faces come in three shapes:

- The six red octagons are the faces of type $I^{(1)} = \{2, 3\}$, i.e. the sets having the form $w(F^{(1)}) = \text{conv}(wW^{(1)} \cdot h)$ for some $w \in W$, where the standard subgroup $W^{(1)} \subset W$ is isomorphic to the dihedral group D_8 and its Coxeter diagram $\circ \text{---} \circ^4$ is the subgraph of Δ obtained by removing the vertex on the left.
- The twelve green rectangles are the faces of type $I^{(2)} = \{1, 3\}$, i.e. the sets having the form $w(F^{(2)}) = \text{conv}(wW^{(2)} \cdot h)$ for some $w \in W$, where the standard subgroup $W^{(2)} \subset W$ is isomorphic to the product $\mathbb{Z}_2 \times \mathbb{Z}_2$ and its Coxeter diagram $\circ \text{---} \circ$ is the subgraph of Δ obtained by removing the vertex in the middle.
- The eight blue hexagons are the faces of type $I^{(3)} = \{1, 2\}$, i.e. the sets having the form $w(F^{(3)}) = \text{conv}(wW^{(3)} \cdot h)$ for some $w \in W$, where the standard subgroup $W^{(3)} \subset W$ is isomorphic to the symmetric group S_3 and its Coxeter diagram $\circ \text{---} \circ$ is the subgraph of Δ obtained by removing the vertex on the right.

There are also three types of edges, each occurring 24 times, that we have not distinguished by distinct colours. In this situation, it is apparent that the faces of Ξ_h correspond bijectively to the standard left cosets of (W, S) , as already claimed by Theorem 4.3.21.

In Figure 4.2b, we have used the virtually singular point $h = (4, 2, 2) \in D_{\{2\}} \subset \overline{C}$ whose stabilizer is the standard subgroup $W_{\{2\}} \cong \mathbb{Z}_2$. Hence, the Coxeter diagram $\Delta = \circ \text{---} \bullet \text{---} \circ^4$ is coloured by the set $I^\bullet = I_h = \{2\}$. In this case, Ξ_h only has $[W : W_{\{2\}}] = 24$ vertices and the two-dimensional faces look differently. If we remove the first vertex from Δ , we obtain the diagram $\bullet \text{---} \circ^4$ of D_8 , where one vertex is now black and the corresponding red W -translates of $F^{(1)}$ become rectangles instead of octagons. Likewise, erasing the third vertex results in the diagram $\circ \text{---} \bullet$ of S_3 and this causes the blue W -translates of $F^{(3)}$ to turn from hexagons into triangles. Omitting the second vertex of Δ , however, leads to the diagram $\circ \text{---} \circ$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ where both vertices are white, which is why the green W -translates of $F^{(2)}$ occur as rectangles as in the regular case. Since the red and green rectangles do not have the same type, we again have three types of two-dimensional faces, whereas the edges now only come in types $\{1\}$ and $\{3\}$, each appearing 24 times, because $F_{\{2\}} = \text{conv}(W_{\{2\}} \cdot h) = \{h\}$ has collapsed to a point. In fact, all vertices of Δ are non-degenerate with respect to this colouring and $\{2\}$ is the only subset of I that is not visible.

Figure 4.2c is a visualization of Ξ_h for the point $h = (4, 2, 0) \in D_{\{3\}} \subset \overline{C}$, where $\Delta = \circ \text{---} \circ \text{---} \bullet$ is coloured by $I^\bullet = I_h = \{3\}$. As in the previous case, Ξ_h has 24 vertices and removing the left node of Δ explains the appearance of the red rectangles. If we omit the one on the right, the resulting diagram $\circ \text{---} \circ$ of S_3 only has white vertices, so the blue hexagons occur as in the regular case. Taking out the central vertex, on the other hand, leads to the diagram $\circ \text{---} \bullet$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$ having an isolated black node, which causes the face $F^{(2)} = \text{conv}(W^{(2)} \cdot h) = [h, s_1(h)] = F_{\{1\}}$ to become one-dimensional. Thus, all the green faces have collapsed to edges and there are only two types of two-dimensional faces. This reflects the fact that $I_{\text{nd}} = \{1, 3\}$ and $I_{\text{deg}} = \{2\}$ because the sets $I^{(1)} = \{2, 3\}$ and $I^{(3)} = \{1, 2\}$ are visible with respect to this colouring but $I^{(2)} = \{1, 3\}$ is not. The edges also only come in the two types $\{1\}$ and $\{2\}$ since $\{3\}$ is not visible.

Finally, Figure 4.2d depicts the set Ξ_h for the virtually singular point $h = (2, 0, 0) \in D_{\{2,3\}} \subset \overline{C}$, where $\Delta = \circ \text{---} \overset{4}{\bullet} \text{---} \bullet$ is coloured by $I^\bullet = I_h = \{2, 3\}$. In this case, Ξ_h only has $[W : W_{\{2,3\}}] = 6$ vertices. If we remove the left vertex of Δ , the resulting diagram $\bullet \text{---} \overset{4}{\bullet}$ of D_8 is entirely black, so the face $F^{(1)} = \{h\}$ is a point and as before, it follows that $F^{(2)} = [h, s_1(h)] = F_{\{1\}}$ is an edge. Thus, the red and green faces have collapsed to vertices and edges, respectively. Dropping the third vertex, however, results in the diagram $\circ \text{---} \bullet$ of S_3 , which explains why the blue triangles appear exactly as in the second example. Hence, all two-dimensional faces are of type $I^{(3)} = \{1, 2\}$ and all edges are of type $\{1\}$. Equivalently, the only non-trivial subsets of I that are visible with respect to this colouring are $I^{(3)} = \{1, 2\}$ and $\{1\}$, so that $I_{\text{nd}} = \{3\}$ and $I_{\text{deg}} = \{1, 2\}$.

If the group W is finite, it is clear that $\mathcal{F}(\Xi_h)$ is finite as well. Conversely, if W is infinite, the assumption that $|W_h| < \infty$ implies that $\mathcal{F}(\Xi_h)$ contains infinitely many vertices and therefore infinitely many faces. However, as in the regular case, we want to show that every face of Ξ_h is only contained in finitely many other faces and determine this number explicitly. In contrast to the situation in Corollary 4.3.22, this is more difficult if W_h is non-trivial as the faces do not correspond bijectively to the standard left cosets of (W, S) and the equivalence relation (4.3.19) has to be incorporated.

Let F be an arbitrary face of Ξ_h and let $K := \tau(F) \in \mathcal{P}_{\text{vis}}(I)$ be its type. If F' is another face of Ξ_h containing F , then $K \subset \tau(F') \in \mathcal{P}_{\text{vis}}(I)$. Thus, in order to determine the total number of such faces, it suffices to compute the number of faces of type J containing F for every $J \in \mathcal{P}_{\text{vis}}(I)$ with $K \subset J$. Since the W -action on $\mathcal{F}(\Xi_h)$ is type-preserving and transitive on faces of the same type, these values can be calculated using the orbit-stabilizer formula. The details are given in the result below. To simplify the notation, we define

$$\iota_K^J := [W_{K^*} : W_{K^* \cap (J \cup J^*)}] \quad \text{for all } J, K \in \mathcal{P}_{\text{vis}}(I) \text{ with } K \subset J. \quad (4.3.20)$$

This is always a well-defined natural number because W_{K^*} is contained in the finite group W_h and it is clear that $\iota_K^K = \iota_K^{I_{\text{vis}}} = 1$ for all $K \in \mathcal{P}_{\text{vis}}(I)$. Recall from Corollary 4.3.14 and 4.3.17 that W_{K^*} is precisely the subgroup of W that fixes the face F_K of Ξ_h point-wise and $W_{K^* \cap (J \cup J^*)}$ is the subgroup of W_{K^*} which also stabilizes the larger face $F_J \supset F_K$.

Proposition 4.3.34. *Let $J, K \in \mathcal{P}_{\text{vis}}(I)$ be such that $K \subset J$ and let F be a face of Ξ_h of type K . The number of faces of type J containing F is equal to ι_K^J .*

Proof. Since the W -action on $\mathcal{F}(\Xi_h)$ is type-preserving and transitive on faces of the same type, we may assume that $F = F_K$ and claim that W_{K^*} acts transitively on the faces of type J containing F_K . Suppose that $F_K \subset w(F_J)$ for some $w \in W$, then $w^{-1}(F_K) \subset F_J$ and therefore $w^{-1} \in W_J W_{K^*}$ by (4.3.15). If $u \in W_J$ and $v \in W_{K^*}$ are chosen such that $w^{-1} = uv^{-1}$, then $v(F_J) = wu(F_J) = w(F_J)$, so the claim is proved. Because the stabilizer of $F_J \supset F_K$ under the action of W_{K^*} is $W_{K^* \cap (J \cup J^*)}$, the faces of type J containing F_K are in bijective correspondence with the left cosets of $W_{K^* \cap (J \cup J^*)}$ in W_{K^*} , so their number is equal to the index ι_K^J . \square

With the previous proposition at hand, it is now straightforward to generalize Corollary 4.3.22 to the virtually singular case.

Corollary 4.3.35. *If $h \in \overline{C}$ with $|W_h| < \infty$, then every face of Ξ_h only lies in finitely many other faces. More precisely, the following assertions hold:*

- (i) *Let F be a face of Ξ_h of type $K := \tau(F)$. For every $l \in \{|K|, \dots, |I_{\text{vis}}|\}$, the number of l -dimensional faces of Ξ_h containing F is finite and equal to*

$$\sum_{J \in \mathcal{P}_{\text{vis}}(I): K \subset J, |J|=l} \iota_K^J. \quad (4.3.21)$$

- (ii) *For every $l \in \{0, \dots, |I_{\text{vis}}|\}$, the number of l -dimensional faces containing any chosen vertex of Ξ_h is equal to*

$$\sum_{J \in \mathcal{P}_{\text{vis}}(I): |J|=l} [W_h : W_{J_h \cup J^*}]. \quad (4.3.22)$$

(iii) The underlying graph of Ξ_h is locally finite and regular with degree

$$\sum_{i \in I \setminus I_h} [W_h : C_h(s_i)], \quad (4.3.23)$$

where $C_h(s_i) := C_h(W_{\{i\}})$ denotes the centralizer of s_i in W_h .

Proof. Assertion (i) is an immediate consequence of Proposition 4.3.34 and the statement in (ii) is the special case of (i) where $K = \emptyset$. To see this, note that if $K = \emptyset$, then $K^* = I_h$ and hence $K^* \cap (J \cup J^*) = I_h \cap (J \cup J^*) = J_h \cup J^*$, which implies that $\iota_\emptyset^J = [W_h : W_{J_h \cup J^*}]$ for all $J \in \mathcal{P}_{\text{vis}}(I)$. Finally, (iii) is the special case of (ii) where $l = 1$. In fact, given any $i \in I$, the singleton $J = \{i\}$ is visible if and only if $i \in I \setminus I_h$, so that $J_h = \emptyset$. By Proposition 4.3.11, W_{J^*} coincides with the centralizer of W_J in W_h , which is simply $C_h(s_i)$ in this situation. \square

Example 4.3.36. The formulas in Corollary 4.3.35 can be verified explicitly for the polytopes in Example 4.3.33. Using the notation introduced there, let us first consider the case $I_h = \{2\}$. It is apparent from Figure 4.2b that every vertex of Ξ_h lies on four edges. This coincides with the value of (4.3.23) because $C_h(s_1) = C_h(s_3) = \{1\}$ both have index 2 in $W_h = W_{\{2\}} \cong \mathbb{Z}_2$. If we instead take $I_h = \{3\}$, then Figure 4.2c shows that every vertex lies on only three edges, which is caused by the fact that $C_h(s_1) = W_h$ has index 1 and $C_h(s_2) = \{1\}$ has index 2 in $W_h = W_{\{3\}} \cong \mathbb{Z}_2$. Finally, if $I_h = \{2, 3\}$, we observe in Figure 4.2d that every vertex again lies on four edges, but this is also consistent with that formula as $C_h(s_1) = \{1, s_3\}$ has index 4 in $W_h = W_{\{2, 3\}} \cong D_8$. Hence, Corollary 4.3.35 (iii) yields the correct numbers in these cases and the other formulas can be checked similarly.

Remark 4.3.37. Let us compare the previous results with their analogues in the regular case.

- (i) If $h \in C$ is regular, then $W_h = \{1\}$ and every subset of I is visible, so that $\iota_K^J = 1$ for all $J, K \subset I$ with $K \subset J$. Hence, if F is a face of Ξ_h of type $K := \tau(F)$ and dimension $k := |K|$, then for every subset $J \subset I$ with $K \subset J$, there exists a unique face of type J containing F . For every $l \in \{k, \dots, n\}$, the number of l -dimensional faces containing F therefore coincides with the number of subsets of I with cardinality l containing K , which is equal to $\binom{n-k}{l-k}$. The total number of faces containing F is the number of subsets of I containing K , which is equal to 2^{n-k} . In particular, these values only depend on the dimension of F and not on its type. Also, we recover that every vertex of Ξ_h is contained in precisely n edges, n facets and 2^n faces of Ξ_h , so Corollary 4.3.35 is indeed a generalization of Corollary 4.3.22. Moreover, we have already remarked that Theorem 4.3.32 reduces to Theorem 4.3.21 since the equivalence relation \sim on $\Sigma^*(W, S)$ defined in (4.3.19) is trivial in this case.
- (ii) Corollary 4.3.23 asserts that if $h \in C$ is regular, the underlying graph of Ξ_h is isomorphic to the Cayley graph of (W, S) . With this result in mind, it might be tempting to conjecture that whenever $h \in \overline{C}$ is virtually singular, the underlying graph of Ξ_h is isomorphic to the Schreier graph of (W, S) with respect to the subgroup W_h , but this naive guess is not true. Although the vertices of these graphs are both in bijective correspondence with the left cosets of W_h in W , the edges are defined differently. Let $wW_h, w'W_h \subset W$ be arbitrary:
 - In the underlying graph of Ξ_h , these cosets correspond to the vertices $w(h)$ and $w'(h)$ which are connected by an edge if and only if $w^{-1}w'(h)$ is the endpoint of an edge of Ξ_h emanating from h . Since W_h acts transitively on such edges with the same type, this is the case if and only if there exist $w_0 \in W_h$ and $i \in I \setminus I_h$ such that $w^{-1}w'(h) = w_0 s_i(h)$, which is equivalent to $w'W_h = ww_0 s_i W_h$. In particular, the cosets wW_h and $w'W_h$ have to be distinct, but this is also obvious from the fact that edges of Ξ_h are convex.
 - In the Schreier graph of (W, S) with respect to W_h , these cosets have to be identified with the vertices (i.e. right cosets) $W_h w$ and $W_h w'$ which are joined by an edge if and only if $W_h w' = W_h w s_i$ for some $i \in I$. Here, it is possible that these vertices coincide. For example, if w is left I_h -reduced and $i \in I$ is such that $l(w) < l(ws_i)$ but ws_i is not left I_h -reduced, then $ws_i = s_j w$ for some $j \in I_h$, so that $W_h w s_i = W_h s_j w = W_h w$ is connected to itself by a loop. Observe that it was necessary to use right cosets in order to obtain the usual “right” Cayley graph in the case of trivial stabilizer.

In addition, both graphs are regular but their degrees may be different. On the one hand, the degree of the Schreier graph is $n = |I|$, irrespective of W_h , but it may contain loops. The underlying graph of Ξ_h , on the other hand, cannot contain loops and its degree is given by the expression in (4.3.23). Note further that the W -action on \mathfrak{a} provided by the \mathcal{D} -representation induces a left-action of W on the underlying graph of Ξ_h by graph automorphisms, but there is no obvious left-action of W on the Schreier graph of (W, S) with respect to W_h .

The almost virtually singular case

All of the previous results can be generalized to almost virtually singular elements by a simple modification. Throughout this subsection, we assume that $h \in \overline{C}$ is almost virtually singular. By definition, this means that the irreducible factors $(W_1, S_1), \dots, (W_m, S_m)$ of (W, S) have the following property: For every $k \in \{1, \dots, m\}$, the intersection $W_h \cap W_k$ is finite or W_k is entirely contained in W_h . Note that these conditions are not mutually exclusive if W_k is finite. Moreover, h is virtually singular if and only if the former is satisfied whenever W_k is infinite, but if h is only assumed to be almost virtually singular, it may happen that there exists some $k \in \{1, \dots, m\}$ such that W_k is infinite and contained in W_h . In either case, the condition $W_k \subset W_h$ is equivalent to $I_k \subset I_{\text{inv}}$, so if this holds, then W_k acts trivially on $\Xi_h = \text{conv}(W_{\text{vis}} \cdot h) = F_{I_{\text{vis}}}$. More formally, if we consider the I_{vis} -restriction $\mathcal{D}_{I_{\text{vis}}}$ of \mathcal{D} , then h is almost virtually \mathcal{D} -singular if and only if it is virtually $\mathcal{D}_{I_{\text{vis}}}$ -singular and $\Xi_h = F_{I_{\text{vis}}}$ does not change if we replace the \mathcal{D} -representation of W by the $\mathcal{D}_{I_{\text{vis}}}$ -representation. Hence, even though the stabilizer of h need not be finite, the infinite irreducible factors of W_h have no influence on the structure of Ξ_h . Therefore, all results that we proved in the virtually singular case apply verbatim to almost virtually singular elements upon replacing the set I by I_{vis} . In fact, this “reduction process” was also available in the preceding subsection since we have never required I_{inv} to be empty. Previously, the only restriction on this set was that it must not contain a non-spherical connected component of I , but we could have easily allowed this in our discussion by adjusting the notation accordingly. For ease of reference, we summarize the most important results in the following theorem:

Theorem 4.3.38. *Let $h \in \overline{C}$ be almost virtually singular.*

- (i) Ξ_h is a closed convex subset of \mathfrak{a} of dimension $|I_{\text{vis}}|$ and its affine span is $E_h = h + \mathfrak{a}'_{\text{vis}}$.
- (ii) The following map is a well-defined W -equivariant poset isomorphism:

$$\begin{aligned} \tilde{\psi} : \tilde{\Sigma}^*(W, S) &\rightarrow \mathcal{F}(\Xi_h) \\ [wW_J] &\mapsto w(F_J) = \text{conv}(wW_J \cdot h) \end{aligned} \quad (4.3.24)$$

More precisely, for every standard left coset $wW_J \subset W$, the set $w(F_J)$ is an exposed face of Ξ_h of dimension $|J_{\text{vis}}|$. Every face is of this form for some standard left coset of (W, S) and for all $w_1W_J, w_2W_K \subset W$, it holds that:

$$w_1(F_J) = w_2(F_K) \iff J_{\text{vis}} = K_{\text{vis}} \text{ and } W_J w_1^{-1} w_2 W_K \cap W_{J_{\text{vis}}}^* \neq \emptyset \quad (4.3.25)$$

- (iii) There is a well-defined type function on $\mathcal{F}(\Xi_h)$ given by:

$$\begin{aligned} \tau : \mathcal{F}(\Xi_h) &\rightarrow \mathcal{P}_{\text{vis}}(I) \\ w(F_J) &\mapsto J_{\text{vis}} \end{aligned} \quad (4.3.26)$$

The action of W on $\mathcal{F}(\Xi_h)$ is type-preserving and transitive on faces of the same type. For every standard left coset $wW_J \subset W$, the stabilizer of the face $w(F_J)$ under the action of W is the parabolic subgroup $w(W_{J_{\text{vis}}}^* \times W_{J_{\text{vis}}})w^{-1}$ and its point-wise fixer is $wW_{J_{\text{vis}}}^* w^{-1}$.

- (iv) Every face of Ξ_h only lies in finitely many other faces. More precisely, Corollary 4.3.35 continues to hold. In particular, the underlying graph of Ξ_h is locally finite and regular.
- (v) A point $h' \in \mathfrak{a}$ is contained in Ξ_h if and only if it lies in E_h and satisfies

$$\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle \quad \text{for all } w \in W \text{ and } i \in I. \quad (4.3.27)$$

Proof. The argument merely consists of collecting results that we have already established and modifying them in a trivial way. Let $\mathcal{D}_{I_{\text{vis}}}$ denote the I_{vis} -restriction of \mathcal{D} and view $h \in \overline{C}$ as a virtually $\mathcal{D}_{I_{\text{vis}}}$ -singular element of \mathfrak{a} .

- (i) We have shown in Proposition 4.3.2 that $\dim(\Xi_h) = |I_{\text{vis}}|$, so its affine span is $E_h = h + \mathfrak{a}'_{\text{vis}}$, and Corollary 4.3.31 further implies that $\Xi_h = \text{conv}(W_{\text{vis}} \cdot h) = F_{I_{\text{vis}}}$ is closed.
- (ii) First, it follows from Theorem 4.3.32 that $\tilde{\psi}$ defines a W_{vis} -equivariant poset isomorphism $\tilde{\Sigma}^*(W_{\text{vis}}, S_{\text{vis}}) \rightarrow \mathcal{F}(\Xi_h)$. Second, it is not difficult to check that $\tilde{\Sigma}^*(W_{\text{vis}}, S_{\text{vis}}) = \tilde{\Sigma}^*(W, S)$ and that W_{inv} acts trivially both on $\tilde{\Sigma}^*(W, S)$ and $\mathcal{F}(\Xi_h)$, so $\tilde{\psi}$ vacuously extends to the required W -equivariant poset isomorphism.
- (iii) It is a consequence of Proposition 4.3.13 that τ is well-defined. By definition, the W -action on $\mathcal{F}(\Xi_h)$ is type-preserving and transitive on faces of the same type. The assertions about the stabilizer and point-wise fixer were established in Corollary 4.3.14 and 4.3.17.
- (iv) This is immediate from Corollary 4.3.35 which does not change if I is replaced by I_{vis} .
- (v) The assumption $h' \in E_h$ implies that W_{inv} stabilizes h' , so (4.3.27) has to be an equality whenever $w \in W_{\text{inv}}$ and $i \in I_{\text{inv}}$. Hence, this condition is equivalent to the equalities in the second and third line of (4.3.18) and the inequalities in the first line are unchanged after replacing I by I_{vis} . Therefore, the statement follows from Proposition 4.3.28. \square

Finally, if $h \in \mathfrak{a}$ is an arbitrary almost virtually singular element, not necessarily contained in \overline{C} , there exists $\varepsilon \in \{1, -1\}$ and some $w_0 \in W$ such that $\varepsilon w_0(h) \in \overline{C}$ and Theorem 4.3.38 applies to that point. Since $\Xi_{\varepsilon w_0(h)} = \varepsilon \Xi_h$, the assertions of (i) and (iv) hold verbatim for Ξ_h once Δ is endowed with the colouring defined by $I^\bullet = I_{w_0(h)}$. In (ii), it is necessary to change $\tilde{\psi}$ to the map $[wW_J] \mapsto \varepsilon \text{conv}(wW_J \cdot w_0(h))$ and then (iii) continues to hold. In (v), it suffices to replace h by $w_0(h)$ and flip the inequality sign if $\varepsilon = -1$. In particular, we have proved Theorem E. We close this subsection with one more simple but noteworthy consequence of the previous theorem.

Corollary 4.3.39. *If $h_1, h_2 \in \mathfrak{a}$ are two almost virtually singular elements, then $\Xi_{h_1} = \Xi_{h_2}$ holds if and only if $W \cdot h_1 = W \cdot h_2$.*

Proof. By Theorem 4.3.38 (ii), the sets Ξ_{h_1} and Ξ_{h_2} coincide if and only if they have the same vertices, which are precisely the respective W -orbits of h_1 and h_2 . \square

The essentially singular case

To conclude our study of internal Weyl group orbits, it remains to treat the essentially singular case. Hence, in this subsection, we suppose that $h \in \overline{C}$ is essentially singular. By definition, this means that there exists some $k \in \{1, \dots, m\}$ such that $W_h \cap W_k$ is an infinite proper subgroup of W_k or, equivalently, that I_{vis} contains a non-spherical subset of I_h . In this situation, we will see that many assertions of Theorem 4.3.38 are no longer valid. In particular, we will establish that Ξ_h is not closed and therefore not defined by weak linear inequalities as in Theorem 4.3.38 (v). Before we prove this abstractly, let us illustrate the central idea in a concrete example.

Example 4.3.40. Consider the following GCM of indefinite type:

$$\mathbb{A} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \iff \Gamma(\mathbb{A}) = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ \swarrow \quad \searrow \\ \bullet \end{array}$$

Since \mathbb{A} is invertible, the simply connected Kac-Moody root datum $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is free and cofree. We identify $\mathfrak{a} = \Lambda_{\text{sc}}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with \mathbb{R}^3 by mapping $\mathcal{B} = \{\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee\} \subset \mathfrak{a}$ to the standard basis of \mathbb{R}^3 and endow $\mathfrak{a} \cong \mathbb{R}^3$ with the Euclidean norm $\|\cdot\|$ via this isomorphism. Under this identification, the fundamental weights $\omega_1, \omega_2, \omega_3 \in \mathfrak{a}^*$ correspond to the standard coordinate functionals. The point $h := (-\frac{1}{2}, -\frac{1}{2}, 0)$ satisfies $\langle \alpha_1, h \rangle = \langle \alpha_2, h \rangle = 0$ and $\langle \alpha_3, h \rangle = 1$. Thus, it is contained in \overline{C} and its stabilizer W_h under the \mathcal{D} -representation of $W = W(\mathbb{A})$ is the infinite standard subgroup $W_{\{1,2\}} \cong D_\infty$, so $h \in \overline{C}$ is essentially singular. We are going to show that Ξ_h is not closed by constructing a convergent sequence in Ξ_h whose limit point does not lie in that set.

To this end, we write $J := \{1, 2\}$ and denote by \mathcal{D}_J the J -restriction of \mathcal{D} . Then $W_h = W_J$ and the point $p := s_3(h) = h - \alpha_3^\vee = (-\frac{1}{2}, -\frac{1}{2}, -1) \in \Xi_h$ now satisfies $\langle \alpha_1, p \rangle = \langle \alpha_2, p \rangle = 1$, so it is contained in the open fundamental \mathcal{D}_J -chamber $C_J \subset \mathfrak{a}$ and therefore \mathcal{D}_J -regular. In particular, the set $\Xi_p^J := \text{conv}(W_J \cdot p) \subset \mathfrak{a}$ is closed and, by construction, a subset of Ξ_h . We claim that Ξ_p^J contains the infinite ray $p - \mathbb{R}_{\geq 0}(\alpha_1^\vee + \alpha_2^\vee) = \{(t, t, -1) \mid t \leq -\frac{1}{2}\} \subset p + \mathfrak{a}'_J$. Since $\langle \alpha_1, \alpha_1^\vee + \alpha_2^\vee \rangle = \langle \alpha_2, \alpha_1^\vee + \alpha_2^\vee \rangle = 0$, it follows that $w(\alpha_1^\vee + \alpha_2^\vee) = \alpha_1^\vee + \alpha_2^\vee$ for all $w \in W_J$. Given any $\lambda \in \mathbb{R}_{\geq 0}$, the fact that $p \in C_J$ now implies that

$$\langle \omega_j, w(p - \lambda(\alpha_1^\vee + \alpha_2^\vee)) \rangle = \langle \omega_j, w(p) \rangle - \lambda \leq \langle \omega_j, p \rangle \quad \text{for all } w \in W_J \text{ and } j \in J.$$

Hence, Corollary 4.3.27 shows that $p - \lambda(\alpha_1^\vee + \alpha_2^\vee) \in \Xi_p^J$ and the claim is proved. For every $l \in \mathbb{N}$, the point $y_l := p - l(\alpha_1^\vee + \alpha_2^\vee)$ now lies in $\Xi_p^J \subset \Xi_h$, so the closed line segment $[h, y_l]$ is a subset of Ξ_h as well, and the vector $v_l := h - y_l = l(\alpha_1^\vee + \alpha_2^\vee) + \alpha_3^\vee = (l, l, 1)$ satisfies $\|v_l\| \geq 1$, so this line segment intersects the unit sphere in \mathfrak{a} centered at h in the point $x_l := h - v_l/\|v_l\| \in [h, y_l]$. On the one hand, it follows that the resulting sequence $(x_l)_{l \in \mathbb{N}}$ is contained in Ξ_h and converges to $x := h - (\alpha_1^\vee + \alpha_2^\vee)/\sqrt{2}$. On the other hand, this limit point does not lie in Ξ_h since $x \neq h$ and $\langle \omega_3, x \rangle = \langle \omega_3, h \rangle$, but $\Xi_h \cap E^{(3)} = F^{(3)} = \{h\}$ by Proposition 4.3.6.

Thus, we have proved that Ξ_h is not closed. Consequently, it cannot be written as an intersection of closed half-spaces and is therefore not characterized by a family of weak linear inequalities. While the same argument as in the almost virtually singular case still shows that every point $h' \in \Xi_h$ is contained in $E_h = h + \mathfrak{a}'_{\text{vis}}$ (which is all of \mathfrak{a} in our current example) and satisfies $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ for all $w \in W$ and $i \in I$, the converse is not true. For example, every point of the form $h' = (t, t, 0)$ for some $t < -\frac{1}{2}$ is contained in \overline{C} and $\langle \omega_i, h' \rangle \leq \langle \omega_i, h \rangle$ holds for all $i \in I$, so it fulfils all these inequalities, but no such point lies in Ξ_h .

Using a simple property of unbounded convex sets, the same argument as in the previous example applies in the general case.

Proposition 4.3.41. *If $h \in \overline{C}$ is essentially singular, then Ξ_h is not closed.*

Proof. Throughout this proof, we endow $\mathfrak{a} \cong \mathbb{R}^d$ with the Euclidean norm $\|\cdot\|$ induced by the isomorphism that maps the basis $\mathcal{B} = \{\alpha_1^\vee, \dots, \alpha_n^\vee, h_{n+1}, \dots, h_d\} \subset \mathfrak{a}$ to the standard basis of \mathbb{R}^d in the natural order. To simplify the notation, we first assume that \mathbb{A} is indecomposable, so that (W, S) is irreducible and W_h is an infinite proper subgroup of W . Let J be a connected component of I_h such that the irreducible factor $W_J \subset W_h$ is infinite. Because Δ is connected and $I_h \neq I$, there exists a white vertex $i \in I \setminus I_h$ that is joined to some black vertex of $J \subset I_h$. Thus, the point $p := s_i(h) \in \Xi_h$ does not coincide with h and satisfies $\langle \alpha_j, p \rangle \geq 0$ for all $j \in J$, where the inequality is strict for at least one $j \in J$. Consequently, p is contained in the closed fundamental \mathcal{D}_J -chamber $\overline{C}_J \subset \mathfrak{a}$, but it is not a global fixed-point for the W_J -action on \mathfrak{a} . We set $J_p := \{j \in J : \langle \alpha_j, p \rangle = 0\} \subsetneq J$, so that W_{J_p} is the stabilizer of p under the W_J -action on \mathfrak{a} . As (W_J, S_J) is infinite and irreducible, it follows from Theorem 1.1.6 that $W_{J_p} \subsetneq W_J$ has infinite index in W_J . Therefore, $W_J \cdot p \subset \mathfrak{a}$ is an infinite set and Corollary 4.3.4 implies that its convex hull $\Xi_p^J := \text{conv}(W_J \cdot p) \subset \Xi_h$ is unbounded.

By [Roc70, Corollary 8.3.1 and Theorem 8.4], every unbounded convex set contains an infinite half-line. More precisely, since $\Xi_p^J \subset p + \mathfrak{a}'_J$, it implies that there exists a point $q \in \Xi_p^J$ and a vector $v \in \mathfrak{a}'_J \setminus \{0\}$ such that the ray $q + \mathbb{R}_{\geq 0}v$ is contained in Ξ_p^J . We set $y_l := q + lv$ for $l \in \mathbb{N}$. Let $S_1(h)$ denote the unit sphere in \mathfrak{a} centered at h . We may w.l.o.g. assume that $\|h - y_l\| \geq 1$ for all $l \in \mathbb{N}$, which ensures that the closed line segment $[h, y_l] \subset \Xi_h$ intersects that sphere in a unique point $x_l \in \Xi_h$. As $S_1(h)$ is compact, the sequence $(x_l)_{l \in \mathbb{N}}$ converges to a point $x \in S_1(h)$. We are going to show that $x \notin \Xi_h$, which will conclude the proof in the irreducible case.

To this end, we consider the sequence of vectors $(v_l)_{l \in \mathbb{N}}$ defined by $v_l := h - y_l = h - q - lv$. Then $x_l = h - \frac{v_l}{\|v_l\|}$ for all $l \in \mathbb{N}$ and the sequence $(\frac{v_l}{\|v_l\|})_{l \in \mathbb{N}}$ converges to $-\frac{v}{\|v\|}$, so that

$$x = \lim_{l \rightarrow \infty} x_l = h - \lim_{l \rightarrow \infty} \frac{v_l}{\|v_l\|} = h + \frac{v}{\|v\|}.$$

It follows that x lies in the affine subspace $h + \mathfrak{a}'_J \subset \mathfrak{a}$, but the only point of Ξ_h with that property is h itself. To see this, note that W_h fixes h and if $w \in W \setminus \{1\}$ is right I_h -reduced, there exists some $r \in I(w) \setminus I_h \subset I \setminus J$ and Lemma 4.3.5 implies that $\langle \omega_r, w(h) \rangle < \langle \omega_r, h \rangle$, so $w(h)$ is not contained in that subspace. Hence, every non-trivial convex combination of W -translates of h involves a point that does not lie in $h + \mathfrak{a}'_J$. As $x \neq h$, we conclude that $x \notin \Xi_h$.

If (W, S) is not necessarily irreducible, there exists some $k \in \{1, \dots, m\}$ such that $W_h \cap W_k$ is an infinite proper subgroup of W_k , so the irreducible case implies that $F_{I_k} = \text{conv}(W_k \cdot h) \subset \Xi_h$ is not closed. More precisely, the proof constructs a sequence $(x_l)_{l \in \mathbb{N}}$ in F_{I_k} that converges to a point $x \notin F_{I_k}$, but the same argument as in the previous paragraph shows that x is not contained in Ξ_h either. Hence, Ξ_h is not closed, as required. \square

Remark 4.3.42. In the setting of Example 4.3.40, the reader may have noticed that the submatrix $\mathbb{A}_J = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ of \mathbb{A} is of affine type, which is a very special case in Kac-Moody theory where it is not uncommon to encounter phenomena that have no analogues in the “generic” indefinite case. The fact that Ξ_h is not closed, however, is not related to this and we could have easily replaced \mathbb{A}_J by any other GCM of size 2×2 that is not spherical. The only reason for this particular choice is that a GCM of affine type is not invertible, which made it very simple to find an infinite ray that is contained in Ξ_p^J and write down an explicit convergent sequence in Ξ_h whose limit point does not belong to Ξ_h . But as the proof of Proposition 4.3.41 shows, the existence of such a ray is guaranteed by a general property of unbounded convex sets. Apart from this minor difference, the argument in the general case is almost verbatim to the concrete example before.

Since Ξ_h is not closed if $h \in \overline{C}$ is essentially singular, it cannot be reconstructed from its affine span and its proper faces as the intersection of all its supporting half-spaces. Therefore, Ξ_h is not characterized by a family of weak linear inequalities as in the almost virtually singular case and the face poset $\mathcal{F}(\Xi_h)$ does not describe Ξ_h in a satisfactory way. More precisely, recall from Corollary 4.3.7 that every standard coset of (W, S) defines an exposed face of Ξ_h , even in the essentially singular case, so Ξ_h indeed has many proper faces. In particular, its relative boundary is not empty. What we cannot prove at the moment is that every face arises in this way because the proof of Corollary 4.3.31 does not apply in this setting. The author is not aware whether the map $\psi : \Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$ can be shown to be surjective in the essentially singular case by an alternative argument. However, even if it was true, it would not allow us to extend the convexity theorems in Chapter 5 to this situation since their proofs crucially hinge on the property that Ξ_h is an intersection of closed half-spaces.

4.3.3 Parallel slices

Altogether, we have arrived at a satisfactory and meaningful description of the face poset of Ξ_h whenever $h \in \overline{C}$ is almost virtually singular. In this section, we investigate those convex sets that arise as intersections of Ξ_h with parallel translates of the supporting hyperplanes $E^{(1)}, \dots, E^{(n)}$. As we shall see, every such “parallel slice” is the convex hull of a finite union of orbits of a maximal proper standard subgroup of (W, S) . This property will be essential to the proofs of the convexity theorems as it will allow us to apply inductive arguments. The results of this section crucially rely on Theorem 4.3.38, so we have to invoke the following restriction:

Assumption 4.3.43. Throughout Section 4.3.3, we suppose that $h \in \overline{C}$ is almost virtually singular or, equivalently, that $W_h \cap W_{\text{vis}}$ is finite.

Recall from Proposition 4.3.6 that for every $i \in I$, the set $\Xi_h \cap E^{(i)} = F^{(i)} = \text{conv}(W^{(i)} \cdot h)$ is an exposed face of Ξ_h and the following trichotomy holds:

- a) If $i \in I_{\text{inv}}$ is invisible, then $E^{(i)} = E_h \supset \Xi_h$ is a trivial supporting hyperplane and $F^{(i)} = \Xi_h$ is the trivial face.
- b) If $i \in I_{\text{nd}} \subset I_{\text{vis}}$ is non-degenerate, then $E^{(i)}$ is a defining hyperplane and $F^{(i)}$ a facet of Ξ_h , i.e. $\dim(F^{(i)}) = \dim(\Xi_h) - 1$.
- c) If $i \in I_{\text{deg}} \subset I_{\text{vis}}$ is degenerate, then $E^{(i)}$ is a supporting hyperplane and $F^{(i)}$ a face of Ξ_h that satisfies $\dim(F^{(i)}) < \dim(\Xi_h) - 1$.

For every $i \in I$ and $t \in \mathbb{R}$, we define

$$E^{(i)}(t) := \{y \in E_h : \langle \omega_i, y \rangle = t\}. \quad (4.3.28)$$

This extends the notation for the supporting hyperplanes in the sense that $E^{(i)}(\langle \omega_i, h \rangle) = E^{(i)}$. The purpose of this section is to study sets of the form

$$P^{(i)}(t) := \Xi_h \cap E^{(i)}(t) \quad \text{for some } i \in I \text{ and } t \in \mathbb{R}. \quad (4.3.29)$$

If $i \in I_{\text{inv}}$, then ω_i is constant on $E_h = h + \mathfrak{a}'_{\text{vis}}$, which implies that $E^{(i)}(t) = \emptyset$ for $t \neq \langle \omega_i, h \rangle$, so it suffices to consider the visible case. If $i \in I_{\text{vis}}$, then ω_i is not constant on E_h , so given any $t \in \mathbb{R}$, the affine hyperplane $E^{(i)}(t)$ separates E_h into two closed half-spaces. We denote the resulting “positive side” by

$$E_+^{(i)}(t) := \{y \in E_h : \langle \omega_i, y \rangle \geq t\}. \quad (4.3.30)$$

Both the hyperplane $E^{(i)}(t)$ and the half-space $E_+^{(i)}(t)$ are invariant under the standard subgroup $W^{(i)} \subset W$ since this is the stabilizer of ω_i under the W -action on \mathfrak{a}^* . Similarly, the functional $\chi := \sum_{i \in I_{\text{vis}}} \omega_i \in \mathfrak{a}^*$ is not constant on E_h either and for every $t \in \mathbb{R}$, we set

$$E_+(t) := \{y \in E_h : \langle \chi, y \rangle \geq t\}. \quad (4.3.31)$$

Our first objective is to determine the set $T_h^{(i)} := \{t \in \mathbb{R} : P^{(i)}(t) \neq \emptyset\}$ for an arbitrary $i \in I_{\text{vis}}$. On the one hand, Proposition 4.3.6 implies that $P^{(i)}(t) = \emptyset$ if $t > \langle \omega_i, h \rangle$. On the other hand, as $P^{(i)}(\langle \omega_i, h \rangle) = F^{(i)} \neq \emptyset$ and Ξ_h is closed and convex, this set is either a finite closed interval $[t_0, \langle \omega_i, h \rangle]$ for some $t_0 \in \mathbb{R}$ with $t_0 \leq \langle \omega_i, h \rangle$ or the half-infinite interval $(-\infty, \langle \omega_i, h \rangle]$. As we shall see, the existence of a lower bound depends on whether the index $[W : W^{(i)}]$ is finite. In order to prove this, we are going to show that the closed half-space $E_+^{(i)}(t)$ contains only finitely many $W^{(i)}$ -orbits of vertices of Ξ_h . These orbits are in bijective correspondence with the elements of the double quotient space $W^{(i)} \backslash W / W_h$. Every double coset $W^{(i)} w W_h \subset W$ with $w \in W$ has a unique representative of minimal length that is characterized by the property of being $(I^{(i)}, I_h)$ -reduced. To prepare for the proof of Proposition 4.3.47, we first establish a simple but slightly technical result about this quotient space.

Lemma 4.3.44. *Let $i \in I_{\text{vis}}$ be arbitrary. The index $[W : W^{(i)}]$ is finite if and only if the double quotient space $W^{(i)} \backslash W / W_h$ is finite.*

Proof. Using that $i \in I_{\text{vis}}$, $W \cong W_{\text{vis}} \times W_{\text{inv}}$ and $W_{\text{inv}} \subset W_h$, we find that

$$W^{(i)} \cong (W^{(i)} \cap W_{\text{vis}}) \times W_{\text{inv}} \quad \text{and} \quad W_h \cong (W_h \cap W_{\text{vis}}) \times W_{\text{inv}}.$$

This implies that the following quotient spaces coincide:

$$\begin{aligned} W^{(i)} \backslash W &= (W^{(i)} \cap W_{\text{vis}}) \backslash W_{\text{vis}} \\ W^{(i)} \backslash W / W_h &= (W^{(i)} \cap W_{\text{vis}}) \backslash W_{\text{vis}} / (W_h \cap W_{\text{vis}}) \end{aligned}$$

As h is assumed to be almost virtually singular, the group $W_h \cap W_{\text{vis}}$ is finite. Hence, it acts on the right quotient space $(W^{(i)} \cap W_{\text{vis}}) \backslash W_{\text{vis}}$ by right-multiplication such that every orbit is finite. But the double quotient space on the right in the second line is the orbit space for this action, so on the one hand, this set is finite if and only if the quotient space on the right in the first line is already finite, which is the case if and only if $[W : W^{(i)}] < \infty$. On the other hand, it is also finite if and only if $|W^{(i)} \backslash W / W_h| < \infty$. \square

The next lemma is the essential step towards the anticipated result about $T_h^{(i)}$. Its proof requires one more bit of terminology: For an element $\alpha = \sum_{i \in I} k_i \alpha_i \in Q$ (resp. $\alpha = \sum_{i \in I} k_i \alpha_i^\vee \in Q^\vee$) with $k_1, \dots, k_n \in \mathbb{Z}$ and a subset $J \subset I$, we define the J -height of α by

$$\text{ht}_J(\alpha) := \sum_{j \in J} k_j. \quad (4.3.32)$$

Lemma 4.3.45.

- (i) For every $i \in I_{\text{vis}}$ and $t \in \mathbb{R}$, there exist only finitely many $w \in W$ that are $(I^{(i)}, I_h)$ -reduced and satisfy $w(h) \in E_+^{(i)}(t)$.
- (ii) For every $t \in \mathbb{R}$, there exist only finitely many $w \in W$ that are right I_h -reduced and satisfy $w(h) \in E_+(t)$.

Proof. (i) For every $w \in W$, we define $\beta_i(w) := \omega_i - w^{-1}(\omega_i) \in Q_+$, so that

$$\langle \omega_i, w(h) \rangle \geq t \iff \langle \beta_i(w), h \rangle \leq \langle \omega_i, h \rangle - t.$$

Note that these inequalities are impossible if $t > \langle \omega_i, h \rangle$. Since the stabilizer of ω_i under the W -action on \mathfrak{a}^* is the standard subgroup $W^{(i)} \subset W$, we have $\beta_i(w_1) = \beta_i(w_2)$ for two elements $w_1, w_2 \in W$ if and only if $W^{(i)}w_1 = W^{(i)}w_2$. Thus, the set $\{\beta_i(w) \mid w \in W\} \subset Q_+$ is in bijective correspondence with the right cosets of $W^{(i)}$ in W and when writing $\beta_i(w)$, we may assume that w is left $I^{(i)}$ -reduced. Moreover, the following transformation rule for right-multiplication holds for all $w \in W$ and $j \in I$:

$$\beta_i(ws_j) = \omega_i - s_j w^{-1}(\omega_i) = \beta_i(w) + \langle \omega_i, w(\alpha_j^\vee) \rangle \alpha_j$$

If $w \in W$ is left $I^{(i)}$ -reduced and $j \in I$ is such that $l(w) < l(ws_j)$, the folding condition states that ws_j is either left $I^{(i)}$ -reduced as well, and hence $\beta_i(ws_j) \neq \beta_i(w)$, or $ws_j = s_{j'}w$ for some $j' \in I^{(i)}$ and $\beta_i(ws_j) = \beta_i(w)$. In both cases, we have $w(\alpha_j^\vee) \in \Phi_+^\vee$ by (2.5.4) and $\langle \omega_i, w(\alpha_j^\vee) \rangle \geq 0$. If equality holds, then $\beta_i(w) = \beta_i(ws_j)$, implying that w and ws_j lie in the same right $W^{(i)}$ -coset and the latter case occurs. Therefore, if ws_j is left $I^{(i)}$ -reduced too, we have $\langle \omega_i, w(\alpha_j^\vee) \rangle > 0$ and $\text{ht}(\beta_i(w)) < \text{ht}(\beta_i(ws_j))$.

Suppose now that $w \in W$ is left $I^{(i)}$ -reduced and set $l := l(w)$. Since $W_{\text{inv}} \subset W^{(i)}$, this assumption implies that $w \in W_{\text{vis}}$, so if $w = s_{i_1} \dots s_{i_l}$ is a reduced decomposition, then $i_1, \dots, i_l \in I_{\text{vis}}$. In addition, for every $k \in \{0, \dots, l\}$, the element $w_k := s_{i_1} \dots s_{i_k}$ is also left $I^{(i)}$ -reduced, so we conclude from the previous paragraph that $\text{ht}(\beta_i(w_k)) \geq \text{ht}(\beta_i(w_0)) + k$, where $\beta_i(w_0) = \beta_i(1) = 0$. Moreover, since $W_h \cap W_{\text{vis}}$ is finite, the maximal length of a subword of this reduced decomposition that only involves the simple reflections in $S_{I_h \cap I_{\text{vis}}}$ is bounded by $l(w_+)$, where w_+ denotes the longest word of $W_h \cap W_{\text{vis}}$. Since no simple factor of w lies in S_{inv} , it follows that if $l \geq l_0(l(w_+) + 1)$ for some $l_0 \in \mathbb{N}$, then at least l_0 of these reflections must be contained in $S_{I_{\text{vis}} \setminus (I_h \cap I_{\text{vis}})} = S_{I \setminus I_h}$, so we even find that $\text{ht}_{I \setminus I_h}(\beta_i(w)) \geq \text{ht}_{I \setminus I_h}(\beta_i(w_0)) + l_0 = l_0$. Because $\langle \alpha_j, h \rangle > 0$ for all $j \in I \setminus I_h$, every left $I^{(i)}$ -reduced element $w \in W$ with sufficiently large length satisfies $\langle \beta_i(w), h \rangle > \langle \omega_i, h \rangle - t$, so there are only finitely many such elements with $\langle \omega_i, w(h) \rangle \geq t$ and the same statement then holds for $(I^{(i)}, I_h)$ -reduced elements as well. In fact, the value of $\langle \omega_i, w(h) \rangle$ is constant on the double coset $W^{(i)}wW_h$.

- (ii) The proof is similar to (i). For every $w \in W$, we define $\beta(w) := \chi - w^{-1}(\chi) \in Q_+$, so that

$$\langle \chi, w(h) \rangle \geq t \iff \langle \beta(w), h \rangle \leq \langle \chi, h \rangle - t.$$

As $\langle \chi, \alpha_j^\vee \rangle = 1$ for $j \in I_{\text{vis}}$ and $\langle \chi, \alpha_j^\vee \rangle = 0$ for $j \in I_{\text{inv}}$, the same argument as in the proof of Lemma 4.1.8 shows that the stabilizer of χ under the W -action on \mathfrak{a}^* is the standard subgroup $W_{\text{inv}} \subset W$. Hence, the set $\{\beta(w) \mid w \in W\} \subset Q_+$ is in bijective correspondence with the right cosets of W_{inv} in W and when writing $\beta(w)$, we may assume that w is left I_{inv} -reduced. Now the following transformation rule holds for all $w \in W$ and $j \in I$:

$$\beta(ws_j) = \chi - s_j w^{-1}(\chi) = \beta(w) + \langle \chi, w(\alpha_j^\vee) \rangle \alpha_j$$

Exactly as in (i), it follows that if w and ws_j are both left I_{inv} -reduced and $l(w) < l(ws_j)$, then $\text{ht}(\beta(w)) < \text{ht}(\beta(ws_j))$ and this again implies that there are only finitely many left I_{inv} -reduced elements $w \in W$ that satisfy $\langle \chi, w(h) \rangle \geq t$, so the same statement holds for (I_{inv}, I_h) -reduced elements. To conclude the proof, it suffices to note that for every $w \in W$, there exist unique $w_1 \in W_{\text{vis}}$ and $w_2 \in W_{\text{inv}} \subset W_h$ such that $w = w_1 w_2$, so we have $W_{\text{inv}} w W_h = W_{\text{inv}} w_1 W_h = w_1 W_{\text{inv}} W_h = w_1 W_h = w W_h$. Therefore, w is (I_{inv}, I_h) -reduced if and only if it is right I_h -reduced. \square

Given any $i \in I_{\text{vis}}$ and $t \in \mathbb{R}$, the closed half-space $E_+^{(i)}(t) \subset E_h$ is $W^{(i)}$ -invariant, which implies that a vertex of Ξ_h is contained in $E_+^{(i)}(t)$ if and only if its entire $W^{(i)}$ -orbit has this property. The vertices of Ξ_h are in bijective correspondence with the left W_h -cosets in W or, equivalently, the right I_h -reduced elements of W , and their $W^{(i)}$ -orbits with the $(I^{(i)}, I_h)$ -reduced elements. Therefore, Lemma 4.3.45 immediately leads to the following geometric statement:

Corollary 4.3.46.

- (i) For every $i \in I_{\text{vis}}$ and $t \in \mathbb{R}$, the closed half-space $E_+^{(i)}(t) \subset E_h$ contains only a finite number of $W^{(i)}$ -orbits of vertices of Ξ_h .
- (ii) For every $t \in \mathbb{R}$, the closed half-space $E_+(t) \subset E_h$ contains only a finite number of vertices of Ξ_h . \square

We are now in a position to determine the set $T_h^{(i)} = \{t \in \mathbb{R} : P^{(i)}(t) \neq \emptyset\}$ for any fixed $i \in I_{\text{vis}}$. To formulate the next result, recall from Theorem 1.1.6 that the index $[W : W^{(i)}]$ is finite if and only if $i \in I \setminus I_\infty$, where I_∞ denotes the union of all non-spherical connected components of I . Its complement is spherical, so $W_{I \setminus I_\infty}$ is finite and contains a unique element w_+ of maximal length. Conjugation in $W_{I \setminus I_\infty}$ by $w_+ = w_+^{-1}$ maps $S_{I \setminus I_\infty}$ to itself and induces a diagram automorphism of $\Delta_{I \setminus I_\infty}$. When w_+ is considered as an element of the ambient group W , it commutes with W_{I_∞} , so conjugation in W by w_+ induces an involutive automorphism of the entire Coxeter diagram Δ that is trivial on the subgraph Δ_{I_∞} .

Proposition 4.3.47. *Let $i \in I_{\text{vis}}$ be arbitrary.*

- (i) If $i \in I \setminus I_\infty$, then $[W : W^{(i)}]$ is finite. In this case, let w_+ be the longest word of $W_{I \setminus I_\infty}$, then $\langle \omega_i, w_+(h) \rangle < \langle \omega_i, h \rangle$ and $P^{(i)}(t) \neq \emptyset$ if and only if $t \in [\langle \omega_i, w_+(h) \rangle, \langle \omega_i, h \rangle]$.
- (ii) If $i \in I_\infty$, then $[W : W^{(i)}]$ is infinite and $P^{(i)}(t) \neq \emptyset$ for all $t \in (-\infty, \langle \omega_i, h \rangle]$.

Proof. To simplify the notation in this proof, we set $\lambda_k := \langle \omega_k, h \rangle \in \mathbb{R}$ for all $k \in I$.

- (i) Let $\hat{w}_+ := \pi_{I_h}^r(w_+)$ be the unique element of minimal length in w_+W_h . To prove the asserted inequality, we claim that $i \in I(\hat{w}_+)$. Our argument uses basic properties of the Bruhat order on a finite Coxeter group. Suppose to the contrary that $i \notin I(\hat{w}_+)$ and let $w \in W_{I \setminus I_\infty}$ be arbitrary. As $w \leq w_+$ in the Bruhat order on $(W_{I \setminus I_\infty}, S_{I \setminus I_\infty})$, and thus in that of (W, S) , and $\pi_{I_h}^r$ is order-preserving (cf. [BB05, Proposition 2.3.1 (i) and 2.5.1]), it follows that $\pi_{I_h}^r(w) \leq \hat{w}_+$ and hence $\pi_{I_h}^r(w) \in W^{(i)}$. To derive a contradiction, it suffices to construct a right I_h -reduced element of $W_{I \setminus I_\infty}$ that contains s_i as a simple factor.

To achieve this, let $I_p \subset I \setminus I_\infty$ for some $p \in \{1, \dots, m\}$ be the connected component of I containing i . Since $i \in I_{\text{vis}}$, this set is visible, so Lemma 1.2.5 ensures that there exists an exposing forest $\Delta_{I_p}^\circ$ for Δ_{I_p} . Set $\delta := \delta_{I_p}(i)$ and let $\gamma = (i_0, \dots, i_\delta = i)$ be the unique path in $\Delta_{I_p}^\circ$ connecting i with a white vertex $i_0 \in I_p \setminus I_h$, then the element $w_i := s_i s_{i_{\delta-1}} \dots s_{i_0}$ from the exposing set defined by $\Delta_{I_p}^\circ$ is contained in $W_{I_p} \subset W_{I \setminus I_\infty}$ and satisfies $i \in I(w_i)$. We are going to show that w_i is right I_h -reduced. Once this is established, it will follow that $w_i = \pi_{I_h}^r(w_i) \notin W^{(i)}$, a contradiction.

We prove this by induction on δ . If $\delta = 0$, then $i = i_0 \in I_p \setminus I_h$, so that $w_i = s_{i_0} \notin W_h$ is obviously right I_h -reduced. If $\delta > 0$, then $w' := s_{i_{\delta-1}} \dots s_{i_0}$ is right I_h -reduced by the induction hypothesis. Thus, the folding condition yields that $w_i = s_i w'$ is either also right I_h -reduced or $w_i = s_i w' = w' s_k$ for some $k \in I_h$. If the latter case occurred, it would follow that $s_i = w' s_k w'^{-1} \in W_{I(w')} s_k W_{I(w')}$ and as $i \notin I(w')$, this would only be possible if $i = k$ and $w_i = s_i w' = w' s_i$. However, by definition of an exposing forest, s_i commutes with s_{i_q} for every $q \in \{0, \dots, \delta-2\}$, so the previous equation is equivalent to $s_i s_{i_{\delta-1}} = s_{i_{\delta-1}} s_i$, which contradicts the fact that s_i does not commute with $s_{i_{\delta-1}}$ as i and $i_{\delta-1}$ are adjacent in Δ_{I_p} . Hence, w_i has to be right I_h -reduced and the proof of the claim is finished. We conclude that $i \in I(\hat{w}_+)$, so Lemma 4.3.5 implies that $h - \hat{w}_+(h) \in \alpha'$ is a linear combination of simple coroots in which α_i^\vee occurs with a positive coefficient and the desired inequality follows:

$$\langle \omega_i, h - w_+(h) \rangle = \langle \omega_i, h - \hat{w}_+(h) \rangle > 0$$

It remains to prove the second statement. Let σ denote the involutive automorphism of Δ induced by conjugation with w_+ , then $w_+ s_k w_+ = s_{\sigma(k)}$ and thus $w_+(\alpha_k^\vee) \in \{\alpha_{\sigma(k)}^\vee, -\alpha_{\sigma(k)}^\vee\}$ for all $k \in I$. As $l(w_+) < l(w_+ s_k)$ holds if and only if $k \in I_\infty$ and σ is the identity on I_∞ , it follows that $w_+(\alpha_k^\vee) = \alpha_k^\vee$ for all $k \in I_\infty$ and $w_+(\alpha_k^\vee) = -\alpha_{\sigma(k)}^\vee$ for all $k \in I \setminus I_\infty$, so if $j := \sigma(i) \in I \setminus I_\infty$, the functionals $w_+(\omega_i)$ and $-\omega_j$ coincide on the linear subspace $\mathfrak{a}' \subset \mathfrak{a}$ and if we define $c := \langle w_+(\omega_i) + \omega_j, h \rangle \in \mathbb{R}$, then $w_+(\omega_i)$ is equal to $-\omega_j + c$ on the affine subspace $h + \mathfrak{a}' \subset \mathfrak{a}$ containing Ξ_h .

Given any $h' \in \Xi_h$, we also have $w_+(h') \in \Xi_h$ by W -invariance of Ξ_h , so Proposition 4.3.6 implies that $\langle \omega_k, h' \rangle \leq \lambda_k$ and $\langle \omega_k, w_+(h') \rangle \leq \lambda_k$ for all $k \in I$. In particular, we must have $\langle \omega_i, h' \rangle \leq \lambda_i$ and

$$\langle \omega_i, h' \rangle = \langle w_+(\omega_i), w_+(h') \rangle = \langle -\omega_j, w_+(h') \rangle + c \geq c - \lambda_j,$$

so it follows that $P^{(i)}(t)$ can only be non-empty if $t \in [c - \lambda_j, \lambda_i]$. Moreover, we know that $P^{(i)}(\lambda_i) = F^{(i)}$ is a face of Ξ_h and using the same reasoning, one verifies that

$$P^{(i)}(c - \lambda_j) = \text{conv}(W^{(i)} w_+ \cdot h) = \text{conv}(w_+ W^{(j)} \cdot h) = w_+(F^{(j)})$$

is a face of Ξ_h as well. Since Ξ_h is convex, we conclude that $P^{(i)}(t)$ is non-empty for all $t \in [c - \lambda_j, \lambda_i]$. As $c - \lambda_j = \langle \omega_i, w_+(h) \rangle$, the proof of (i) is complete.

- (ii) Let $t_0 \in (-\infty, \lambda_i]$ be arbitrary. Since $[W : W^{(i)}] = \infty$, we also have $|W^{(i)} \backslash W / W_h| = \infty$ by Lemma 4.3.44, so there are infinitely many orbits for the $W^{(i)}$ -action on the vertices of Ξ_h . By Corollary 4.3.46 (i), only a finite number of them is contained in $E_+^{(i)}(t_0)$, so there exists some $w \in W$ such that $t_1 := \langle \omega_i, w(h) \rangle < t_0$. Hence, $P^{(i)}(t_1)$ contains $w(h)$ and convexity of Ξ_h implies that $P^{(i)}(t)$ is non-empty for all $t \in [t_1, \lambda_i]$. As $t_0 \leq \lambda_i$ was arbitrary and $t_1 < t_0$, the assertion follows. \square

Consequently, if $i \in I_{\text{vis}}$, the interval $T_h^{(i)} = \{t \in \mathbb{R} : P^{(i)}(t) \neq \emptyset\}$ is given by

$$T_h^{(i)} = \begin{cases} [\langle \omega_i, w_+(h) \rangle, \langle \omega_i, h \rangle] & \text{if } i \in I \setminus I_\infty \text{ and } w_+ \text{ is the longest word of } W_{I \setminus I_\infty} \\ (-\infty, \langle \omega_i, h \rangle] & \text{if } i \in I_\infty \end{cases} \quad (4.3.33)$$

and we have

$$\Xi_h = \bigcup_{t \in T_h^{(i)}} P^{(i)}(t). \quad (4.3.34)$$

Remark 4.3.48. In the proof of Proposition 4.3.47 (i), readers familiar with classical Lie theory might be tempted to assume that $w_+(\omega_k) = -\omega_{\sigma(k)}$ should hold on all of \mathfrak{a} for every $k \in I \setminus I_\infty$, but this need not be true if \mathfrak{a}' is a proper subspace of \mathfrak{a} . In this case, the fundamental weights $\omega_1, \dots, \omega_n \in \mathfrak{a}^*$ are not uniquely determined by the condition $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for all $i, j \in I$ and depend on the chosen complement $\mathfrak{a}'' := \bigoplus_{i=n+1}^d \mathbb{R} h_i$. For $k \in I \setminus I_\infty$, the functional $-\omega_{\sigma(k)}$ vanishes on \mathfrak{a}'' but $w_+(\omega_k)$ may be non-zero on this space. Indeed, we have $w_+(\omega_k) = \omega_k - \alpha$ for some $\alpha \in Q_+$ with $I(\alpha) \subset I \setminus I_\infty$, so this functional vanishes on \mathfrak{a}'' if the simple roots $(\alpha_l)_{l \in I \setminus I_\infty}$ all have this property, but this is not necessarily the case without a more careful choice of \mathfrak{a}'' .

For the rest of Section 4.3.3, we suppose that $i \in I_{\text{vis}}$ and $t \in T_h^{(i)}$ are arbitrary but fixed. We are now going to study the “geometry” of $P^{(i)}(t)$ in more detail. Intuitively, we shall see that $P^{(i)}(t)$ “almost looks like” the face $P^{(i)}(\langle \omega_i, h \rangle) = F^{(i)} = \text{conv}(W^{(i)} \cdot h)$ of Ξ_h and has a similar face poset. The main difference is that $P^{(i)}(t)$ need not be the convex hull of a single $W^{(i)}$ -orbit, but it is always the convex hull of a finite union of such orbits. As a first step in this direction, let us note two simple properties:

- a) Since Ξ_h is closed and convex, the set $P^{(i)}(t)$ is a closed convex subset of E_h whose faces are precisely the non-empty intersections of $E^{(i)}(t)$ with the faces of Ξ_h (cf. Lemma 4.2.4). Hence, all faces of $P^{(i)}(t)$ are exposed and every vertex of $P^{(i)}(t)$ is either a vertex of Ξ_h that happens to be contained in $E^{(i)}(t)$ or lies in the relative interior of a unique edge of Ξ_h . The first case is only possible if $t = \langle \omega_i, w(h) \rangle$ for some $w \in W$.

- b) Let $\mathcal{D}^{(i)} := \mathcal{D}_{I^{(i)}}$ denote the $I^{(i)}$ -restriction of \mathcal{D} , $C^{(i)}$ (resp. $\overline{C^{(i)}}$) the open (resp. closed) fundamental $\mathcal{D}^{(i)}$ -chamber in \mathfrak{a} and $X_+^{(i)}$ the $\mathcal{D}^{(i)}$ -Tits cone. Since $P^{(i)}(t)$ is $W^{(i)}$ -invariant, the same is true for its face poset, so from $P^{(i)}(t) \subset \Xi_h \subset X_+ \subset X_+^{(i)}$ (cf. Corollary 2.6.9), it follows that every face of $P^{(i)}(t)$ is $W^{(i)}$ -equivalent to a face that meets $\overline{C^{(i)}}$.

In order to obtain more precise information about the faces of $P^{(i)}(t)$, we will make use of the following rather technical lemma, which is well-known if $P^{(i)}(t)$ happens to be a convex polytope, but does not appear to be obvious without this assumption.

Lemma 4.3.49. *Every face of $P^{(i)}(t)$ is equal to the convex hull of its vertices.*

Proof. Let F' be an arbitrary face of $P^{(i)}(t)$. By Lemma 4.2.4, there exists a unique smallest face F of Ξ_h such that $F' = F \cap E^{(i)}(t)$. The faces of F are all exposed as they are simply the faces of Ξ_h that are contained in F . Hence, by the same result, every face of F' is exposed as well and every vertex of F' is either a vertex of F that is contained in $E^{(i)}(t)$ or lies in the relative interior of a unique edge of F . Let Y denote the set of vertices of F' , then Y is not empty and $\text{conv}(Y) \subset F'$ holds by convexity of F' . To show the converse, we claim that for every $x \in F'$, there exists a finite subset $Y_0 \subset Y$ such that $x \in \text{conv}(Y_0)$. To this end, set $t_x := \langle \chi, x \rangle \in \mathbb{R}$ and consider the closed half-space

$$E_+(t_x) = \{y \in E_h : \langle \chi, y \rangle \geq \langle \chi, x \rangle\}.$$

Then $F'_+ := F' \cap E_+(t_x)$ is compact as $F' = F \cap E^{(i)}(t)$ is closed and all functionals in the basis $\mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\} \subset \mathfrak{a}^*$ are bounded on F'_+ . To see this, note that the functionals $\phi_{n+1}, \dots, \phi_d$ and each ω_i with $i \in I_{\text{inv}}$ are constant on $E_h \supset F'_+$ and every ω_i with $i \in I_{\text{vis}}$ is bounded from above by $\langle \omega_i, h \rangle \in \mathbb{R}$ on $\Xi_h \supset F'_+$. By definition, $\chi = \sum_{i \in I_{\text{vis}}} \omega_i$ is bounded from below by t_x on F'_+ , so each ω_i with $i \in I_{\text{vis}}$ is bounded from below on F'_+ as well.

Since every face of F' is exposed, the same also holds for F'_+ . In particular, every “extreme point” (i.e. 0-dimensional face) of F'_+ is a vertex, so [Roc70, Corollary 18.5.1] yields that F'_+ is the convex hull of its vertices. Now every vertex of F'_+ is either a vertex of F' contained in $E_+(t_x)$ or lies in the relative interior of a unique edge of F' . Let $Y_+ \subset Y$ be the set of vertices of F' contained in $E_+(t_x)$ and let $Y_- \subset Y$ be the set of all endpoints of edges of F' which start in $E_+(t_x)$ and end in $E_h \setminus E_+(t_x)$. Then Y_+ is finite by Corollary 4.3.46 (ii) and Y_- is finite by the same result and the fact that the underlying graph of Ξ_h is locally finite by Theorem 4.3.38 (iv). As every vertex of F'_+ is either contained in $Y_0 := Y_+ \cup Y_-$ or a convex combination of two such points, it follows that $x \in F'_+ \subset \text{conv}(Y_0)$, as claimed. Since $x \in F'$ was arbitrary, we conclude that $F' \subset \text{conv}(Y)$ and the proof is complete. \square

This lemma now enables us to show that $P^{(i)}(t)$ is the convex hull of a finite union of $W^{(i)}$ -orbits.

Proposition 4.3.50. *The vertices of $P^{(i)}(t)$ that are contained in $\overline{C^{(i)}}$ form a non-empty finite set $\{x_1, \dots, x_q\}$. Each of these vertices is almost virtually $\mathcal{D}^{(i)}$ -singular and*

$$P^{(i)}(t) = \text{conv} \left(\bigcup_{p=1}^q W^{(i)} \cdot x_p \right). \quad (4.3.35)$$

Proof. Replacing \mathcal{D} by its I_{vis} -restriction $\mathcal{D}_{I_{\text{vis}}}$ does not change $\Xi_h = F_{I_{\text{vis}}}$ and $P^{(i)}(t)$ in any way, so we may assume that $h \in \overline{C}$ is virtually \mathcal{D} -singular, i.e. $|W_h| < \infty$. For every $x \in \mathfrak{a}$, we denote by $W_x^{(i)}$ the stabilizer of x under the $W^{(i)}$ -action on \mathfrak{a} .

Applying Lemma 4.3.49 to the trivial face yields that $P^{(i)}(t)$ is the convex hull of its vertices. Furthermore, every vertex is $W^{(i)}$ -equivalent to one in $\overline{C^{(i)}}$, so $P^{(i)}(t)$ also coincides with the convex hull of the union of all $W^{(i)}$ -orbits of the vertices that are contained in $\overline{C^{(i)}}$. In particular, the set of such vertices is not empty, so in order to prove the first statement, it remains to show that it is finite. Recall that every vertex of $P^{(i)}(t)$ belongs to one of two types: It is either a vertex of Ξ_h contained in $E^{(i)}(t)$, i.e. equal to $w(h)$ for some $w \in W$, or it lies in the relative interior of a unique edge of Ξ_h .

Let $x \in \overline{C^{(i)}}$ be a vertex of $P^{(i)}(t)$. Corollary 4.3.46 (i) implies that $\overline{E^{(i)}(t)}$ contains only a finite number of $W^{(i)}$ -orbits of vertices of Ξ_h . Since every such orbit meets $\overline{C^{(i)}}$ in a unique point, there are only finitely many possibilities for x to be of the first type. If x is of the second type, it lies in the relative interior of an edge of Ξ_h connecting a vertex on the positive side of $E^{(i)}(t)$ to one on the negative side, which is $W^{(i)}$ -equivalent to an edge that starts in $\overline{C^{(i)}} \cap E_+^{(i)}(t)$. By the same argument, there are only finitely many possible choices for the initial point of this edge and by Corollary 4.3.35 (iii), the number of edges of Ξ_h emanating from any particular vertex is finite. Hence, there are also only finitely many possibilities for x to be of the second type and the proof of the first assertion is complete.

If $\{x_1, \dots, x_q\}$ denotes the finite set of vertices of $P^{(i)}(t)$ that are contained in $\overline{C^{(i)}}$, then we have also proved (4.3.35). Given any $p \in \{1, \dots, q\}$, we now show that $|W_{x_p}^{(i)}| \leq |W_h|$ which will imply that x_p is virtually $\mathcal{D}^{(i)}$ -singular. Recall that we have replaced \mathcal{D} by $\mathcal{D}_{I_{\text{vis}}}$ (and so W by W_{vis}) in the beginning of the proof to make h virtually \mathcal{D} -singular, but the subgroup $W_{\text{inv}} \subset W$ fixes Ξ_h point-wise. Thus, once the inequality has been shown under the assumption that $|W_h| < \infty$, it will follow immediately that if h is only required to be almost virtually \mathcal{D} -singular, then x_p is almost virtually $\mathcal{D}^{(i)}$ -singular as well and the proof will be finished.

If $x_p = w(h)$ for some $w \in W$, then $W_{x_p}^{(i)} = wW_hw^{-1} \cap W^{(i)} = w(W_h \cap w^{-1}W^{(i)}w)w^{-1}$ is conjugate to a subgroup of W_h and the inequality is clear. Otherwise, x_p lies in the relative interior of an edge L of Ξ_h which has the form $L = [w(h), ws_k(h)]$ for some $w \in W$ and $k \in I \setminus I_h$, where we may assume that w and ws_k are both right I_h -reduced. We claim that $W_{x_p}^{(i)} = W_{w(h)}^{(i)} \cap W_{ws_k(h)}^{(i)}$. Once this is established, the required inequality will follow from the previous case. The inclusion “ \supset ” is obvious and it remains to prove the converse.

Set $J := \{j \in I^{(i)} : \langle \alpha_j, x_p \rangle = 0\}$. As the $W^{(i)}$ -stabilizer of $x_p \in \overline{C^{(i)}}$ coincides with the standard subgroup $W_J \subset W^{(i)}$, it suffices to show that every simple reflection in S_J fixes $w(h)$ and $ws_k(h)$. Suppose for a contradiction that there exists some $j \in J$ such that s_j does not fix both of these points or, equivalently, that L is not contained in the wall $H_j = \ker(\alpha_j)$. As $x_p \in H_j$ belongs to the relative interior of L , its endpoints lie in the two opposite open half-spaces defined by H_j and we may w.l.o.g. assume that $\langle \alpha_j, w(h) \rangle > 0$ and $\langle \alpha_j, ws_k(h) \rangle < 0$. Lemma 4.3.3 now implies that $l(w) < l(s_jw)$ and $l(s_jws_k) < l(ws_k)$. In other words, w is left $\{j\}$ -reduced but ws_k is not, so that $ws_k = s_jw$ by the folding condition. But then we have $\langle \omega_i, ws_k(h) \rangle = \langle \omega_i, s_jw(h) \rangle = \langle \omega_i, w(h) \rangle$, so ω_i is constant along L . Because $x_p \in E^{(i)}(t)$ lies on L , it follows that $L \subset E^{(i)}(t)$. Consequently, L is an edge of Ξ_h contained in $E^{(i)}(t)$ and thus an edge of $P^{(i)}(t)$, but x_p is a vertex of $P^{(i)}(t)$ and belongs to the relative interior of L , a contradiction. Therefore, the claim holds and the proof is complete. \square

Remark 4.3.51. The proof of Proposition 4.3.50 is more precise than the statement itself in the following sense: In order to show that the vertices $x_1, \dots, x_q \in P^{(i)}(t) \cap \overline{C^{(i)}}$ are almost virtually $\mathcal{D}^{(i)}$ -singular, we reduced the argument to the case where $h \in \overline{C}$ is virtually \mathcal{D} -singular and then established that $|W_{x_p}^{(i)}| \leq |W_h| < \infty$ holds for all $p \in \{1, \dots, q\}$. Intuitively, this inequality states that if we ignore all walls of X_+ that do not already belong to $X_+^{(i)}$, i.e. we only consider the walls of the form $w(H_j)$ for some $w \in W^{(i)}$ and $j \in I^{(i)}$, then x_1, \dots, x_q are “at most as singular” as h . In particular, it implies that if $h \in C$ is \mathcal{D} -regular, then $x_1, \dots, x_q \in C^{(i)}$ are $\mathcal{D}^{(i)}$ -regular. For the other walls of X_+ , however, we cannot make a similar statement. It is possible that such a wall contains some of these vertices but not h . For example, if $i \notin I_h$, then $[h, s_i(h)]$ is an edge of Ξ_h which intersects the wall H_i in a unique point $x \in \overline{C^{(i)}}$. Since ω_i is not constant along this edge, x is a vertex of $P^{(i)}(\langle \omega_i, x \rangle)$ that lies on H_i but h does not.

There is even a combinatorial way to determine the $W^{(i)}$ -stabilizers of x_1, \dots, x_q more precisely. Its formulation requires the notions of *Bruhat order* and its associated *Bruhat graph* that already briefly appeared in the proofs of Proposition 4.3.13 and 4.3.47. We do not introduce them here in detail and refer the interested reader to [BB05, Chapter 2]. Let Γ denote the (left) Bruhat graph of the Coxeter system (W, S) . The vertices of Γ are the elements of W and two vertices $w_1, w_2 \in W$ are connected by a directed edge pointing from w_1 to w_2 if and only if there exists a reflection $r \in R \subset W$ (which is not required to be simple) such that $w_2 = rw_1$ and $l(w_1) < l(w_2)$.

In this situation, the edge is labelled by r . We colour the vertices of Γ according to the left cosets of W_h in W , i.e. two vertices are assigned the same colour if and only if they lie in the same left W_h -coset. For $p \in \{1, \dots, q\}$, the stabilizer of x_p under the $W^{(i)}$ -action on \mathfrak{a} can be read off from this colouring of Γ .

As x_p is contained in $\overline{C^{(i)}}$, the stabilizer $W_{x_p}^{(i)}$ is generated by the simple reflections in $S^{(i)}$ that fix x_p . Suppose first that $x_p = w(h)$ for some $w \in W$ that we may assume to be right I_h -reduced. In this case, Lemma 4.3.3 states that the equality $\langle \alpha_j, w(h) \rangle = 0$ with $j \in I^{(i)}$ holds if and only if $l(w) < l(s_j w)$ and $s_j w$ is not right I_h -reduced. Since $\langle \alpha_j, w(h) \rangle \geq 0$ for all $j \in I^{(i)}$, the same result also shows that w is left $I^{(i)}$ -reduced, so the first condition is automatically satisfied and the second implies that $s_j w = ws_l$ for some $l \in I_h$. Hence, the reflection $s_j \in S^{(i)}$ fixes $x_p = w(h)$ if and only if the unique edge of Γ with label s_j containing w is directed away from w but ends in a vertex with the same colour. Thus, the stabilizer $W_{x_p}^{(i)}$ can be determined by finding the unique right I_h -reduced element $w \in W$ such that $x_p = w(h)$ and investigating which edges of Γ with labels in $S^{(i)}$ connect w to a vertex with the same colour.

Otherwise, x_p lies in the relative interior of an edge of Ξ_h having the form $[w(h), ws_k(h)]$ for some $w \in W$ and $k \in I \setminus I_h$, where w and ws_k may be chosen to be right I_h -reduced. In this case, we proved that a simple reflection in $S^{(i)}$ fixes x_p if and only if it fixes both $w(h)$ and $ws_k(h)$. The $W^{(i)}$ -stabilizers of these two endpoints can be determined by the procedure just described and $W_{x_p}^{(i)}$ is their intersection. Note that this method does not require the assumption that $|W_h| < \infty$. Moreover, if $h \in C$ is \mathcal{D} -regular, then no two vertices of Γ have the same colour and we again conclude that $x_1, \dots, x_q \in C^{(i)}$ are $\mathcal{D}^{(i)}$ -regular, as expected.

Proposition 4.3.50 states that $P^{(i)}(t)$ can be constructed by first forming the $W^{(i)}$ -orbits of the vertices $x_1, \dots, x_q \in P^{(i)}(t) \cap \overline{C^{(i)}}$, then unifying these sets and finally taking the convex hull of the resulting union. Our next goal is to explain that the last two steps can also be carried out in the opposite order, but it is necessary to replace $\{x_1, \dots, x_q\}$ by a larger set which is uncountable unless $q = 1$. More precisely, we will show that $P^{(i)}(t)$ can equivalently be constructed by first forming the convex hulls of the $W^{(i)}$ -orbits of all points in a suitable subset of $P^{(i)}(t)$ and then taking the union of these convex hulls. Trivially, the entire set $P^{(i)}(t)$ would have this property, but we will be interested in a certain “small” subset that only consists of proper faces of $P^{(i)}(t)$ and exploits its $W^{(i)}$ -invariance.

Definition 4.3.52. We call the union of all faces of $P^{(i)}(t)$ that are fully contained in $\overline{C^{(i)}}$ the *essential part* of $P^{(i)}(t)$ and denote this set by $P_{\text{ess}}^{(i)}(t)$.

Remark 4.3.53. As every face of $P^{(i)}(t)$ is the convex hull of its vertices, an arbitrary point $x \in P^{(i)}(t)$ lies in the essential part if and only if all vertices of the unique face F of $P^{(i)}(t)$ with $x \in \text{relint}(F)$ belong to the set $\{x_1, \dots, x_q\}$. In almost every situation, this implies that $F \subsetneq P^{(i)}(t)$ is a proper face. In view of (4.3.33), it suffices to distinguish three cases:

- If $i \in I_\infty$ and $t = \langle \omega_i, h \rangle$, then $P^{(i)}(t) = F^{(i)} = \text{conv}(W^{(i)} \cdot h)$ and $P_{\text{ess}}^{(i)}(t) = \{h\}$. Therefore, $P^{(i)}(t)$ coincides with its essential part if and only if $W^{(i)} \subset W_h$. Since $i \in I_{\text{vis}}$ requires $I_h \neq I$, this inclusion already implies $W^{(i)} = W_h$, so that i is the only white vertex of Δ . Combined with the assumption that h is almost virtually singular, this is a very restrictive condition. Let I_k for some $k \in \{1, \dots, m\}$ be the connected component of I containing i , then I_k is not spherical but $I_k \cap I_h = I_k \setminus \{i\}$ has to be.
- If $i \in I \setminus I_\infty$ and $t \in \{\langle \omega_i, h \rangle, \langle \omega_i, w_+(h) \rangle\}$, where w_+ denotes the longest word of $W_{I \setminus I_\infty}$, then $P^{(i)}(t)$ is either equal to $F^{(i)} = \text{conv}(W^{(i)} \cdot h)$ or $w_+(F^{(i)}) = \text{conv}(w_+ W^{(i)} \cdot h)$, where $j \in I \setminus I_\infty$ is determined by $s_j = w_+ s_i w_+$. In the first case, the same statement as in a) holds, but I_k is now already spherical and thus also $I_k \cap I_h = I_k \setminus \{i\}$ if $W^{(i)} = W_h$. In the second case, we have $P_{\text{ess}}^{(i)}(t) = \{w_+ w'_+(h)\}$, where w'_+ is the longest word of $W^{(j)} \cap W_{I \setminus I_\infty}$, and this single point is equal to $P^{(i)}(t)$ if and only if $W^{(j)} = W_h$.
- In any other situation, Ξ_h intersects both open half-spaces defined by $E^{(i)}(t)$, so we find $y_1, y_2 \in \Xi_h$ with $\langle \omega_i, y_1 \rangle > t$ and $\langle \omega_i, y_2 \rangle < t$. Suppose that there exists some $l \in I^{(i)}$ such that the simple reflection $s_l \in S^{(i)}$ does not fix Ξ_h point-wise. As these two half-spaces are $W^{(i)}$ -invariant and $\Xi_h \not\subset H_l$, we may choose y_1, y_2 in such a way that $\langle \alpha_l, y_1 \rangle, \langle \alpha_l, y_2 \rangle > 0$.

By forming a convex combination of y_1 and y_2 , we can construct a point $y \in P^{(i)}(t)$ with $\langle \alpha_l, y \rangle > 0$, which implies that $s_l(y) \neq y$. If $P^{(i)}(t)$ was equal to its essential part, then all its vertices would have to be contained in $\overline{C^{(i)}}$ and $W^{(i)}$ would fix $P^{(i)}(t)$ point-wise, but this is impossible now. The only situation in which such a reflection does not exist occurs when $W^{(i)} = W_{\text{inv}}$ or, equivalently, i is the only white vertex of Δ and an isolated node. Here, $\Xi_h = F_{\{i\}} = [h, s_i(h)]$ is a closed line segment and $P^{(i)}(t)$ a single point.

Altogether, $P_{\text{ess}}^{(i)}(t)$ only coincides with $P^{(i)}(t)$ if the latter is a single point and this happens in precisely three degenerate situations. In general, it is a union of proper faces of $P^{(i)}(t)$.

If $q = 1$, then $P_{\text{ess}}^{(i)}(t) = \{x_1\}$ consists of a single almost virtually $\mathcal{D}^{(i)}$ -singular point $x_1 \in \overline{C^{(i)}}$. In this situation, the face poset of $P^{(i)}(t) = \text{conv}(W^{(i)} \cdot x_1)$ can be described as in Theorem 4.3.38 upon replacing h by x_1 and all notions associated to \mathcal{D} by their analogues for $\mathcal{D}^{(i)}$. For example, if $x_1 \in C^{(i)}$ is $\mathcal{D}^{(i)}$ -regular, then $\mathcal{F}(P^{(i)}(t))$ is isomorphic to $\Sigma^*(W^{(i)}, S^{(i)})$. If $q > 1$, however, the essential part is an uncountable set as it contains a non-trivial line segment. In this case, $P^{(i)}(t)$ is the convex hull of the union of at least two $W^{(i)}$ -orbits and its face poset is more complicated. We will not attempt to describe $\mathcal{F}(P^{(i)}(t))$ in the same level of detail as in the case of a single orbit. For our purposes, it suffices to establish the following result which generalizes Proposition 4.3.50. On the one hand, it shows that the faces of $P^{(i)}(t)$ are still related to the standard left cosets of $(W^{(i)}, S^{(i)})$ if the vertices x_1, \dots, x_q are incorporated appropriately. On the other hand, it states that a face is either equivalent to one in the essential part or has a certain degree of symmetry.

Proposition 4.3.54. *If F is an arbitrary face of $P^{(i)}(t)$, there exist integers $s \in \{1, \dots, q\}$ and $p_1, \dots, p_s \in \{1, \dots, q\}$, an element $w \in W^{(i)}$ and a subset $J \subset I^{(i)}$ such that*

$$F = \text{conv} \left(\bigcup_{r=1}^s w W_J \cdot x_{p_r} \right). \quad (4.3.36)$$

Moreover, if F is not $W^{(i)}$ -equivalent to a face in $P_{\text{ess}}^{(i)}(t)$, there exists a parabolic subgroup of $(W^{(i)}, S^{(i)})$ that stabilizes F but does not fix it point-wise.

Proof. First, we assume that $\text{relint}(F) \cap \overline{C^{(i)}} \neq \emptyset$ and denote by $\{x_{p_1}, \dots, x_{p_s}\} \subset \{x_1, \dots, x_q\}$ the set of vertices of F that are contained in $\overline{C^{(i)}}$. In this situation, we claim that $s \geq 1$ and there exists some $J \subset I^{(i)}$ such that the set of all vertices of F is equal to the union $\bigcup_{r=1}^s W_J \cdot x_{p_r}$. Because F is the convex hull of its vertices by Lemma 4.3.49, this will prove the first statement under that assumption by taking $w = 1$.

We turn to the proof of the claim. Let $j \in I^{(i)}$ be arbitrary. Since $\mathcal{F}(P^{(i)}(t))$ is $W^{(i)}$ -invariant, $F' := s_j(F)$ is also a face of $P^{(i)}(t)$ and exactly one of the following three possibilities occurs for the relative position of F with respect to the wall $H_j = \ker(\alpha_j)$:

- a) If F is contained in H_j , then s_j fixes $F = F'$ point-wise.
- b) If $\text{relint}(F)$ meets both open half-spaces defined by H_j , then s_j does not fix F point-wise, but there still exists some $z \in \text{relint}(F)$ with $z = s_j(z)$. Hence, the relative interiors of F and F' have a point in common, which implies that $F = F'$ is stabilized by s_j .
- c) If $\text{relint}(F)$ is contained in one of the open half-spaces defined by H_j , then $\text{relint}(F')$ lies in the opposite half-space. As the relative interiors of F and F' are disjoint, it follows that $F \neq F'$ is not stabilized by s_j .

We denote by J_1 (resp. J_2) the set of all $j \in I^{(i)}$ for which a) (resp. b)) is satisfied and define $J := J_1 \cup J_2$. Given any $j \in I^{(i)}$, it follows that F is invariant under the simple reflection $s_j \in S^{(i)}$ if and only if $j \in J$ and the action of s_j on F is trivial if and only if $j \in J_1 \subset J$. Consequently, F is invariant under the standard subgroup $W_J \subset W^{(i)}$, so the union $\bigcup_{r=1}^s W_J \cdot x_{p_r}$ also consists of vertices of F . It remains to show that every vertex of F is contained in this set. In particular, this will prove that $s \geq 1$.

Assume to the contrary that there exists a vertex y of F that is not W_J -equivalent to one in $\overline{C^{(i)}}$. Let $w \in W^{(i)}$ be such that $w^{-1}(y) \in \overline{C^{(i)}}$ and write $w = uv$, where $u \in W_J$ and $v \in W^{(i)}$ is left J -reduced. By assumption on y , we have $w \notin W_J$ and hence $v \neq 1$. As F is W_J -invariant,

we further have $y' := u^{-1}(y) \in F$ and this point lies in $v(\overline{C^{(i)}})$ but not in $\overline{C^{(i)}}$. It follows that $\langle \alpha_j, y' \rangle < 0$ for some $j \in I^{(i)}$, so F meets the open half-space defined by H_j where α_j is negative. At the same time, it also has to intersect the half-space where α_j is positive because if this was not the case, then $\text{relint}(F)$ would be contained in the negative open half-space, contradicting the assumption that $\text{relint}(F) \cap \overline{C^{(i)}} \neq \emptyset$. Thus, F meets both open half-spaces, so that $j \in J_2 \subset J$. However, we also have $0 > \langle \alpha_j, y' \rangle = \langle v^{-1}(\alpha_j), v^{-1}(y') \rangle$ with $v^{-1}(y') \in \overline{C^{(i)}}$, so that $v^{-1}(\alpha_j) \in \Phi_-$ by (4.3.2) and $l(s_j v) < l(v)$ by (2.5.4), which contradicts the condition that v is left J -reduced. The proof of the claim is therefore complete.

The previous argument even shows that $F \subset P_{\text{ess}}^{(i)}(t)$ if and only if $J = J_1$, so if this does not hold, then $J_2 \neq \emptyset$ and the standard subgroup $W_{J_2} \subset W^{(i)}$ stabilizes F but does not fix it point-wise. The second statement is therefore also proved under the assumption that $\text{relint}(F) \cap \overline{C^{(i)}} \neq \emptyset$. In general, if F is an arbitrary face of $P^{(i)}(t)$, we may replace F by $w^{-1}(F)$ for a suitable $w \in W^{(i)}$ to ensure that $\text{relint}(w^{-1}(F)) \cap \overline{C^{(i)}} \neq \emptyset$. By applying the same reasoning to $w^{-1}(F)$, it follows that F has the form in (4.3.36) and is invariant under the parabolic subgroup $wW_J w^{-1} \subset W^{(i)}$ which acts trivially on F if and only if $w^{-1}(F)$ is contained in the essential part. \square

Informally, Proposition 4.3.54 states that if $q > 1$, the face poset of $P^{(i)}(t)$ only differs from that of one $W^{(i)}$ -orbit in the $W^{(i)}$ -translates of the “additional” faces formed by the vertices x_1, \dots, x_q which all lie in the closed fundamental $\mathcal{D}^{(i)}$ -chamber and do not cross any of its walls. In fact, $P^{(i)}(t)$ is completely determined by its essential part in the following sense:

Proposition 4.3.55. *Every point in $P^{(i)}(t)$ lies in the convex hull of the $W^{(i)}$ -orbit of a point in the essential part:*

$$P^{(i)}(t) = \bigcup_{y \in P_{\text{ess}}^{(i)}(t)} \text{conv}(W^{(i)} \cdot y) \quad (4.3.37)$$

Proof. Let $x \in P^{(i)}(t)$ be arbitrary and let F be the unique face of $P^{(i)}(t)$ such that $x \in \text{relint}(F)$. We prove the claim by induction on $\dim(F)$. Since $P^{(i)}(t)$ and its face poset are $W^{(i)}$ -invariant, we may w.l.o.g. assume that $x \in \overline{C^{(i)}}$. In this situation, the proof of Proposition 4.3.54 implies that at least one of the vertices of F is contained in $\{x_1, \dots, x_q\}$. Let x_p for some $p \in \{1, \dots, q\}$ be such a vertex.

If $\dim(F) = 0$, then $x = x_p \in P_{\text{ess}}^{(i)}(t)$ and there is nothing to prove, so we may suppose that $\dim(F) > 0$. If all vertices of F are contained in $\{x_1, \dots, x_q\}$, then $F \subset P_{\text{ess}}^{(i)}(t)$ and we are done. Otherwise, Proposition 4.3.54 yields the existence of a non-empty subset $J \subset I^{(i)}$ such that F is invariant under the standard subgroup $W_J \subset W^{(i)}$. Assume that x is not a global fixed-point of W_J and choose any $j \in J$ such that $s_j(x) \neq x$, then the fundamental weight $\omega_j \in \mathfrak{a}^*$ is not constant on the infinite line $L_0 \subset \mathfrak{a}$ spanned by x and $s_j(x)$. But it is bounded from above by $\langle \omega_j, h \rangle \in \mathbb{R}$ on $\Xi_h \supset F$, so L_0 intersects the relative boundary of the closed set F in at least one point $x_+ \in \partial_{\text{rel}}(F)$ which is contained in a proper face of F . Hence, the induction hypothesis asserts that there exists a point $y \in P_{\text{ess}}^{(i)}(t)$ with $x_+ \in \text{conv}(W^{(i)} \cdot y)$. Since x and $s_j(x)$ are contained in the relative interior of $F = s_j(F)$ and x_+ in its relative boundary, it follows that x_+ cannot lie on the closed line segment $[x, s_j(x)] \subset L_0$. This implies that $x \in [x_+, s_j(x_+)]$ is a convex combination of x_+ and $s_j(x_+)$ and we conclude that $x \in \text{conv}(W^{(i)} \cdot y)$.

If x is a global fixed-point of W_J , then L_0 cannot be defined, but this issue is easily dealt with. Let $V' \subset \mathfrak{a}$ be the affine span of F and $V := V' - x \subset \mathfrak{a}$ its parallel translate through the origin. Since we assume that F is not contained in the essential part, the proof of Proposition 4.3.54 shows that there exists some $j \in J$ such that the simple reflection s_j does not fix F point-wise. Thus, it restricts to a non-trivial involution on V , so -1 is an eigenvalue of $s_j|_V$ and there exists a non-zero eigenvector $v \in V \setminus \{0\}$ with $s_j(v) = -v$. Because $x \in \text{relint}(F)$, we may choose $v \neq 0$ sufficiently small to ensure that $x' := x + v \in F$ and write x as the non-trivial convex combination

$$x = \frac{1}{2}(x + v + x - v) = \frac{1}{2}(x' + s_j(x')).$$

Applying the argument above to the line spanned by x' and $s_j(x')$ yields that x' (and hence x) lies in the convex hull of $W^{(i)} \cdot y$ for some $y \in P_{\text{ess}}^{(i)}(t)$. \square

Example 4.3.56. Figure 4.3 illustrates the statement of Proposition 4.3.55 in two examples. Suppose that the Kac-Moody root datum \mathcal{D} , the element $h \in \mathfrak{a}$, the index $i \in I_{\text{vis}}$ and the value $t \in T_h^{(i)}$ are chosen in such a way that $P^{(i)}(t)$ is the convex hull of the union of three $\mathcal{D}^{(i)}$ -regular $W^{(i)}$ -orbits, where $W^{(i)}$ is the symmetric group S_3 in the first situation and the abelian group $(\mathbb{Z}_2)^3$ in the second. The respective convex hulls of the three individual $W^{(i)}$ -orbits are indicated by the overlapping blue, green and orange regions. The entire set $P^{(i)}(t)$ is the convex hull of these three subsets, which is not included in the figure, and its essential part is highlighted in red. In the first image, it is made up of two adjacent closed line segments and in the second, it consists of a single triangle. In both cases, it is easy to imagine that as a point moves over the red region, the convex hulls of the corresponding $W^{(i)}$ -orbits fill up all of $P^{(i)}(t)$. Figure 4.3a also includes the walls of the $\mathcal{D}^{(i)}$ -chambers. In 4.3b, these would be the three standard coordinate hyperplanes in \mathbb{R}^3 and the $\mathcal{D}^{(i)}$ -chambers the eight octants bounded by them.

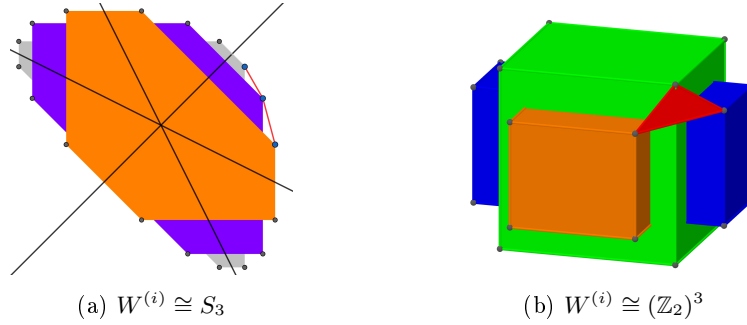


Figure 4.3: Two possible configurations of $P_{\text{ess}}^{(i)}(t)$

All results of this section were proved under the assumption that h is almost virtually singular and contained in \bar{C} , but they easily extend to an arbitrary almost virtually singular point $h \in \mathfrak{a}$ after replacing h by $\varepsilon w_0(h)$, where $\varepsilon \in \{1, -1\}$ and $w_0 \in W$ are chosen such that $\varepsilon w_0(h) \in \bar{C}$. In particular, Proposition 4.3.55 implies that we have established Theorem F. The property that $P^{(i)}(t)$ can be reconstructed from its essential part will be a crucial ingredient in proving the anticipated convexity theorems in Chapter 5.

4.4 Other orbit types

Up to now, most of our results about Ξ_h required the assumption that $h \in \mathfrak{a}$ is almost virtually singular. In the final section of this chapter, we take a brief look at the remaining orbit types. As a motivation, recall that we have already shown that Ξ_h is not closed if h is essentially singular, but in that case, it was still possible to construct some faces from standard left cosets of (W, S) . If h is critical or external, Ξ_h appears to be even less well-behaved and we will not attempt to establish any general results. Instead, we content ourselves with some interesting examples that illustrate the difficulties that arise in these situations. For simplicity, we restrict our attention to indecomposable GCM's in order to avoid possibly unpleasant mixing phenomena caused by their indecomposable components. We begin with a result that is well-known in Kac-Moody theory, but we have reformulated and simplified it to suit our purposes.

Proposition 4.4.1. *Let \mathbb{A} be an indecomposable GCM, \mathcal{D} a free and cofree Kac-Moody root datum associated to \mathbb{A} and consider the \mathcal{D} -representation of $W = W(\mathbb{A})$ on $\mathfrak{a} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$.*

- (i) *If \mathbb{A} is of finite type, every element of \mathfrak{a} is virtually singular.*
- (ii) *If \mathbb{A} is of affine type, every element of \mathfrak{a} is almost virtually singular or critical, but there exist critical elements.*
- (iii) *If \mathbb{A} is of indefinite type, there exist external elements in \mathfrak{a} .*

Proof. These assertions essentially follow from [Kac90, Proposition 5.8], but we add a few details and simplifications for the reader's convenience.

- (i) If \mathbb{A} is of finite type, then $X_+ = \mathfrak{a}$ and $|W| < \infty$ by Proposition 2.6.5 (vi), as required.
- (ii) Suppose that $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is of affine type and set $\delta := \sum_{i \in I} a_i \alpha_i \in Q_+ \subset \mathfrak{a}^*$, where $a_0, \dots, a_n \in \mathbb{N}$ are the labels of the vertices in $\Gamma(\mathbb{A})$. Using the well-known structure of the root systems of affine GCM's (cf. [Kac90, Theorem 5.6 b) and Proposition 6.3]), it holds that if \mathbb{A} is of untwisted type, then

$$\Phi^\infty = \Phi \cup \Phi^{\text{im}} = \{k\delta + \beta \mid k \in \mathbb{Z}, \beta \in \check{\Phi}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\},$$

where $\check{\Phi} := \Phi(\check{\mathbb{A}})$ denotes the finite root system of the reduced matrix $\check{\mathbb{A}} \in \mathbb{Z}^{n \times n}$. Such a root is positive if and only if $k > 0$ or $k = 0$ and $\beta \in \check{\Phi}_+$. Thus, Proposition 2.6.5 (iv) implies that a point $h \in \mathfrak{a}$ lies in X_+ if and only if it satisfies $\langle \delta, h \rangle > 0$ or $\langle \delta, h \rangle = 0$ and $\langle \beta, h \rangle = 0$ for all $\beta \in \check{\Phi}$. In particular, $X_+ \cup X_-$ contains the two open half-spaces where $\delta \in \mathfrak{a}^*$ is non-zero. It follows that $\overline{X_+ \cup X_-} = \mathfrak{a}$, so every $h \in \mathfrak{a}$ is internal or critical. If $\langle \delta, h \rangle \neq 0$, then $h \in X_+^\circ \cup X_-^\circ$ is virtually singular. If $\langle \delta, h \rangle = 0$, then h is internal if and only if $\langle \alpha_i, h \rangle = 0$ for all $i \in \check{I}$ and as $a_0 \neq 0$, these two conditions are equivalent to $\langle \alpha_i, h \rangle = 0$ for all $i \in I$. In this case, h is a global fixed-point for the W -action on \mathfrak{a} and hence almost virtually singular. If $\langle \delta, h \rangle = 0$ but $\langle \alpha_i, h \rangle \neq 0$ for some $i \in I$, then h is critical. For example, as $\langle \delta, \alpha_i^\vee \rangle = 0$ for every $i \in I$, the simple coroots $\alpha_0^\vee, \dots, \alpha_n^\vee \in \mathfrak{a}$ are all critical.

If \mathbb{A} is of twisted type, its real roots look slightly differently. If \mathbb{A} is not of type $A_{2n}^{(2)}$ or $D_4^{(3)}$, one further has to impose that k is even if β is a “long” root in order for $k\delta + \beta$ to define a real root. For type $D_4^{(3)}$, it must be divisible by three if β is long and for type $A_{2n}^{(2)}$, it may be chosen arbitrarily if β is long but it is only true that $\frac{1}{2}(k\delta + \beta) \in \Phi$ if k is odd. Apart from this subtlety, the argument above applies verbatim to the twisted types.

- (iii) Let $\mathbb{A} \in \mathbb{Z}^{n \times n}$ be of indefinite type and $i \in I$ be arbitrary. We claim that the simple coroot $\alpha_i^\vee \in \mathfrak{a}$ is external. On the one hand, as the set of positive imaginary roots is W -invariant, every such root is non-negative on $\overline{X_+}$. On the other hand, [Kac90, Theorem 5.6 c)] asserts that there exists some $\gamma \in \Phi_+^{\text{im}}$ that satisfies $\langle \gamma, \alpha_j^\vee \rangle < 0$ for every $j \in I$. In particular, $\langle \gamma, \alpha_i^\vee \rangle < 0$ implies that $\alpha_i^\vee \notin \overline{X_+}$. At the same time, since we have $\gamma' := s_i(\gamma) \in \Phi_+^{\text{im}}$ and $\langle \gamma', -\alpha_i^\vee \rangle = \langle \gamma, \alpha_i^\vee \rangle < 0$, we also conclude that $-\alpha_i^\vee \notin \overline{X_+}$ and the claim is proved. \square

Observe that Proposition 4.4.1 does not make any statement about essentially singular points. They certainly cannot exist for the finite and affine types, but even in the indefinite case, their existence is not guaranteed. For this, it is a necessary condition that the Weyl group has a proper infinite standard subgroup, but there are many examples of $(n \times n)$ -GCM's of indefinite type that are $(n-1)$ -spherical. To find critical elements, however, it suffices to turn to the affine case.

Example 4.4.2. The simplest example is the GCM of untwisted affine type $A_1^{(1)}$:

$$\mathbb{A} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \iff \Gamma(\mathbb{A}) = \underset{1}{\circ} \rightleftharpoons \underset{1}{\bullet}$$

Its Weyl group $W = W(\mathbb{A})$ is isomorphic to the infinite dihedral group D_∞ . We consider the free and cofree Kac-Moody root datum $\mathcal{D} = \mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ of rank 3 and the \mathcal{D} -representation of W on the three-dimensional real vector space $\mathfrak{a} = \Lambda_{\text{Kac}}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with basis $\mathcal{B} = \{\alpha_0^\vee, \alpha_1^\vee, h_2\}$, where $\langle \alpha_0, h_2 \rangle = 0$ and $\langle \alpha_1, h_2 \rangle = 1$. We denote by $\mathcal{B}^* = \{\omega_0, \omega_1, \phi_2\}$ the corresponding dual basis of \mathfrak{a}^* , so that $\alpha_0 = 2\omega_0 - 2\omega_1$ and $\alpha_1 = -2\omega_0 + 2\omega_1 + \phi_2$. If we identify \mathfrak{a} with \mathbb{R}^3 by mapping \mathcal{B} to the standard basis, the closed fundamental chamber is given by

$$\overline{C} = \{(x, y, z) \in \mathbb{R}^3 : 2y \leq 2x \leq 2y + z\}.$$

In the notation of the proof of Proposition 4.4.1, we have $\delta = \alpha_0 + \alpha_1 = \phi_2$, so the Tits cone of the \mathcal{D} -representation is the union of the open upper half-space and a line in the xy -plane where all roots vanish:

$$X_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\} \cup \mathbb{R} \cdot (1, 1, 0)$$

Thus, a point $(x, y, z) \in \mathbb{R}^3$ is internal if and only if $z \neq 0$ or $z = 0$ and $x = y$. In the first case, it is virtually singular and in the second, it is a global fixed-point for the induced W -action on \mathbb{R}^3 . It is critical if and only if $z = 0$ and $x \neq y$.

The blue region in Figure 4.4a depicts the set Ξ_h for the critical element $h := -\alpha_0^\vee = (-1, 0, 0)$ whose affine span is the xy -plane. We observe that Ξ_h is a closed subset of \mathfrak{a} , but it has a rather poor face poset. The relative boundary of Ξ_h consists of two parallel lines defined by the equations $y = x \pm 1$ and these are the only non-trivial faces. In particular, Ξ_h does not have any vertices. Furthermore, the closed line segments $[h, s_0(h)]$ and $[h, s_1(h)]$ run through its relative interior, so standard left cosets of (W, S) do not define faces of Ξ_h as in the almost virtually singular case. More generally, given any $w \in W$, the closed line segment $[h, w(h)]$ is contained in the relative boundary of Ξ_h if and only if w belongs to the subgroup of W generated by $s_0 s_1$, which is the translation subgroup of $W \cong D_\infty \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$. In terms of inequalities, it is clear that a point $h' = (x', y', z') \in \mathbb{R}^3$ is contained in Ξ_h if and only if $z' = 0$ and $|y' - x'| \leq 1$. As the elements of \mathcal{B}^* correspond to the standard coordinate functionals under the identification $\mathfrak{a} \cong \mathbb{R}^3$, the former condition is equivalent to $\langle \phi_2, h' \rangle = 0 = \langle \phi_2, h \rangle$ and the latter to

$$\langle \omega_1 - \omega_0, h' \rangle \leq 1 = \langle \omega_1 - \omega_0, h \rangle \quad \text{and} \quad \langle \omega_0 - \omega_1, h' \rangle \leq 1 = \langle \omega_0 - \omega_1, h \rangle.$$

It is simple to compute that if $\langle \phi_2, h' \rangle = 0$, then $\langle \omega_0 - \omega_1, h' \rangle = \langle \omega_1 - \omega_0, s_1(h') \rangle$, so the previous three conditions can be rewritten as follows:

$$h' \in \Xi_h \iff \begin{cases} \langle \omega_1 - \omega_0, h' \rangle \leq \langle \omega_1 - \omega_0, h \rangle \\ \langle \omega_1 - \omega_0, s_1(h') \rangle \leq \langle \omega_1 - \omega_0, h \rangle \\ \langle \phi_2, h' \rangle = \langle \phi_2, h \rangle \end{cases} \quad (4.4.1)$$

In addition, it is not difficult to verify that the generator $s_0 s_1$ of the translation subgroup of W acts on the xy -plane as the map $h' \mapsto h' - 2\langle \omega_1 - \omega_0, h' \rangle(\alpha_0^\vee + \alpha_1^\vee)$, so it shifts every point in that plane by a scalar multiple of the vector $\alpha_0^\vee + \alpha_1^\vee = (1, 1, 0)$ which lies in the kernel of $\omega_1 - \omega_0$. Because every $w \in W$ can be uniquely written as $w = s_1^\varepsilon (s_0 s_1)^k$ for some $\varepsilon \in \{0, 1\}$ and $k \in \mathbb{Z}$, the three conditions in (4.4.1) are already equivalent to the following:

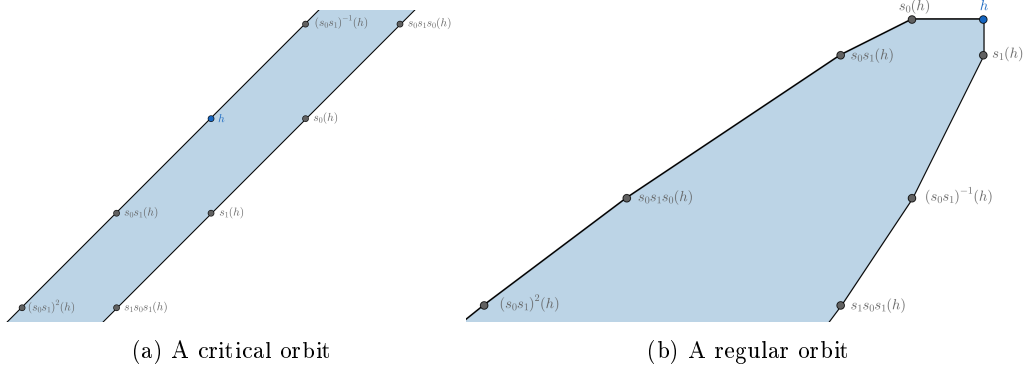
$$h' \in \Xi_h \iff \begin{cases} \langle \omega_1 - \omega_0, w(h') \rangle \leq \langle \omega_1 - \omega_0, h \rangle & \text{for all } w \in W \\ \langle \phi_2, h' \rangle = \langle \phi_2, h \rangle \end{cases} \quad (4.4.2)$$

This is now an infinite set of defining conditions resembling those in the almost virtually singular case (cf. Theorem 4.3.38 (v)), but there are two noteworthy differences. First, the inequalities in (4.4.2) only have to be verified for $w \in \{1, s_1\} \cong \mathbb{Z}_2$ and the remaining ones are all redundant. Second, $\omega_1 - \omega_0 \in \mathfrak{a}^*$ is not a dominant integral weight, so the irreducible highest-weight module $L(\omega_1 - \omega_0)$ of the Kac-Moody algebra of type \mathcal{D} over a field \mathbb{K} is not integrable. This is in sharp contrast to the almost virtually singular case where the supporting hyperplanes were level sets of dominant integral weights.

In the current example, the set Ξ_h can be constructed by translating the closed line segment $[h, s_1(h)]$ along the infinite line spanned by $\alpha_0^\vee + \alpha_1^\vee$ and its faces also arise from the three faces of that segment in this way. A more precise description of the face poset of Ξ_h is the following: Let $\tilde{\mathcal{D}}$ be the \tilde{I} -restriction of \mathcal{D} and consider the $\tilde{\mathcal{D}}$ -representation of $\tilde{W} = W(\tilde{\mathbb{A}}) \cong \mathbb{Z}_2$ on \mathfrak{a} . The face poset of Ξ_h is isomorphic to $\Sigma^*(\tilde{W}, \tilde{S})$, but the isomorphism is not the familiar map which takes a standard left coset of (\tilde{W}, \tilde{S}) to the convex hull of its action on h . Instead, it is defined as follows: Since $\langle \alpha_1, h \rangle = 2 > 0$, this point lies in the open fundamental $\tilde{\mathcal{D}}$ -chamber, so $\tilde{\Xi}_h := \text{conv}(\tilde{W} \cdot h) = [h, s_1(h)]$ is a convex polytope whose three faces are precisely the sets of the form $\text{conv}(wW_J \cdot h) \subset \tilde{\Xi}_h$, where $wW_J \subset \tilde{W}$ runs over the three standard left cosets of (\tilde{W}, \tilde{S}) . The three faces of Ξ_h are obtained by extending them along the infinite line spanned by $\alpha_0^\vee + \alpha_1^\vee$. More formally, the map

$$wW_J \mapsto \text{conv}(wW_J \cdot h) + \mathbb{R}(\alpha_0^\vee + \alpha_1^\vee)$$

defines a poset isomorphism $\Sigma^*(\tilde{W}, \tilde{S}) \rightarrow \mathcal{F}(\Xi_h)$. The line $\mathbb{R}(\alpha_0^\vee + \alpha_1^\vee)$ is the kernel of the matrix $\mathbb{A} = \mathbb{A}^T$ and coincides with the set of global fixed-points for the W -action on \mathfrak{a} .

Figure 4.4: Two configurations of Ξ_h in the two-dimensional affine case

Altogether, it appears that the geometry of Ξ_h does not reflect the combinatorial structure of the entire reversed Coxeter complex $\Sigma^*(W, S)$, but only that of the underlying finite subcomplex $\Sigma^*(\check{W}, \check{S})$ in the sense that $\Xi_h = \check{\Xi}_h + \mathbb{R}(\alpha_0^\vee + \alpha_1^\vee)$ and adding the line $\mathbb{R}(\alpha_0^\vee + \alpha_1^\vee)$ defines a poset isomorphism $\mathcal{F}(\check{\Xi}_h) \rightarrow \mathcal{F}(\Xi_h)$. Moreover, the defining inequalities for Ξ_h in (4.4.2) only have to be verified for the finite subgroup $\check{W} \cong \mathbb{Z}_2$.

To conclude this example, it is interesting to note that if $h \in \mathfrak{a}$ is any other critical element, the set Ξ_h looks almost the same, only the distance between the parallel lines might change because Ξ_h only depends on the value $|\langle \omega_1 - \omega_0, h \rangle| \in \mathbb{R}_{>0}$. In particular, we observe that Corollary 4.3.39 is not valid in the critical case. However, if we consider the point $h := \alpha_0^\vee + 3h_2 = (1, 0, 3)$ instead, then $h \in C$ is regular and the convex hull of its Weyl group orbit is shown in Figure 4.4b. Here, we recover the expected phenomenon that the faces of Ξ_h are in bijective correspondence with the standard left cosets of (W, S) and it is defined by the familiar (in-)equalities:

$$h' \in \Xi_h \iff \begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W \text{ and } i \in I \\ \langle \phi_2, h' \rangle = \langle \phi_2, h \rangle \end{cases}$$

The crucial difference to the critical case is that the affine span of Ξ_h is now the affine hyperplane defined by the equation $z = 3$ and the generator s_0s_1 of the translation subgroup of W does not simply shift every point in that plane by a scalar multiple of one fixed vector. This has the effect that the W -translates of h do not lie on straight lines but “curve away from h ”. Also, the picture suggests that Ξ_h is not defined by any finite family of (in-)equalities.

Example 4.4.3. Next, we apply the same considerations to the GCM of twisted affine type $D_3^{(2)}$ in order to verify the previous observations in a three-dimensional situation:

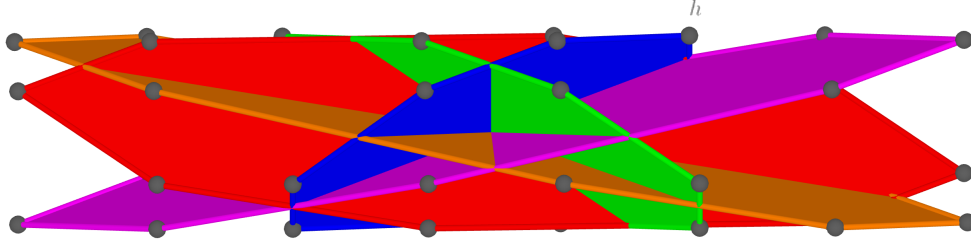
$$\mathbb{A} = \begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \iff \Gamma(\mathbb{A}) = \begin{array}{c} \circ \text{---} \bullet \text{---} \bullet \\ 1 \quad 1 \quad 1 \end{array}$$

Its Weyl group $W = W(\mathbb{A})$ is isomorphic to the Coxeter group with the following presentation:

$$\langle s_0, s_1, s_2 \mid s_0^2, s_1^2, s_2^2, (s_0s_1)^4, (s_0s_2)^2, (s_1s_2)^4 \rangle$$

Parallel to Example 4.4.2, we consider the free and cofree Kac-Moody root datum $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ of rank 4 and the \mathcal{D} -representation of W on the four-dimensional real vector space $\mathfrak{a} = \Lambda_{\text{Kac}}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with basis $\mathcal{B} = \{\alpha_0^\vee, \alpha_1^\vee, \alpha_2^\vee, h_3\}$, where $\langle \alpha_0, h_3 \rangle = \langle \alpha_1, h_3 \rangle = 0$ and $\langle \alpha_2, h_3 \rangle = 1$. As before, we denote by $\mathcal{B}^* = \{\omega_0, \omega_1, \omega_2, \phi_3\}$ the dual basis of \mathfrak{a}^* , so that $\alpha_0 = 2\omega_0 - \omega_1$, $\alpha_1 = -2\omega_0 + 2\omega_1 - 2\omega_2$ and $\alpha_2 = -\omega_1 + 2\omega_2 + \phi_3$, and we identify \mathfrak{a} with \mathbb{R}^4 by mapping \mathcal{B} to the standard basis of \mathbb{R}^4 . Hence, the closed fundamental chamber is now given by

$$\overline{C} = \{(x, y, z, u) \in \mathbb{R}^4 : 2x \geq y, y \geq x + z, 2z + u \geq y\}.$$

Figure 4.5: A visualization of Ξ_h for a critical element in the three-dimensional affine case

As $\delta = \alpha_0 + \alpha_1 + \alpha_2 = \phi_3$, the Tits cone is again the union of the open “upper half-space” and a line in the “ xyz -hyperplane” where all roots are identically zero:

$$X_+ = \{(x, y, z, u) \in \mathbb{R}^4 : u > 0\} \cup \mathbb{R} \cdot (1, 2, 1, 0)$$

Thus, a point $(x, y, z, u) \in \mathbb{R}^4$ is internal if and only if $u \neq 0$ or $u = 0$ and $2x = y = 2z$. In the first case, it is virtually singular and in the second, it is a global fixed-point for the induced W -action on \mathbb{R}^4 . It is critical if and only if $u = 0$ and $2x \neq y$ or $2z \neq y$.

Figure 4.5 is an attempt to visualize the set Ξ_h for the critical point $h := \frac{3}{2}\alpha_1^\vee + \alpha_2^\vee = (0, \frac{3}{2}, 1, 0)$, but requires some explanation. Let $\check{\mathcal{D}}$ be the \check{I} -restriction of \mathcal{D} and consider the $\check{\mathcal{D}}$ -representation of $\check{W} = W(\check{\mathbb{A}}) \cong \langle s_1, s_2 \mid s_1^2, s_2^2, (s_1 s_2)^4 \rangle \cong D_8$ on \mathfrak{a} . Since $\langle \alpha_1, h \rangle = 1 > 0$ and $\langle \alpha_2, h \rangle = \frac{1}{2} > 0$, this point lies in the open fundamental $\check{\mathcal{D}}$ -chamber, so $\check{\Xi}_h := \text{conv}(\check{W} \cdot h)$ is a convex polytope whose face poset is isomorphic to $\Sigma^*(\check{W}, \check{S})$. In fact, the dihedral group D_8 is the symmetry group of a regular octagon and the five octagons in Figure 4.5 are the following subsets of Ξ_h :

blue	green	red	orange	pink
$\check{\Xi}_h$	$s_0(\check{\Xi}_h)$	$s_1 s_0(\check{\Xi}_h)$	$s_0 s_1 s_0(\check{\Xi}_h)$	$s_2 s_1 s_0(\check{\Xi}_h)$

The entire set Ξ_h , which is not included in the figure, can be constructed as the convex hull of the union of all octagons of the form $w(\check{\Xi}_h)$, where w runs over the infinite set of right \check{I} -reduced elements of W . If we enumerate them by word length, the image shows the octagons associated to the shortest five elements $1, s_0, s_1 s_0, s_0 s_1 s_0, s_2 s_1 s_0$ and it highlights phenomena that are very similar to the previous two-dimensional example.

First of all, we again observe that standard left cosets of (W, S) do not define faces of Ξ_h in the usual way. For instance, the left \check{W} -cosets generate the octagons, but none of them is a face. Second, by rotating the image, it appears that all these octagons lie in a closed “octagonal tube” that is bounded by eight hyperplanes and whose “central axis” is the infinite line spanned by $\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee = (1, 2, 1, 0)$. Every such octagon $w(\check{\Xi}_h) \subset \Xi_h$, where $w \in W$ is right \check{I} -reduced, “fills out” that tube in the sense that every infinite line that is parallel to the central axis and lies inside the tube also intersects the octagon. Moreover, increasing the word length of w increases the diameter of $w(\check{\Xi}_h)$. For every $k \in \mathbb{N}_0$, we denote by P_k the convex hull of the union of all octagons $w(\check{\Xi}_h)$, where $w \in W$ is right \check{I} -reduced with $l(w) \leq k$. The figure suggests that, as k tends to infinity, the sequence $(P_k)_{k=0}^\infty$ of convex polytopes “converges to” the octagonal tube in the sense that $P_k \subset P_{k+1}$ holds for all $k \in \mathbb{N}_0$ and $\Xi_h = \bigcup_{k=0}^\infty P_k$ coincides with the entire tube. If this visual conclusion is correct, then Ξ_h is closed and has precisely one three-dimensional face, eight two-dimensional faces, eight edges, but no vertices.

The face poset of Ξ_h seems to be related to that of $\check{\Xi}_h$ exactly as in Example 4.4.2 in the sense that Ξ_h is obtained by translating $\check{\Xi}_h$ along the line spanned by $\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee$ and the map

$$wW_J \mapsto \text{conv}(wW_J \cdot h) + \mathbb{R}(\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee)$$

defines a poset isomorphism $\Sigma^*(\check{W}, \check{S}) \rightarrow \mathcal{F}(\Xi_h)$. The line $\mathbb{R}(\alpha_0^\vee + 2\alpha_1^\vee + \alpha_2^\vee)$ is the kernel of \mathbb{A}^T and coincides with the set of global fixed-points for the W -action on \mathfrak{a} . It is also not difficult to

determine the hyperplanes bounding the tube explicitly, which leads to a characterization of Ξ_h by linear (in-)equalities. The result is that a point $h' = (x', y', z', u') \in \mathbb{R}^4$ lies in Ξ_h if and only if $u' = 0$ and $|z' - x'| \leq 1$, $|y' - 2x'| \leq \frac{3}{2}$, $|2z' - y'| \leq \frac{3}{2}$ and $|y' - x' - z'| \leq 1$. Moreover, by computing the action of \check{W} on the xyz -hyperplane, one can show that these five conditions are equivalent to the following criterion:

$$h' \in \Xi_h \iff \begin{cases} \langle \omega_2 - \omega_0, w(h') \rangle \leq \langle \omega_2 - \omega_0, h \rangle & \text{for all } w \in \check{W} \\ \langle \omega_1 - 2\omega_0, w(h') \rangle \leq \langle \omega_1 - 2\omega_0, h \rangle & \text{for all } w \in \check{W} \\ \langle \phi_3, h' \rangle = \langle \phi_3, h \rangle \end{cases} \quad (4.4.3)$$

In the first two lines, there are 16 inequalities, but 8 of them are redundant because s_1 (resp. s_2) stabilizes $\omega_2 - \omega_0$ (resp. $\omega_1 - 2\omega_0$), so replacing w by $s_1 w$ (resp. $s_2 w$) in the first (resp. second) line results in the same inequality. This redundancy was not visible in (4.4.1) since \mathbb{Z}_2 does not have any non-trivial standard subgroups. Figure 4.5 also suggests that these conditions are sufficient to imply that the inequalities in the first two lines of (4.4.3) hold for all $w \in W$. Thus, we observe the same phenomenon as in Example 4.4.2, i.e. that Ξ_h is defined by an infinite set of inequalities involving the full Weyl group W , but they only have to be verified for the finite subgroup \check{W} . Furthermore, the functionals $\omega_2 - \omega_0$ and $\omega_1 - 2\omega_0$ whose level sets yield the defining hyperplanes of Ξ_h are not dominant integral weights.

The similarities between the previous two examples definitely motivate to study the set Ξ_h in the critical affine case in more detail. In particular, it would be desirable to prove our observations rigorously and generalize them to arbitrary GCM's of affine type. For the purposes of this thesis, however, we content ourselves with these examples and leave the general case open as an interesting problem for further research. To conclude this chapter, we briefly turn to GCM's of indefinite type in order to look at external elements. Here, we restrict to a two-dimensional example because the Tits cone is rather difficult to compute in higher dimensions.

Example 4.4.4. For arbitrary $a, b \in \mathbb{N}$ with $ab > 4$, we consider the GCM $\mathbb{A} = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ of indefinite type whose Weyl group $W = W(\mathbb{A})$ is isomorphic to the infinite dihedral group D_∞ . We use the free and cofree Kac-Moody root datum $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ of rank 2 and we investigate the \mathcal{D} -representation of W on the two-dimensional real vector space $\mathfrak{a} = \Lambda_{\text{sc}}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. We take the dual bases $\mathcal{B} = \{\alpha_1^\vee, \alpha_2^\vee\} \subset \mathfrak{a}$ and $\mathcal{B}^* = \{\omega_1, \omega_2\} \subset \mathfrak{a}^*$, so that $\alpha_1 = 2\omega_1 - b\omega_2$ and $\alpha_2 = -a\omega_1 + 2\omega_2$, and we identify \mathfrak{a} with \mathbb{R}^2 by mapping \mathcal{B} to the standard basis, so that the closed fundamental chamber is given by

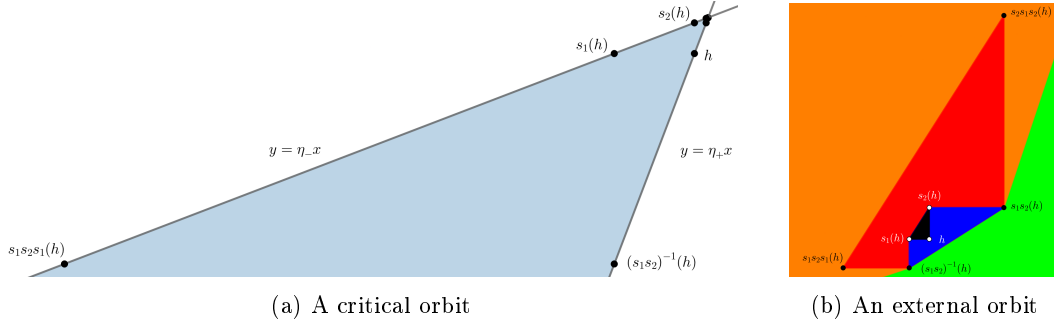
$$\overline{C} = \{(x, y) \in \mathbb{R}^2 : 2x \geq by, 2y \geq ax\}.$$

By explicitly computing the W -translates of \overline{C} (cf. [Mar18, Exercise 4.39]), it is possible to show that the Tits cone is the set

$$X_+ = \{0\} \cup \{(x, y) \in \mathbb{R}^2 : \eta_+ x < y < \eta_- x\} \quad \text{where} \quad \eta_\pm := \frac{ab \pm \sqrt{ab(ab-4)}}{2b}.$$

The two lines defined by the equations $y = \eta_\pm x$ divide \mathbb{R}^2 into four open connected components. Two of them are the sets X_+° and X_-° of virtually singular elements, the other two consist of external elements. The critical elements are precisely the non-zero points on these two lines and the origin is the only almost virtually singular point that is not virtually singular. Observe that there do not exist any essentially singular elements, which reflects the fact that $W \cong D_\infty$ does not have any proper infinite standard subgroups. In the following two concrete examples, we consider the case $a = b = 3$ where $\eta_\pm = \frac{3 \pm \sqrt{5}}{2}$.

- (i) In Figure 4.6a, the author has made a rather poor attempt to visualize the set Ξ_h for the critical point $h := -\alpha_1^\vee - \eta_+ \alpha_2^\vee = (-1, -\eta_+)$. All W -translates of h lie on the two infinite open half-lines $\mathbb{R}_{>0} \cdot (-1, -\eta_\pm)$, whose union is $\overline{X_+} \setminus X_+$, and the picture is intended to suggest that they accumulate at the origin and also diverge to infinity on both rays. Thus, Ξ_h coincides with the convex hull of these open half-lines, which is simply the set $\overline{X_+} \setminus \{0\}$,

Figure 4.6: Two configurations of Ξ_h in the two-dimensional indefinite case

and they are the only non-trivial faces of Ξ_h . In contrast to the critical orbit in Figure 4.4a, these faces are only rays instead of bi-infinite lines and not parallel. Moreover, Ξ_h is not closed in the current situation. Every point $h' = (x', y') \in \Xi_h$ satisfies the inequalities $\eta_+ x' \leq y' \leq \eta_- x'$ but conversely, an element of \mathbb{R}^2 with these properties only lies in Ξ_h if it is non-zero. Hence, Ξ_h is not determined by weak inequalities. What is also striking in this example is that Ξ_h hardly depends on h . Any other critical element is a non-zero scalar multiple of h or $s_1(h)$ and leads either to the same set or its negative. A similar phenomenon occurred in Example 4.4.2 as well, but here it is even stronger. Altogether, it appears that the face poset of Ξ_h does not contain a lot of meaningful information in the critical indefinite case. However, if $h \in C$ is regular, the set Ξ_h looks very similar to the one in Figure 4.4b, which mirrors the fact that the GCM's in these two examples have the same Weyl group, even though their types are different.

- (ii) Figure 4.6b is an illustration of the set Ξ_h for the external element $h := \alpha_1^\vee = (1, 0)$, but the colours require some explanation. For every $k \in \mathbb{N}_0$, we set $U_k := \{w \in W : l(w) \leq k\}$ and $P_k := \text{conv}(U_k \cdot h) \subset \Xi_h$, then P_k is a convex polytope and the five overlapping coloured regions are the following subsets of Ξ_h :

black	blue	red	orange	green
P_1	P_2	P_3	P_4	P_5

The image suggests that the sequence $(P_k)_{k=0}^\infty$ of convex polytopes “converges to” \mathbb{R}^2 in the sense that $P_k \subset P_{k+1}$ for all $k \in \mathbb{N}_0$ and $\Xi_h = \bigcup_{k=0}^\infty P_k = \mathbb{R}^2$. In other words, Ξ_h is obtained by stacking these convex polytopes on top of each other, but it appears that they “grow in all directions” of \mathbb{R}^2 , meaning that for every $k \in \mathbb{N}_0$, there exists some $l \in \mathbb{N}$ such that P_k is contained in the relative interior of P_l . Hence, $\Xi_h = \mathbb{R}^2$ does not have any non-trivial faces, let alone any other meaningful structure. The same phenomenon arises from any other external element, so Ξ_h is even less determined by h than in the critical case. The reader may find it instructive to compare this example with the proof of Proposition 4.3.24, where we have essentially shown that if $h \in C$ is regular, the sequence $(P_k)_{k=0}^\infty$ only “grows in one direction” in the sense that if $k \in \mathbb{N}_0$ is fixed, then “most” faces of P_k are also faces of P_l for all $l \in \mathbb{N}$ with $l > k$.

While it would certainly be desirable to prove that the observations in Example 4.4.4 are generic phenomena that occur for all GCM's of indefinite type, it is now also clear that there is no reason to expect any further interesting general results about Ξ_h when $h \in \mathfrak{a}$ is critical or external. Thus, we will not study these situations beyond the examples of this section.

Altogether, if we return to the fully general situation of an arbitrary free and cofree Kac-Moody root datum \mathcal{D} , the extensive discussion of this chapter can be summarized overly briefly by saying that Ξ_h has a rich and well-behaved facial structure whenever $h \in \mathfrak{a}$ is almost virtually singular. In this case, it is a closed subset of \mathfrak{a} that is characterized by a family of weak linear inequalities and its face poset has an elegant combinatorial description as a quotient of the reversed Coxeter

complex $\Sigma^*(W, S)$. If h is essentially singular, every standard left coset of (W, S) still defines a face of Ξ_h in the same way, but Ξ_h is no longer closed. If \mathbb{A} is of affine type and h is critical, it appears that Ξ_h is again closed and that its face poset is obtained by adding to each face of the convex polytope $\hat{\Xi}_h \subset \Xi_h$ the one-dimensional kernel of \mathbb{A}^T . In this situation, Ξ_h is determined by a family of weak linear inequalities as well, but in contrast to the almost virtually singular case, the functionals appearing in these inequalities are not necessarily dominant integral weights. If \mathbb{A} is of indefinite type and h is critical or external, it seems that Ξ_h does not have any meaningful structure at all.

Chapter 5

Kostant's Convexity Theorems for Kac-Moody Groups

Armed with the rich structure theory of Kac-Moody groups & algebras and the precise geometric information about convex hulls of Weyl group orbits, we are now finally in a position to state and prove the anticipated generalizations of *Kostant's Convexity Theorems* to the Kac-Moody setting. At the heart of this chapter lies the proof of Theorem A, a variant of the “non-linear” convexity theorem for the class of split real and complex Kac-Moody groups. Its precise formulation will be given in Section 5.1 and its proof is contained in Section 5.2. It is divided into three steps that crucially rely on the results of the previous chapters. Afterwards, in Section 5.3 and 5.4, we proceed in the exact same way to prove an analogue of the “linear” convexity theorem. In this situation, our result is closely related to a convexity theorem for Kac-Moody algebras established by Kac & Peterson in [KP84]. We conclude this chapter in Section 5.5 by comparing and contrasting our two theorems with theirs and Kostant's original formulations.

We retain the notation of the previous chapters. We continue to denote by $\mathbb{A} \in \mathbb{Z}^{n \times n}$ a GCM, by $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ an arbitrary Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to \mathbb{A} and we again consider the \mathcal{D} -representation of the Weyl group $W := W(\mathbb{A})$ on the d -dimensional real vector space $\mathfrak{a} := \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$ with open (resp. closed) fundamental chamber C (resp. \overline{C}) and Tits cone X_+ . Given any point $h \in \mathfrak{a}$, we write $\Xi_h := \text{conv}(W \cdot h) \subset \mathfrak{a}$ for the convex hull of its Weyl group orbit under the \mathcal{D} -representation. For our theorems to be valid, we further have to assume that the following conditions are satisfied throughout Chapter 5:

- The GCM \mathbb{A} is symmetrizable.
- $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is the field of real or complex numbers.

Therefore, we adopt the notational conventions and results following Assumption 2.7.1 and 2.7.6. We also allow \mathbb{A} to be decomposable and apply Notation 4.1.1 to denote all objects associated to its indecomposable components.

5.1 The Non-Linear Convexity Theorem

Let $G := G_{\mathcal{D}}(\mathbb{K})$ be the (split, minimal) Kac-Moody group of type \mathcal{D} over \mathbb{K} with (\mathbb{K} -split) torus $T := T_{\Lambda}(\mathbb{K})$ and root groups $(U_{\alpha})_{\alpha \in \Phi}$ defined via the constructive Tits functor. Let $G = KAU_+$ be an Iwasawa decomposition of G and $\Pi_A : G \rightarrow A$, $g = kau \mapsto a$, the Iwasawa projection onto the A -component. The following analogue of Kostant's Non-Linear Convexity Theorem in the Kac-Moody setting is Theorem A from the introduction and the principal result of this thesis:

Theorem 5.1.1 (Kostant's Non-Linear Convexity Theorem for Kac-Moody groups). *If $h \in \mathfrak{a}$ is almost virtually singular and $a := \exp(h) \in A$, then*

$$\{\Pi_A(ak) \mid k \in K\} = \exp(\Xi_h). \quad (5.1.1)$$

- Remark 5.1.2.* (i) Recall that the exponential map defines a W -equivariant isomorphism $\mathfrak{a} \rightarrow A$ whose inverse is denoted by $\log : A \rightarrow \mathfrak{a}$. Thus, Theorem 5.1.1 can (and should) be viewed as a geometric description of the set $\Pi_A(aK) = \{\Pi_A(ak) \mid k \in K\}$ whenever $a \in A$ is such that $h := \log(a) \in \mathfrak{a}$ is almost virtually singular.
- (ii) In contrast to the abelian group $A = \exp(\mathfrak{a})$, the compact form K depends on the field. It is clear that $K(\mathbb{R}) \subset K(\mathbb{C})$, but the latter is much larger. Hence, it is a surprising consequence of Theorem 5.1.1 that the set $\Pi_A(aK)$ is the same in the real and complex case.
- (iii) The statement of Theorem 5.1.1 requires that $h \in \mathfrak{a}$ is almost virtually singular. The reason for this restriction will be clear from the proof which involves many results of Chapter 4 that are not true without this condition. At the time of writing, the author is not aware whether it can be extended to the essentially singular, critical or external case by a different argument, but Proposition 4.3.41 and the discussion in Section 4.4 imply that Ξ_h is simply not sufficiently well-behaved in these situations, so that our methods do not seem to be a promising approach. Likewise, the proof will clarify why symmetrizability of A is required. The same assumptions are necessary in Theorem 5.3.1 which is proved in a very similar way.

5.2 Proof of the non-linear case

In order to prepare for the proof of Theorem 5.1.1, we fix an arbitrary almost virtually singular element $h \in \mathfrak{a}$, set $a := \exp(h) \in A$ and define the set

$$\Pi_a := \Pi_A(aK) = \{\Pi_A(ak) \mid k \in K\}. \quad (5.2.1)$$

Thus, we need to show that $\Pi_a = \exp(\Xi_h)$. We often tacitly exploit two simple facts:

- a) By definition of Π_A , it holds that $\Pi_A(kg) = \Pi_A(g)$ for all $g \in G$ and $k \in K$.
- b) Let $J \subset I$ be arbitrary and let \mathcal{D}_J denote the J -restriction of \mathcal{D} . By construction, the subgroup G_J of G generated by the entire torus T and the subfamily $(U_\alpha)_{\alpha \in \Phi_J}$ of the root groups can be naturally identified with the Kac-Moody group $G_{\mathcal{D}_J}(\mathbb{K})$ of type \mathcal{D}_J over \mathbb{K} . Therefore, all results of Chapter 3 can also be applied to G_J upon replacing all notions associated to \mathcal{D} by their appropriate analogues for \mathcal{D}_J .

Furthermore, the following restrictions will simplify the argument and introduce some notation that will be used afterwards.

Lemma 5.2.1. *It is sufficient to prove Theorem 5.1.1 under the additional assumptions that h is contained in $\overline{C} \subset X_+$ and not a global fixed-point of any irreducible factor of (W, S) .*

Proof. As $h \in \mathfrak{a}$ is assumed to be almost virtually singular, at least one of the points h and $-h$ is contained in X_+ and we claim that we may w.l.o.g. suppose that $h \in X_+$. To this end, observe that $\Xi_{-h} = -\Xi_h$ holds by linearity of the W -action and $\exp(\Xi_{-h}) = \exp(-\Xi_h) = \exp(\Xi_h)^{-1}$ by definition of the exponential map. Likewise, we also have $\exp(-h) = \exp(h)^{-1} = a^{-1}$ and $\Pi_{a^{-1}} = \Pi_a^{-1}$ holds by the following argument:

If $a' \in \Pi_a$, there exist $k, k' \in K$ and $u' \in U_+$ such that $ak = k'a'u'$. This can be rewritten as $a'u' = k'^{-1}ak = (k'^{-1}a^{-1}k')^{-1}$, which implies $u'^{-1}a'^{-1} = k'^{-1}a^{-1}k'$. Since $A \subset T$ normalizes U_+ , we have $u'^{-1}a'^{-1} = a'^{-1}u''$ for some $u'' \in U_+$ and $ka'^{-1}u'' = a^{-1}k'$, showing that $a'^{-1} \in \Pi_{a^{-1}}$. We find that $\Pi_a^{-1} \subset \Pi_{a^{-1}}$ and the converse inclusion follows by reversing the argument. Hence, it suffices to prove Theorem 5.1.1 either for h or $-h$ and it is no restriction to assume that $h \in X_+$ and $a \in \exp(X_+)$.

Next, we claim that Ξ_h (resp. Π_a) does not depend on the W -translate of h (resp. a). For Ξ_h , this is obvious and for Π_a , we may argue as follows: Let $w \in W$ be arbitrary and let $\tilde{w} \in \tilde{W} \subset K$ be the canonical representative of w in the extended Weyl group. By the defining relations of G , conjugating the elements of A by \tilde{w} induces the natural action of w on $A \cong \mathfrak{a}$, so that

$$\Pi_{w(a)} = \Pi_A(w(a)K) = \Pi_A(\tilde{w}a\tilde{w}^{-1}K) = \Pi_A(a\tilde{w}^{-1}K) = \Pi_A(aK) = \Pi_a.$$

Since $\exp(w(h)) = w(\exp(h)) = w(a)$ holds for all $w \in W$, we may replace h and a by a suitable common W -translate to ensure that $h \in \overline{C}$ and $a \in \exp(\overline{C})$.

Once this is satisfied, write $I_h := \{i \in I : \langle c_i, h \rangle = 0\}$, so that h is contained in the cell $D_{I_h} \subset \overline{C}$ and its stabilizer $W_h \subset W$ is the standard subgroup $W_{I_h} \subset W$. We endow the Coxeter diagram $\Delta := \Delta(W, S)$ with the colouring defined by the set $I^\bullet = I_h$ and apply all notions introduced in Section 1.2 (e.g. visibility) to this situation. With this terminology, the assumption that no irreducible factor of (W, S) fixes h is equivalent to the condition that no connected component of Δ only consists of black vertices or, equivalently, that $I = I_{\text{vis}}$. To justify this restriction, let G_{vis} (resp. G_{inv}) be the subgroup of G generated by the torus T and all root groups U_α with $\alpha \in \Phi_{I_{\text{vis}}}$ (resp. $\alpha \in \Phi_{I_{\text{inv}}}$) and set $K_{\text{vis}} := G_{\text{vis}} \cap K$ (resp. $K_{\text{inv}} := G_{\text{inv}} \cap K$). We claim that $K = K_{\text{inv}} K_{\text{vis}}$ and we only need to prove the inclusion “ \subset ”.

Let K' be the subgroup of K generated by the K^0 -images $(K_i)_{i \in I}$, then $K = MK'$ holds by Corollary 3.4.11. Applying the same result to the I_{vis} -restriction of \mathcal{D} implies that $K_{\text{vis}} = MK'_{\text{vis}}$, where K'_{vis} is the subgroup of K_{vis} generated by $(K_i)_{i \in I_{\text{vis}}}$, and the obvious analogue of this statement is true for K_{inv} . If $i \in I_{\text{vis}}$ and $j \in I_{\text{inv}}$, the G^0 -images $G_i = \varphi_i(G^0) = \langle U_i, U_{-i} \rangle$ and $G_j = \varphi_j(G^0) = \langle U_j, U_{-j} \rangle$ commute by Remark 3.1.2 (ii), so the same holds for $K_i = \varphi_i(K^0) \subset G_i$ and $K_j = \varphi_j(K^0) \subset G_j$. Hence, the decomposition $K = MK'$ can be refined to $K = MK'_{\text{vis}} K'_{\text{inv}}$. By taking inverses, we conclude that $K = K'_{\text{inv}} K'_{\text{vis}} M \subset K_{\text{inv}} K_{\text{vis}}$, as claimed.

Given any $i \in I$, the condition $\langle c_i, h \rangle = 0$ is equivalent to $a(c_i) = 1$ by Lemma 3.4.15, so (3.1.8) shows that $a \in A \subset T$ commutes with U_α whenever $\alpha \in \Phi_{I_{\text{inv}}}$ and thus with all of G_{inv} . An arbitrary $k \in K$ can now be written as $k = k_1 k_2$ with $k_1 \in K_{\text{inv}}$ and $k_2 \in K_{\text{vis}}$, so that $\Pi_A(ak) = \Pi_A(ak_2)$ and hence $\Pi_A(aK) = \Pi_A(aK_{\text{vis}})$. At the same time, we also have $\Xi_h = F_{I_{\text{vis}}}$, so it is enough to prove Theorem 5.1.1 for the I_{vis} -restriction of \mathcal{D} . \square

In particular, the assumption on the stabilizer of h has the effect that $\dim(\Xi_h) = |I_{\text{vis}}| = |I| = n$ by Proposition 4.3.2. Moreover, it implies that h is virtually singular, i.e. $|W_h| < \infty$, since an almost virtually singular element can only have infinite stabilizer if it is a global fixed-point of some infinite irreducible factor of (W, S) .

The proof of Theorem 5.1.1 is divided into three steps. In the first two, we additionally require the Kac-Moody root datum \mathcal{D} to be free, cofree and cotorsion-free. Under this assumption, we show that $\Pi_a \subset \exp(\Xi_h)$ by verifying certain inequalities in highest-weight representations of G . Afterwards, we prove that $\exp(\Xi_h) \subset \Pi_a$ by induction on the dimension of Ξ_h or, equivalently, the size of \mathbb{A} . In the third step, we generalize the result to arbitrary root data using the functoriality developed in Section 3.6. For Section 5.2.1 and 5.2.2, however, we apply Assumption 2.7.5 and the related notational conventions. In particular, $d \geq n$ and we choose the usual dual \mathbb{Z} -bases

$$\mathcal{B} = \{\alpha_1^\vee, \dots, \alpha_n^\vee, h_{n+1}, \dots, h_d\} \subset \Lambda^\vee \quad \text{and} \quad \mathcal{B}^* = \{\omega_1, \dots, \omega_n, \phi_{n+1}, \dots, \phi_d\} \subset \Lambda.$$

In contrast to Chapter 4, it is really necessary that \mathcal{D} is cotorsion-free as well, so that \mathcal{B}^* consists of proper dominant integral weights. The reason for this will be apparent from the first step below. Altogether, we invoke the following restrictions:

Assumption 5.2.2. Throughout the proof of Theorem 5.1.1, we may w.l.o.g. assume that $h \in \overline{C}$ is virtually singular and that $I = I_{\text{vis}}$, which implies $\dim(\Xi_h) = n$. In Step 1 and 2, we further suppose that \mathcal{D} is free, cofree and cotorsion-free. In Step 3, \mathcal{D} is allowed to be arbitrary.

5.2.1 Step 1: Highest-weight representations

Having fixed the setup, we begin with the proof of Theorem 5.1.1 by showing that $\Pi_a \subset \exp(\Xi_h)$. Let $k \in K$ be arbitrary and write $ak \in G$ in Iwasawa form as

$$ak = k'a'u' \quad \text{with} \quad k' \in K, a' \in A, u' \in U_+.$$

We need to prove that $h' := \log(a') \in \mathfrak{a}$ is contained in Ξ_h . Since $h \in \overline{C}$ is virtually singular, Corollary 4.3.30 implies that it suffices to verify the following (in-)equalities:

$$\begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W \text{ and } i \in I \\ \langle \phi_i, h' \rangle = \langle \phi_i, h \rangle & \text{for all } i \in \{n+1, \dots, d\} \end{cases}$$

We will achieve this by comparing the actions of $ak = k'a'u'$ arising from the two expressions in suitable highest-weight representations of G . Let $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ denote the Kac-Moody algebra of type \mathcal{D} over \mathbb{K} . By Proposition 2.7.8 and 3.4.18, the basis $\mathcal{B}^* \subset \Lambda$ consists of proper dominant integral weights and the group G acts on the real or complex irreducible highest-weight \mathfrak{g} -modules $L(\omega_1), \dots, L(\omega_n), L(\phi_{n+1}), \dots, L(\phi_d)$. Since \mathbb{A} is symmetrizable, each of these modules carries a positive-definite, contravariant, symmetric bilinear or Hermitian form H such that weight spaces belonging to different weights are H -orthogonal and the action of K is unitary. Every element of A preserves the weight space decomposition and acts on each weight space by multiplication with a positive real number. The action of U_+ satisfies (3.2.3). Moreover, the modules $L(\phi_{n+1}), \dots, L(\phi_d)$ are one-dimensional and U_+ acts trivially on them. We write $\hat{\rho}_1, \dots, \hat{\rho}_d$ (resp. ρ_1, \dots, ρ_d) for the homomorphisms describing the actions of G (resp. \mathfrak{g}) in all these representations.

First, we fix an arbitrary $i \in \{n+1, \dots, d\}$ and show that $\langle \phi_i, h' \rangle = \langle \phi_i, h \rangle$. Let $v \in L(\phi_i) \setminus \{0\}$ be any non-zero vector. As $L(\phi_i)$ is one-dimensional and the K -action is unitary, it follows that $\hat{\rho}_i(k)$ acts on $L(\phi_i)$ by multiplication with a complex number of the form $e^{i\gamma}$ for some $\gamma \in [0, 2\pi)$. If $\mathbb{K} = \mathbb{R}$, it is necessary that $\gamma \in \{0, \pi\}$ and $\hat{\rho}_i(k) \in \{\text{id}_{L(\phi_i)}, -\text{id}_{L(\phi_i)}\}$. Irrespective of the real or complex case, we find that

$$\hat{\rho}_i(ak)v = e^{i\gamma}\hat{\rho}_i(a)v = e^{i\gamma}a(\phi_i)v.$$

Likewise, $\hat{\rho}_i(k')$ acts on $L(\phi_i)$ by multiplication with $e^{i\gamma'}$ for some $\gamma' \in [0, 2\pi)$ and U_+ acts trivially, so we also have

$$\hat{\rho}_i(k'a'u')v = \hat{\rho}_i(k'a')v = a'(\phi_i)\hat{\rho}_i(k')v = e^{i\gamma'}a'(\phi_i)v.$$

As $ak = k'a'u'$, the two resulting vectors must coincide. Now $a(\phi_i)$ and $a'(\phi_i)$ are both positive real numbers and $v \neq 0$, so equality is only possible if $e^{i\gamma} = e^{i\gamma'}$ and $a(\phi_i) = a'(\phi_i)$. The former condition is equivalent to $\gamma = \gamma'$ and the latter to $e^{\langle \phi_i, h \rangle} = e^{\langle \phi_i, h' \rangle}$ by Lemma 3.4.14 and hence to $\langle \phi_i, h \rangle = \langle \phi_i, h' \rangle$ as the exponents are real numbers.

Second, we let $w \in W$ and $i \in I$ be arbitrary and prove that $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$. Let $\|\cdot\|$ be the norm on $L(\omega_i)$ induced by H . Since \mathcal{D} is free, every weight of $L(\omega_i)$ has the form $\omega_i - \alpha$ for some $\alpha \in Q_+$ and the eigenvalues of the diagonal map $\hat{\rho}_i(a) \in \text{GL}(L(\omega_i))$ are precisely the exponentials of that of $\rho_i(h) \in \mathfrak{gl}(L(\omega_i))$. As $\langle \alpha, h \rangle \geq 0$ for all $\alpha \in Q_+$, the largest eigenvalue is $a(\omega_i) = e^{\langle \omega_i, h \rangle}$. Moreover, the set of weights of $L(\omega_i)$ is W -invariant, so $\mu := w^{-1}(\omega_i)$ is also a weight and we may choose a vector $v \in L(\omega_i)_\mu$ with $\|v\| = 1$. Using that $u' \in U_+$ is a product of elements in positive root groups, it follows from (3.2.3) that for every $\nu \in \text{wt}(L(\omega_i))$ with $\mu < \nu \leq \omega_i$, there exists a suitable vector $v_\nu \in L(\omega_i)_\nu$ such that

$$\hat{\rho}_i(u')v = v + \sum_{\mu < \nu \leq \omega_i} v_\nu.$$

On the one hand, this justifies the following computation:

$$\begin{aligned} \|\hat{\rho}_i(ak)v\| &= \|\hat{\rho}_i(k'a'u')v\| = \left\| \hat{\rho}_i(k'a') \left(v + \sum_{\mu < \nu \leq \omega_i} v_\nu \right) \right\| \\ &= \left\| \hat{\rho}_i(k') \left(a'(\mu)v + \sum_{\mu < \nu \leq \omega_i} a'(\nu)v_\nu \right) \right\| \\ &= \left\| a'(\mu)v + \sum_{\mu < \nu \leq \omega_i} a'(\nu)v_\nu \right\| \\ &\geq \|a'(\mu)v\| = a'(\mu) = a'(w^{-1}(\omega_i)) = w(a')(\omega_i) \end{aligned}$$

Here, the equalities in the second and third line are satisfied since the action of A is diagonal with respect to the weight space decomposition and that of K is unitary. The inequality in the last line follows from the fact that different weight spaces are orthogonal and the equality thereafter holds as $\|v\| = 1$ and $a'(\mu) = e^{\langle \mu, h' \rangle} > 0$.

On the other hand, the vector $\hat{\rho}_i(ak)v$ is contained in the image of the unit sphere in $L(\omega_i)$ under the diagonal map $\hat{\rho}_i(a)$. More precisely, since $L(\omega_i)$ is the direct sum of all weight spaces and only finitely many of them contain a non-zero component of $\hat{\rho}_i(k)v$, there exists some $N \in \mathbb{N}_0$ with

$$\hat{\rho}_i(k)v \in \bigoplus_{\alpha \in Q_+ : \text{ht}(\alpha) \leq N} L(\omega_i)_{\omega_i - \alpha} =: V_0.$$

This is a finite-dimensional real or complex vector space and $\hat{\rho}_i(k)v$ lies on the unit sphere of V_0 . As $\hat{\rho}_i(a)$ acts diagonally on $L(\omega_i)$, it preserves V_0 and the restriction $\hat{\rho}_i(a)|_{V_0}$ is represented by a finite diagonal matrix whose non-zero entries are the values $a(\omega_i - \alpha) \in \mathbb{R}_{>0}$, where α runs over all elements of Q_+ that satisfy $\text{ht}(\alpha) \leq N$ and $\omega_i - \alpha \in \text{wt}(L(\omega_i))$. Hence, $\hat{\rho}_i(ak)v$ is contained in the image of the unit sphere of V_0 under the diagonal map $\hat{\rho}_i(a)|_{V_0}$. This set is an ellipsoid, where the lengths of its half-axes are equal to the absolute values of the eigenvalues of $\hat{\rho}_i(a)|_{V_0}$. Since these are all positive real numbers, the absolute values can be omitted, so the length of the longest half-axis is the maximal eigenvalue $a(\omega_i) \in \mathbb{R}_{>0}$ and this is an upper bound for the norm of any vector on the ellipsoid. Thus, we conclude that

$$w(a')(\omega_i) \leq \|\hat{\rho}_i(ak)v\| \leq a(\omega_i).$$

As before, this is equivalent to $e^{\langle \omega_i, w(h') \rangle} \leq e^{\langle \omega_i, h \rangle}$ and hence to $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ because the exponents are real numbers. The proof of the inclusion $\Pi_a \subset \exp(\Xi_h)$ is therefore finished.

5.2.2 Step 2: Induction

In order to establish the converse inclusion $\exp(\Xi_h) \subset \Pi_a$ as well, we argue by induction on $\dim(\Xi_h) = |I_{\text{vis}}| = |I| = n$. Suppose that $n = 1$, then $\mathbb{A} = \mathbb{A}^0 = (2)$ and $W = W^0 = \{1, s_1\} \cong \mathbb{Z}_2$, so G is a Kac-Moody group of rank 1 and we need to show that $a = \exp(h)$ satisfies

$$\Pi_a \supset \exp(\Xi_h) = \exp(\text{conv}(\{h, s_1(h)\})) = \exp([h, s_1(h)]). \quad (5.2.2)$$

Recall from Corollary 3.5.2 that we already established this inclusion for the group $G^0 = \text{SL}_2(\mathbb{K})$ (i.e. the root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$), where s_1 acts as $-\text{id}$ on the one-dimensional space $\mathfrak{a}^0 \subset \mathfrak{g}^0 = \mathfrak{sl}_2(\mathbb{K})$, and our goal is to reduce the general rank-1 case to this situation.

By the discussion in Section 3.3, the group G is isomorphic to a semidirect extension of $\text{SL}_2(\mathbb{K})$ or $\text{PGL}_2(\mathbb{K})$ and as \mathcal{D} is assumed to be cofree and cotorsion-free, Remark 3.2.1 (i) implies that $(-1)^{\alpha_1^\vee} \neq 1$ and the first case occurs. More precisely, (3.3.2) asserts that $G \cong G_1 \rtimes T_2$, where $G_1 = \varphi_1(G^0) = \langle U_1, U_{-1} \rangle \cong \text{SL}_2(\mathbb{K})$ and T_2 is a $(d-1)$ -dimensional complement to the subtorus $T_1 := \{r^{\alpha_1^\vee} \mid r \in \mathbb{K}^*\} \subset G_1$. Let $G^0 = K^0 A^0 U_+^0$ be the usual Iwasawa decomposition of $\text{SL}_2(\mathbb{K})$. By Example 3.4.12, the injective group homomorphism $\varphi_1 : G^0 \rightarrow G$ intertwines the compact involutions of G^0 and G and induces an Iwasawa decomposition $G_1 = K_1 A_1 U_1$ on its image, where the three subgroups are the respective intersections of G_1 with K, A and U_+ . In particular, the associated Iwasawa projections satisfy $\varphi_1 \circ \Pi_{A^0} = \Pi_A \circ \varphi_1$.

We can now show that (5.2.2) holds. By Lemma 3.4.16, there exist $a_1, a_2 \in A$ such that $a = a_1 a_2$, $a_1 \in A_1$ and a_2 commutes with G_1 or, equivalently, $a_2(\alpha_1) = 1$. This implies that a_2 is central in $G \cong G_1 \rtimes T_2$ and hence $\Pi_A(ak) = \Pi_A(a_1 k) a_2$ for all $k \in K$. If we restrict this to the subgroup $K_1 \subset K$ and define $a^0 := \varphi_1^{-1}(a_1) \in A^0$, it follows that

$$\Pi_a \supset \Pi_A(a K_1) = \Pi_A(a_1 K_1) a_2 = \Pi_A(\varphi_1(a^0 K^0)) a_2 = \varphi_1(\Pi_{A^0}(a^0 K^0)) a_2.$$

Upon setting $h^0 := \log(a^0) \in \mathfrak{a}^0$, Corollary 3.5.2 states that $\Pi_{A^0}(a^0 K^0)$ is the exponential of the closed line segment in \mathfrak{a}^0 between h^0 and $-h^0$ or, equivalently, the exponential of the convex hull of the orbit $W^0 \cdot h^0 \subset \mathfrak{a}^0$. Moreover, we have also seen in Example 3.4.12 that the restriction to \mathfrak{a}^0 of the injective Lie algebra homomorphism $\varphi_1' : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ is W^0 -equivariant and satisfies $\exp \circ \varphi_1'|_{\mathfrak{a}^0} = \varphi_1 \circ \exp^0|_{\mathfrak{a}^0}$, where \exp^0 denotes the standard exponential map of (2×2) -matrices.

We set $h_1 := \log(a_1)$ and $h_2 := \log(a_2)$, so that $h = h_1 + h_2$, $h_1 = \varphi'_1(h^0) \in \mathfrak{a}_1$ and $\langle \alpha_1, h_2 \rangle = 0$. Putting all this information together leads to the following computation:

$$\begin{aligned} \Pi_a \supset \varphi_1(\Pi_{A^0}(a^0 K^0))_{a_2} &= \varphi_1(\exp^0(\operatorname{conv}(W^0 \cdot h^0)))_{a_2} = \exp(\varphi'_1(\operatorname{conv}(W^0 \cdot h^0)))_{a_2} \\ &= \exp(\operatorname{conv}(\varphi'_1(W^0 \cdot h^0)))_{a_2} = \exp(\operatorname{conv}(W^0 \cdot h_1))_{a_2} = \exp(\operatorname{conv}(W^0 \cdot h_1) + h_2) \\ &= \exp(\operatorname{conv}(W^0 \cdot (h_1 + h_2))) = \exp(\operatorname{conv}(W^0 \cdot h)) = \exp(\Xi_h) \end{aligned}$$

The first two equalities in the second line are justified by linearity and W^0 -equivariance of $\varphi'_1|_{\mathfrak{a}^0}$, the third by definition of the exponential map, and the first in the third line by the fact that $W^0 \cong \mathbb{Z}_2$ fixes h_2 because $\langle \alpha_1, h_2 \rangle = 0$. Thus, the proof of the case $n = 1$ is finished.

Suppose from now on that $n > 1$. Before proceeding with the main argument, it is useful to spell out the induction hypothesis in detail. Let $J \subsetneq I$ be any proper subset, let \mathcal{D}_J denote the J -restriction of \mathcal{D} and consider the \mathcal{D}_J -representation of $W_J = W(\mathbb{A}_J)$ on \mathfrak{a} . The Kac-Moody group of type \mathcal{D}_J over \mathbb{K} can be identified with the subgroup $G_J \subset G$ generated by T and the root groups $(U_\alpha)_{\alpha \in \Phi_J}$. As discussed in Notation 3.4.13, its Iwasawa decomposition has the form $G_J = K_J A U_{J,+}$, where the three factors are the respective intersections of G_J with K, A and U_+ . In particular, its Iwasawa projection onto the A -component coincides with the restriction $\Pi_A|_{G_J}$. Consequently, the induction hypothesis states that whenever $a' \in A$ is such that $h' := \log(a') \in \mathfrak{a}$ is almost virtually \mathcal{D}_J -singular, then¹

$$\Pi_A(a' K_J) \supset \exp(\operatorname{conv}(W_J \cdot h')). \quad (5.2.3)$$

Note that this can be applied to $a' = a$ because $h = \log(a)$ is assumed to be virtually \mathcal{D} -singular which is a stronger condition. We are going to need a minor generalization of (5.2.3). Let $w \in W$ be arbitrary and let $\tilde{w} \in \tilde{W} \subset K$ be the canonical representative of w in the extended Weyl group. The subgroup $G_{w(J)} := \tilde{w} G_J \tilde{w}^{-1} \subset G$ is isomorphic to G_J and inherits an Iwasawa decomposition

$$G_{w(J)} = K_{w(J)} A U_{w(J),+},$$

where $K_{w(J)} := \tilde{w} K_J \tilde{w}^{-1}$ and $U_{w(J),+} := \tilde{w} U_{J,+} \tilde{w}^{-1}$. It is clear that $K_{w(J)} \subset K$ and $U_{w(J),+}$ is the subgroup of G generated by all $U_{w(\alpha)}$ with $\alpha \in \Phi_{J,+}$, but it is not necessarily true that $U_{w(J),+} \subset U_+$ because it is possible that w maps a positive root in $\Phi_{J,+}$ to a negative one. However, the definition of $G_{w(J)}$ only depends on the left coset $w W_J \subset W$, so we may w.l.o.g. assume that w is right J -reduced. With this condition, we have $l(w) < l(ws_j)$ and so $w(\alpha_j) \in \Phi_+$ for all $j \in J$ by (2.5.4), which implies that $w(\alpha) \in \Phi_+$ for all $\alpha \in \Phi_{J,+}$ and $U_{w(J),+} \subset U_+$. In this situation, the Iwasawa decomposition of $G_{w(J)}$ is the restriction of that of G . In particular, $G_{w(J)}$ contains the Iwasawa components of all its elements, the projection onto the A -component is simply the restriction $\Pi_A|_{G_{w(J)}}$ and $\Pi_A(\tilde{w} g' \tilde{w}^{-1}) = \tilde{w} \Pi_A(g') \tilde{w}^{-1}$ holds for all $g' \in G_J$. Thus, whenever $a' \in A$ is such that $w^{-1}(h') := \log(w^{-1}(a')) \in \mathfrak{a}$ is almost virtually \mathcal{D}_J -singular, we obtain the following generalization of (5.2.3):

$$\begin{aligned} \Pi_A(a' K_{w(J)}) &= \Pi_A(a' \tilde{w} K_J \tilde{w}^{-1}) = \Pi_A(\tilde{w} w^{-1}(a') K_J \tilde{w}^{-1}) = \tilde{w} \Pi_A(w^{-1}(a') K_J) \tilde{w}^{-1} \\ &\supset \tilde{w} \exp(\operatorname{conv}(W_J \cdot w^{-1}(h'))) \tilde{w}^{-1} = w[\exp(\operatorname{conv}(W_J \cdot w^{-1}(h')))] \\ &= \exp[w(\operatorname{conv}(W_J \cdot w^{-1}(h')))] = \exp(\operatorname{conv}(w W_J w^{-1} \cdot h')) \end{aligned} \quad (5.2.4)$$

This holds for the point $a' = w(a)$ since $h = \log(w^{-1}(w(a)))$ is virtually \mathcal{D} -singular.

We now return to the proof of the inclusion $\exp(\Xi_h) \subset \Pi_a$. Our strategy is to “sweep out” Ξ_h by parallel translates of its supporting hyperplanes and apply the induction hypothesis to every such “parallel slice”. We employ the notation and results of Section 4.3.3. Let $i \in I = I_{\text{vis}}$ be arbitrary. In view of (4.3.34), it suffices to show the following:

$$\exp(P^{(i)}(t)) \subset \Pi_a \quad \text{for all } t \in T_h^{(i)} \quad (5.2.5)$$

¹In order to simplify the argument, we have imposed Assumption 5.2.2 on h in the beginning of the proof, but this is not a restriction, so a similar condition on h' does not have to be included in the induction hypothesis.

Here, $P^{(i)}(t) = \Xi_h \cap E^{(i)}(t) = \{y \in \Xi_h : \langle \omega_i, y \rangle = t\}$ and $T_h^{(i)} = \{t \in \mathbb{R} : P^{(i)}(t) \neq \emptyset\}$. Let $t \in T_h^{(i)}$ be arbitrary but fixed. As i is visible, Ξ_h is not contained in the affine hyperplane $E^{(i)}(t) \subset E_h$, which implies that $\dim(P^{(i)}(t)) \leq n - 1$. We prove (5.2.5) in two steps.

First, we show that $\exp(F') \subset \Pi_a$ holds whenever F' is a face of $P^{(i)}(t)$ that is contained in the essential part. By Remark 4.3.53, F' is a proper face, so that $\dim(F') < \dim(P^{(i)}(t)) \leq n - 1$, unless $P^{(i)}(t)$ is a single point, and Lemma 4.2.4 states that there is a unique minimal face F of Ξ_h with $F' = F \cap E^{(i)}(t)$ which satisfies $\dim(F) \leq \dim(F') + 1 < n$ in both cases. Hence, $F \subsetneq \Xi_h$ is a proper face, so Theorem 4.3.38 implies that its type $J := \tau(F)$ is a proper visible subset of I and there exists some $w \in W$, which may be assumed to be right J -reduced, such that

$$F = w(F_J) = \text{conv}(wW_J \cdot h) = \text{conv}(wW_J w^{-1} \cdot w(h)).$$

Applying (5.2.4) to the subgroup $G_{w(J)} \subset G$ and the point $w(a) \in A$ yields

$$\exp(F') \subset \exp(F) = \exp(\text{conv}(wW_J w^{-1} \cdot w(h))) \subset \Pi_A(w(a)K_{w(J)}) = \Pi_A(aK_J \tilde{w}^{-1}) \subset \Pi_a,$$

where $\tilde{w} \in \tilde{W} \subset K$ denotes the canonical representative of w in the extended Weyl group. At this point, we have proved that $\exp(P_{\text{ess}}^{(i)}(t)) \subset \Pi_a$ but for the second step, the more precise inclusion $\exp(F') \subset \Pi_A(w(a)K_{w(J)})$ will be important.

Suppose now that $a'' \in \exp(P^{(i)}(t))$ is arbitrary and set $h'' := \log(a'') \in P^{(i)}(t)$. It follows from Proposition 4.3.55 that there exists a point $h' \in P_{\text{ess}}^{(i)}(t)$ such that $h'' \in \text{conv}(W^{(i)} \cdot h')$. We denote by $F' \subset P_{\text{ess}}^{(i)}(t)$ the smallest face of the essential part containing h' and write $a' := \exp(h')$. The previous argument implies that if F is the minimal face of Ξ_h with $F' = F \cap E^{(i)}(t)$, its type $J := \tau(F)$ is a proper visible subset of I and there exists a right J -reduced element $w \in W$ such that $F = w(F_J)$ and $a' \in \Pi_A(w(a)K_{w(J)})$. Hence, we find $k_1, k' \in K_{w(J)}$ and $u' \in U_{w(J),+}$ with

$$w(a)k_1 = k'a'u'.$$

Moreover, we deduce from (the proofs of) Proposition 4.3.50 and 4.3.54 that the face $F' \subset P_{\text{ess}}^{(i)}(t)$ is the convex hull of finitely many virtually $\mathcal{D}^{(i)}$ -singular points in $\overline{C^{(i)}}$. In particular, its vertices are all contained in the topological interior of the $\mathcal{D}^{(i)}$ -Tits cone, so the same is true for h' which is therefore virtually $\mathcal{D}^{(i)}$ -singular as well. Thus, we may apply (5.2.3) to the subgroup $G^{(i)} \subset G$ and the point $a' \in A$ to conclude that $a'' = \exp(h'') \in \exp(\text{conv}(W^{(i)} \cdot h')) \subset \Pi_A(a'K^{(i)})$. This implies that there exist some $k_2, k'' \in K^{(i)}$ and $u'' \in U_+^{(i)}$ such that

$$a'k_2 = k''a''u''.$$

Combining the previous two equations leads to

$$w(a)k_1k_2 = k'a'u'k_2 = k'a'k_2(k_2^{-1}u'k_2) = k'k''a''u''(k_2^{-1}u'k_2). \quad (5.2.6)$$

The crucial step towards finishing the proof is to show that $k_2^{-1}u'k_2 \in U_+$, which is a rather surprising property. In general, it is usually not the case that a conjugate of some element in U_+ is also contained in that subgroup, but in our situation, it is a consequence of the construction. Indeed, we even prove a stronger statement.

Claim. If $g \in G^{(i)}$, then $gU_{w(J),+}g^{-1} \subset U_+$.

Proof of Claim. Recall from (3.1.13) that $G^{(i)}$ is the Levi factor of the standard parabolic subgroup $P^{(i)} \cong G^{(i)} \ltimes U_+^{(i)\perp}$ of type $I^{(i)}$ and that its unipotent radical is generated by all root groups U_α with $\alpha \in \Phi_+^{(i)\perp} = \Phi_+ \setminus \Phi_+^{(i)}$. Since $U_{w(J),+}$ is generated by all $U_{w(\alpha)}$ with $\alpha \in \Phi_{J,+}$, it suffices to prove that the g -conjugate of every such root group lies in U_+ . To this end, our goal is to show that $w(\alpha) \in \Phi_+^{(i)\perp}$ for all $\alpha \in \Phi_{J,+}$. Once this is established, it will follow that $U_{w(\alpha)}$ is contained in the unipotent radical of $P^{(i)}$. As $g \in G^{(i)}$ lies in its Levi factor, this will imply that $gU_{w(\alpha)}g^{-1}$ is also a subgroup of the unipotent radical $U_+^{(i)\perp} \subset U_+$ and the proof will be complete.

Let $\alpha \in \Phi_{J,+}$ be arbitrary. First of all, we have $w(\alpha) \in \Phi_+$ because w is right J -reduced, so to prove that $w(\alpha) \in \Phi_+^{(i)\perp}$, we only need to show that $i \in I(w(\alpha))$. Second, since α is an \mathbb{N}_0 -linear combination of the simple roots $(\alpha_j)_{j \in J}$, we may w.l.o.g. assume that $\alpha = \alpha_j$ for some $j \in J$. Suppose to the contrary that $\beta := w(\alpha_j) \in \Phi_+^{(i)}$, i.e. $i \notin I(\beta)$, then the reflection $s_\beta = ws_jw^{-1}$ is an element of $W^{(i)}$, so it stabilizes the affine hyperplane $E^{(i)}(t)$ and the closed half-space $E_+^{(i)}(t)$. In addition, $s_j \in W_J$ preserves F_J , so s_β stabilizes $w(F_J) = F$ and therefore also $F \cap E^{(i)}(t) = F'$. As $s_\beta \in W^{(i)}$ and $F' \subset P_{\text{ess}}^{(i)}(t) \subset \overline{C^{(i)}}$, this implies that F' is contained in the wall $H_\beta = \ker(\beta)$. Hence, s_β fixes F' point-wise and there are two possibilities for $F = w(F_J)$:

- If F is entirely contained in $E^{(i)}(t)$, then $F = F'$ and s_β fixes F point-wise.
- If F is not a subset of $E^{(i)}(t)$, then $E := \text{aff}(F) \subset E_h$ meets both open half-spaces in E_h defined by $E^{(i)}(t)$. Since s_β has order two and preserves F , it restricts to an affine map on E that is either the identity or a reflection at an affine hyperplane. Now on the one hand, s_β fixes the affine hyperplane $\text{aff}(F') = E \cap E^{(i)}(t) \subset E$ point-wise. But on the other hand, it also preserves the “positive side” of that hyperplane, i.e. the closed half-space $E \cap E_+^{(i)}(t)$. Thus, s_β cannot act on E as an affine reflection, so it fixes F point-wise.

In either case, it follows that $s_j \in W_J$ acts trivially on $w^{-1}(F) = F_J$. This is a contradiction to Corollary 4.3.17 because the point-wise fixer of F_J is the standard subgroup $W_{J^*} \subset W$ defined in (4.3.10) which intersects W_J trivially. We conclude that $i \in I(\beta)$, as required.

The previous claim implies that the product on the right-hand side of (5.2.6) yields the Iwasawa decomposition of $w(a)k_1k_2 \in G$ once the factors to the left (resp. right) of a'' are combined, so we obtain

$$a'' = \Pi_A(w(a)k_1k_2) \in \Pi_A(w(a)K) = \Pi_{w(a)} = \Pi_a.$$

Since $a'' \in \exp(P^{(i)}(t))$ was arbitrary, we conclude that $\exp(P^{(i)}(t)) \subset \Pi_a$ holds for all $t \in T_h^{(i)}$, which finishes the proof of Theorem 5.1.1 in the case where \mathcal{D} is free, cofree and cotorsion-free.

5.2.3 Step 3: Generalization to arbitrary root data

Finally, we are going to prove that Theorem 5.1.1 holds for all Kac-Moody root data by using that the definition of Kac-Moody groups is functorial with respect to the underlying root datum. Thus, we use the notation and terminology introduced in Section 3.6. We proceed in two steps: First, we drop the assumption that \mathcal{D} is cofree and cotorsion-free. Afterwards, we apply a similar argument to dispose of freeness as well. In both steps, we employ almost the same reasoning and do not explicitly spell out the definition of every symbol, but use suggestive notation that should be clear from the context.

Claim 1. Theorem 5.1.1 holds whenever \mathcal{D} is free.

Proof of Claim. Let \mathcal{D} be a free Kac-Moody root datum of rank $d \in \mathbb{N}$ and consider its cofree extension $\tilde{\mathcal{D}}$ (cf. Example 3.6.2 (iii)). This is a free, cofree and cotorsion-free Kac-Moody root datum of rank $d + n$ and the dual of its underlying \mathbb{Z} -module is a direct sum $\tilde{\Lambda}^\vee = \Lambda^\vee \oplus \Lambda_0^\vee$, where Λ^\vee is the dual of the underlying \mathbb{Z} -module of \mathcal{D} and Λ_0^\vee is a free \mathbb{Z} -module of rank n . The canonical projection $\tilde{\Lambda}^\vee \rightarrow \Lambda^\vee$ defines a morphism $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$ which induces a surjective Lie algebra homomorphism $p' : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$ with central kernel $\mathfrak{h}_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K} \subset \tilde{\mathfrak{z}}$ and a surjective group homomorphism $p : \tilde{G} \rightarrow G$ with central kernel $T_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}^* \subset \tilde{Z}$. The maps p and p' satisfy (3.6.4)-(3.6.7) upon renaming the symbols appropriately. If $k \in K$ is arbitrary and $p(\tilde{k}) = k$ for some $\tilde{k} \in \tilde{G}$, then \tilde{k} is uniquely determined up to multiplication by an element of $T_0 \subset \tilde{Z}$ which splits as $T_0 \cong M_0 \times A_0$, where M_0 and A_0 are the respective intersections of T_0 with \tilde{K} and \tilde{A} . Upon multiplying by some element of $A_0 \subset \tilde{A}$ if necessary, we may therefore assume that $\tilde{k} \in \tilde{K}$. It follows that $p(\tilde{K}) = K$ and the same argument shows that $p(\tilde{A}) = A$.

Let $h \in \mathfrak{a}$ be an arbitrary almost virtually \mathcal{D} -singular element and set $a := \exp(h) \in A$. As in the previous two steps, we may suppose that $h \in \overline{C}$ is virtually \mathcal{D} -singular. Choose $\tilde{a} \in \tilde{A}$ such that $p(\tilde{a}) = a$ and define $\tilde{h} := \log(\tilde{a}) \in \tilde{\mathfrak{a}}$, so that $p'(\tilde{h}) = h$ and $\langle \tilde{c}_i, \tilde{h} \rangle = \langle c_i, h \rangle$ for all $i \in I$ by (3.6.1).

It follows that $\tilde{h} \in \tilde{C} \subset \tilde{X}_+$ and $W_{\tilde{h}} = W_h$, so \tilde{h} is virtually \tilde{D} -singular. Thus, we deduce from Theorem 5.1.1 for the free, cofree and cotorsion-free case that

$$\Pi_{\tilde{A}}(\tilde{a}\tilde{K}) = \exp(\Xi_{\tilde{h}}).$$

Moreover, since p' restricts to a W -equivariant linear map $\tilde{\mathfrak{a}} \rightarrow \mathfrak{a}$, we also have

$$p'(\Xi_{\tilde{h}}) = p'(\text{conv}(W \cdot \tilde{h})) = \text{conv}(W \cdot p'(\tilde{h})) = \text{conv}(W \cdot h) = \Xi_h.$$

In view of (3.6.6) and (3.6.7), the previous two equations yield

$$\Pi_a = \Pi_A(aK) = \Pi_A(p(\tilde{a}\tilde{K})) = p(\Pi_{\tilde{A}}(\tilde{a}\tilde{K})) = p(\exp(\Xi_{\tilde{h}})) = \exp(p'(\Xi_{\tilde{h}})) = \exp(\Xi_h).$$

Hence, we have proved Theorem 5.1.1 for all free Kac-Moody root data.

Claim 2. Theorem 5.1.1 holds for all Kac-Moody root data.

Proof of Claim. Let \mathcal{D} be an arbitrary Kac-Moody root datum of rank $d \in \mathbb{N}$ and consider its free extension $\tilde{\mathcal{D}}$ (cf. Example 3.6.2 (ii)). This is a free Kac-Moody root datum of rank $d+n$ and the dual of its underlying \mathbb{Z} -module is a direct sum $\hat{\Lambda}^\vee = \Lambda^\vee \oplus \Lambda_0^\vee$, where Λ^\vee is the dual of the underlying \mathbb{Z} -module of \mathcal{D} and Λ_0^\vee is a free \mathbb{Z} -module of rank n . The canonical inclusion $\Lambda^\vee \rightarrow \hat{\Lambda}^\vee$ defines a morphism $\mathcal{D} \rightarrow \tilde{\mathcal{D}}$ which induces an injective Lie algebra homomorphism $i' : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ and an injective group homomorphism $i : G \rightarrow \hat{G}$ satisfying appropriate formulations of (3.6.4)-(3.6.7). The group $\hat{G} = i(G)T_0 \cong G \rtimes T_0$ is a semidirect product, where $T_0 := \Lambda_0^\vee \otimes_{\mathbb{Z}} \mathbb{K}^*$, and with respect to this structure, its compact involution splits as $\hat{\Theta} \equiv \Theta \times \Theta_0$, where Θ is the compact involution of G and Θ_0 is the complex conjugate of the inversion map on $T_0 \subset \hat{T}$. Hence, its fixed-point set is $\hat{K} = i(K)M_0 \cong K \rtimes M_0$ where $M_0 := \hat{K} \cap T_0$.

Let $h \in \mathfrak{a}$ be an arbitrary (almost) virtually \mathcal{D} -singular element, set $a := \exp(h) \in A$ and define $\hat{a} := i(a) \in \hat{A}$ and $\hat{h} := \log(\hat{a}) = i'(h) \in \hat{\mathfrak{a}}$. The same argument as in Claim 1 implies that \hat{h} is (almost) virtually $\tilde{\mathcal{D}}$ -singular, so we deduce from Theorem 5.1.1 for the free case that

$$\Pi_{\hat{A}}(\hat{a}i(K)) = \Pi_{\hat{A}}(\hat{a}\hat{K}) = \exp(\Xi_{\hat{h}}).$$

Here, the first equality is satisfied because $\hat{K} = M_0 i(K)$ and $M_0 \subset \hat{T}$ commutes with $\hat{a} \in \hat{A} \subset \hat{T}$. Likewise, since i' restricts to a W -equivariant linear map $\mathfrak{a} \rightarrow \hat{\mathfrak{a}}$, we again find that $i'(\Xi_h) = \Xi_{\hat{h}}$, so (3.6.6) and (3.6.7) yield

$$i(\Pi_a) = i(\Pi_A(aK)) = \Pi_{\hat{A}}(i(aK)) = \Pi_{\hat{A}}(\hat{a}i(K)) = \exp(\Xi_{\hat{h}}) = \exp(i'(\Xi_h)) = i(\exp(\Xi_h)).$$

As $i : G \rightarrow \hat{G}$ is injective, we conclude that $\Pi_a = \exp(\Xi_h)$. Altogether, we have now proved that Theorem 5.1.1 holds for all Kac-Moody root data. \square

5.3 The Linear Convexity Theorem

While Theorem 5.1.1 can be viewed as the natural variant of Kostant's Non-Linear Convexity Theorem for the class of split real or complex Kac-Moody groups, the linear one can also be generalized to this setting. A close analogue of the next result has already been established by V.G. Kac and D.H. Peterson in [KP84]. It will be discussed in more detail in Section 5.5.1. Their proof relies on transferring concepts from symplectic geometry, especially the moment map, to highest-weight modules of Kac-Moody algebras. An alternative approach that is almost identical to the proof of Theorem 5.1.1 allows us to show a very similar statement. We continue to use the same notation as in the non-linear case. This time, however, we consider the Kac-Moody algebra $\mathfrak{g} := \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$ of type \mathcal{D} over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ and the corresponding linear Iwasawa projection $\pi_{\mathfrak{a}} : \mathfrak{g} \rightarrow \mathfrak{a}$ onto the \mathfrak{a} -component.

Theorem 5.3.1 (Kostant's Linear Convexity Theorem for Kac-Moody groups). *If $h \in \mathfrak{a}$ is almost virtually singular, then*

$$\{\pi_{\mathfrak{a}}(\text{Ad}(k)h) \mid k \in K\} = \Xi_h. \quad (5.3.1)$$

- Remark 5.3.2.* (i) As in the non-linear case, it is a surprising consequence of Theorem 5.3.1 that if $h \in \mathfrak{a}$ is almost virtually singular, the set $\{\pi_{\mathfrak{a}}(\text{Ad}(k)h) \mid k \in K\}$ does not depend on the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
- (ii) In the historical introduction, we mentioned that Kostant's Non-Linear Convexity Theorem for semisimple Lie groups may appear to be nothing but an "exponentiated version" of the linear one, but the two are not related in such a simple way because the exponential map does not intertwine the Iwasawa projections on the group and algebra level. This problem continues to exist in the Kac-Moody setting and it is exacerbated by the fact that the exponential map is not even defined on the entire Kac-Moody algebra \mathfrak{g} .² Furthermore, the Kac-Moody group G does not carry any sort of differentiable structure, so adapting the analytic techniques of [Dui84] that established a link between Kostant's original theorems does not seem feasible in this situation. Despite the apparent absence of a direct connection between the two results, it is possible to prove Theorem 5.3.1 with almost the same methods as Theorem 5.1.1. Therefore, the assumptions that \mathbb{A} is symmetrizable and $h \in \mathfrak{a}$ is almost virtually singular are necessary for the same reasons as in the previous proof.

5.4 Proof of the linear case

In order to establish Theorem 5.3.1, we use the same notation as in the non-linear case and argue in a very similar way. The reader is advised to refer to Section 5.2 to compare the arguments and look up all undefined symbols. Let $h \in \mathfrak{a}$ be a fixed almost virtually singular element and set

$$\pi_h := \pi_{\mathfrak{a}}(\text{Ad}(K)h) = \{\pi_{\mathfrak{a}}(\text{Ad}(k)h) \mid k \in K\}. \quad (5.4.1)$$

We need to show that $\pi_h = \Xi_h$ and we may impose the same conditions as in Lemma 5.2.1.

Lemma 5.4.1. *It is sufficient to prove Theorem 5.3.1 under the additional assumptions that h is contained in $\overline{C} \subset X_+$ and not a global fixed-point of any irreducible factor of (W, S) .*

Proof. Since $h \in \mathfrak{a}$ is almost virtually singular, at least one of the points h and $-h$ lies in X_+ . Because $\Xi_{-h} = -\Xi_h$ and $\pi_{-h} = -\pi_h$, we may assume that $h \in X_+$. Moreover, the definitions of Ξ_h and π_h do not depend on the W -translate of h . For Ξ_h , this is clear and for π_h , we argue as follows: Let $w \in W$ be arbitrary and let $\tilde{w} \in \tilde{W} \subset K$ be the canonical representative of w in the extended Weyl group. As discussed in Section 2.5 and 3.2, the linear map $\text{Ad}(\tilde{w}) = w^* \in \text{Aut}(\mathfrak{g})$ preserves \mathfrak{a} and $\text{Ad}(\tilde{w})|_{\mathfrak{a}}$ coincides with the natural action of w on \mathfrak{a} , which implies that

$$\pi_{w(h)} = \pi_{\mathfrak{a}}(\text{Ad}(K)w(h)) = \pi_{\mathfrak{a}}(\text{Ad}(K\tilde{w})h) = \pi_{\mathfrak{a}}(\text{Ad}(K)h) = \pi_h.$$

Thus, we may even suppose that $h \in \overline{C}$. To justify the final simplification, recall from the proof of Lemma 5.2.1 that $K = K_{\text{vis}}K_{\text{inv}}$. If $h \in \overline{C}$ is fixed by some irreducible factor of (W, S) , then $I_{\text{inv}} \neq \emptyset$ and $\langle c_i, h \rangle = 0$ for all $i \in I_{\text{inv}}$. Applying Lemma 3.4.15 to the Kac-Moody group G_{inv} yields that $\text{Ad}(k)h = h$ for all $k \in K_{\text{inv}}$ and hence $\pi_{\mathfrak{a}}(\text{Ad}(K)h) = \pi_{\mathfrak{a}}(\text{Ad}(K_{\text{vis}})h)$. At the same time, we have $\Xi_h = F_{I_{\text{vis}}}$, so it suffices to prove Theorem 5.3.1 for the I_{vis} -restriction of \mathcal{D} . \square

In the proof of Kostant's Linear Convexity Theorem for Kac-Moody groups, we may therefore invoke the same restrictions as in the non-linear case. Likewise, it is again divided into three steps that are very similar to their counterparts in Section 5.2.

Assumption 5.4.2. Throughout the proof of Theorem 5.3.1, we may w.l.o.g. assume that $h \in \overline{C}$ is virtually singular and that $I = I_{\text{vis}}$, which implies $\dim(\Xi_h) = n$. In Step 1 and 2, we further suppose that \mathcal{D} is free, cofree and cotorsion-free. In Step 3, \mathcal{D} is allowed to be arbitrary.

²We have only defined the exponential map on the Cartan subalgebra $\mathfrak{u}_0 \subset \mathfrak{g}$ which is not the maximal domain where this can be done in a reasonable way. For example, using the algebro-geometric approach to Kac-Moody theory developed by Kumar, it is possible to define the exponential map on a certain subset $\mathfrak{g}_{\text{fin}} \subset \mathfrak{g}$ whose elements act as locally finite operators in every integrable \mathfrak{g} -module (cf. [Kum02, Proposition 6.2.11]).

5.4.1 Step 1: Highest-weight representations

Our first goal in the proof of Theorem 5.3.1 is to establish the inclusion $\pi_h \subset \Xi_h$. Let $k \in K$ be arbitrary and write $\text{Ad}(k)h \in \mathfrak{g}$ in Iwasawa form as

$$\text{Ad}(k)h = x' + h' + z' \quad \text{with} \quad x' \in \mathfrak{k}, h' \in \mathfrak{a}, z' \in \mathfrak{u}_+.$$

We need to show that $h' \in \Xi_h$. Since we assume that $h \in \overline{C}$ is virtually singular, Corollary 4.3.30 asserts that it suffices to verify the following (in-)equalities:

$$\begin{cases} \langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle & \text{for all } w \in W \text{ and } i \in I \\ \langle \phi_i, h' \rangle = \langle \phi_i, h \rangle & \text{for all } i \in \{n+1, \dots, d\} \end{cases}$$

To this end, we compare the actions of $\text{Ad}(k)h = x' + h' + z'$ arising from these two expressions in the irreducible real or complex highest-weight \mathfrak{g} -modules $L(\omega_1), \dots, L(\omega_n), L(\phi_{n+1}), \dots, L(\phi_d)$. Each of these modules is endowed with a positive-definite, contravariant, symmetric bilinear or Hermitian form H and lifts to a representation of the Kac-Moody group G .

First, we fix an arbitrary $i \in \{n+1, \dots, d\}$ and prove that $\langle \phi_i, h' \rangle = \langle \phi_i, h \rangle$ by considering the actions of G and \mathfrak{g} on the highest-weight module $L(\phi_i)$. Lemma 3.6.3 (i) shows that

$$\rho_i(x') + \rho_i(h') + \rho_i(z') = \rho_i(\text{Ad}(k)h) = \hat{\rho}_i(k)\rho_i(h)\hat{\rho}_i(k)^{-1} = \rho_i(h).$$

Here, the last equality holds since $L(\phi_i)$ is one-dimensional, so that all linear operators commute. On the one hand, we now have $\rho_i(z') = 0$ because \mathfrak{u}_+ acts trivially on $L(\phi_i)$ by Proposition 2.7.8. On the other hand, as H is contravariant, the action of \mathfrak{k} is skew-adjoint, which implies that $\rho_i(x')$ acts by multiplication with a purely imaginary number. If $\mathbb{K} = \mathbb{R}$, this directly forces $\rho_i(x') = 0$. If $\mathbb{K} = \mathbb{C}$, the same follows because $h - h' \in \mathfrak{a}$, so that $\rho_i(x') = \rho_i(h - h')$ acts by multiplication with $\langle \phi_i, h - h' \rangle \in \mathbb{R}$. We conclude that $\rho_i(h') = \rho_i(h)$ and hence $\langle \phi_i, h' \rangle = \langle \phi_i, h \rangle$.

Second, we let $w \in W$ and $i \in I$ be arbitrary and show that $\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle$ by investigating the actions of G and \mathfrak{g} on the highest-weight module $L(\omega_i)$. We denote by $\|\cdot\|$ the norm on $L(\omega_i)$ induced by H , write $\mu := w^{-1}(\omega_i) \in \text{wt}(L(\omega_i))$ and choose a vector $v \in L(\omega_i)_\mu$ with $\|v\| = 1$. Using the same arguments as in Section 5.2.1, it follows that there exists some $N \in \mathbb{N}_0$ such that $\hat{\rho}_i(k)^{-1}v$ is contained in the unit sphere of the finite-dimensional real or complex vector space

$$V_0 := \bigoplus_{\alpha \in Q_+ : \text{ht}(\alpha) \leq N} L(\omega_i)_{\omega_i - \alpha}.$$

The diagonal action of $\rho_i(h)$ preserves V_0 and the restriction $\rho_i(h)|_{V_0}$ is represented by a finite diagonal matrix whose diagonal entries are the values $\langle \omega_i - \alpha, h \rangle \in \mathbb{R}$, where α runs over all elements of Q_+ such that $\text{ht}(\alpha) \leq N$ and $\omega_i - \alpha \in \text{wt}(L(\omega_i))$. In contrast to the non-linear case, these numbers need not be positive, but this can easily be fixed. We set

$$\kappa := \min\{\langle \omega_i - \alpha, h \rangle \mid \alpha \in Q_+, \text{ht}(\alpha) \leq N, \omega_i - \alpha \in \text{wt}(L(\omega_i))\} - 1 \in \mathbb{R}$$

and define the linear map

$$f := \rho_i(h) - \kappa \text{id}_{L(\omega_i)} \in \mathfrak{gl}(L(\omega_i)).$$

Then f preserves V_0 as well and the restriction $f|_{V_0}$ is represented by a finite diagonal matrix with positive entries. As $h \in \overline{C}$, the largest entry is $\langle \omega_i, h \rangle - \kappa \geq 1$. On the one hand, the vector $f\hat{\rho}_i(k)^{-1}v$ lies in the image of the unit sphere in V_0 under the diagonal map $f|_{V_0}$, which is an ellipsoid where the lengths of its half-axes are equal to the absolute values of the eigenvalues of $f|_{V_0}$. Since these are all positive, the absolute values can be omitted and the length of the longest half-axis is the maximal eigenvalue $\langle \omega_i, h \rangle - \kappa$, which implies that

$$\|\hat{\rho}_i(k)f\hat{\rho}_i(k)^{-1}v\| = \|f\hat{\rho}_i(k)^{-1}v\| \leq \langle \omega_i, h \rangle - \kappa.$$

On the other hand, Lemma 3.6.3 (i) again yields that

$$\hat{\rho}_i(k)f\hat{\rho}_i(k)^{-1} = \hat{\rho}_i(k)\rho_i(h)\hat{\rho}_i(k)^{-1} - \kappa \text{id}_{L(\omega_i)} = \rho_i(\text{Ad}(k)h) - \kappa \text{id}_{L(\omega_i)}$$

and the first term is equal to $\rho_i(x') + \rho_i(h') + \rho_i(z')$. Now $x' \in \mathfrak{k}$ implies that $\rho_i(x')$ is skew-adjoint with respect to H and hence that $H(v, \rho_i(x')v) = 0$ if $\mathbb{K} = \mathbb{R}$ and $H(v, \rho_i(x')v) \in i\mathbb{R}$ if $\mathbb{K} = \mathbb{C}$. In addition, $z' \in \mathfrak{u}_+$ is a finite sum of vectors in positive root spaces of \mathfrak{g} , so (2.4.2) shows that $\rho_i(z')v$ is a finite sum of vectors in weight spaces of the form $L(\omega_i)_\nu$ for some $\nu \in \text{wt}(L(\omega_i))$ with $\mu < \nu \leq \omega_i$. Because distinct weight spaces are H -orthogonal, we conclude that $H(v, \rho_i(z')v) = 0$. As $H(v, v) = \|v\|^2 = 1$ and $\langle \mu, h' \rangle \in \mathbb{R}$, it follows that

$$H(v, \rho_i(\text{Ad}(k)h)v) = H(v, \rho_i(x')v) + H(v, \rho_i(h')v) = H(v, \rho_i(x')v) + \langle \mu, h' \rangle.$$

The first summand on the right-hand side is a purely imaginary number and the second is real. Moreover, $\theta(h) = -h$ implies that $\rho_i(h)$ is self-adjoint with respect to H and as $\hat{\rho}_i(k)$ is unitary, we find that $\rho_i(\text{Ad}(k)h) = \hat{\rho}_i(k)\rho_i(h)\hat{\rho}_i(k)^{-1}$ is also self-adjoint, so that $H(v, \rho_i(\text{Ad}(k)h)v) \in \mathbb{R}$. Thus, the equality above can only hold if $H(v, \rho_i(x')v) = 0$. Combining all our observations leads to the following computation:

$$\begin{aligned} \langle \mu, h' \rangle - \kappa &= H(v, \langle \mu, h' \rangle v) - \kappa = H(v, \rho_i(h')v) - \kappa \\ &= H(v, [\rho_i(\text{Ad}(k)h) - \rho_i(x') - \rho_i(z')]v) - \kappa \\ &= H(v, \rho_i(\text{Ad}(k)h)v) - \kappa = H(v, \rho_i(\text{Ad}(k)h)v - \kappa v) \\ &= H(v, \hat{\rho}_i(k)f\hat{\rho}_i(k)^{-1}v) = H(\hat{\rho}_i(k)^{-1}v, f\hat{\rho}_i(k)^{-1}v) \end{aligned}$$

The vector in the left entry of the resulting inner product lies on the unit sphere of V_0 while the one on the right is contained in the image of that sphere under the diagonal map $f|_{V_0}$. As the norm of the second vector is at most $\langle \omega_i, h \rangle - \kappa > 0$, the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \langle \mu, h' \rangle - \kappa &\leq |\langle \mu, h' \rangle - \kappa| = |H(\hat{\rho}_i(k)^{-1}v, f\hat{\rho}_i(k)^{-1}v)| \\ &\leq \|\hat{\rho}_i(k)^{-1}v\| \cdot \|f\hat{\rho}_i(k)^{-1}v\| \leq \langle \omega_i, h \rangle - \kappa. \end{aligned}$$

Using that $\langle \mu, h' \rangle = \langle w^{-1}(\omega_i), h' \rangle = \langle \omega_i, w(h') \rangle$, we arrive at the desired inequality. The proof of the inclusion $\pi_h \subset \Xi_h$ is therefore complete.

5.4.2 Step 2: Induction

The converse inclusion $\Xi_h \subset \pi_h$ will be proved by induction on $\dim(\Xi_h) = |I_{\text{vis}}| = |I| = n$. First, suppose that $n = 1$, then $\mathbb{A} = \mathbb{A}^0 = (2)$ and $W = W^0 = \{1, s_1\} \cong \mathbb{Z}_2$, so we need to show that

$$\pi_h \supset \Xi_h = \text{conv}(\{h, s_1(h)\}) = [h, s_1(h)]. \quad (5.4.2)$$

We have already established this result for the group $G^0 = \text{SL}_2(\mathbb{K})$ (i.e. the root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}^0}$) in Corollary 3.5.9 and we are going to reduce the general rank-1 case to this situation.

By Lemma 2.3.5, the Lie algebra \mathfrak{g} is isomorphic to a trivial central extension of $\mathfrak{g}^0 = \mathfrak{sl}_2(\mathbb{K})$. More precisely, we have $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{g}_1 = \varphi'_1(\mathfrak{g}^0) = \mathbb{K}e_1 \oplus \mathbb{K}\alpha_1^\vee \oplus \mathbb{K}f_1 \cong \mathfrak{g}^0$ and $\mathfrak{h}_2 := \ker(\alpha_1)$. Let $\mathfrak{g}^0 = \mathfrak{k}^0 \oplus \mathfrak{a}^0 \oplus \mathfrak{u}_+^0$ denote the usual Iwasawa decomposition of $\mathfrak{sl}_2(\mathbb{K})$. By Example 3.4.12, the injective Lie algebra homomorphism $\varphi'_1 : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ intertwines the compact involutions of \mathfrak{g}^0 and \mathfrak{g} and induces an Iwasawa decomposition $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{u}_1$ on its image, where the three subalgebras are the respective intersections of \mathfrak{g}_1 with \mathfrak{k} , \mathfrak{a} and \mathfrak{u}_+ . In particular, the associated Iwasawa projections satisfy $\varphi'_1 \circ \pi_{\mathfrak{a}^0} = \pi_{\mathfrak{a}} \circ \varphi'_1$.

We can now prove (5.4.2). Lemma 3.4.16 states that there exist $h_1, h_2 \in \mathfrak{a}$ such that $h = h_1 + h_2$, $h_1 \in \mathfrak{a}_1$ and $\langle \alpha_1, h_2 \rangle = 0$ and Lemma 3.4.15 implies that h_2 is a global fixed-point for both the

adjoint action of G on \mathfrak{g} and the W^0 -action on \mathfrak{a} . We define $h^0 := \varphi_1'^{-1}(h_1) \in \mathfrak{a}^0$. By restricting the adjoint action of K to the subgroup $K_1 = \varphi_1(K^0)$, we may perform the following computation:

$$\begin{aligned} \pi_h \supset \pi_{\mathfrak{a}}(\text{Ad}(K_1)h) &= \pi_{\mathfrak{a}}(\text{Ad}(K_1)h_1) + h_2 = \pi_{\mathfrak{a}}(\text{Ad}(\varphi_1(K^0))\varphi_1'(h^0)) + h_2 \\ &= \pi_{\mathfrak{a}}(\varphi_1'(\text{Ad}^0(K^0)h^0)) + h_2 = \varphi_1'(\pi_{\mathfrak{a}^0}(\text{Ad}^0(K^0)h^0)) + h_2 \\ &= \varphi_1'(\text{conv}(W^0 \cdot h^0)) + h_2 = \text{conv}(\varphi_1'(W^0 \cdot h^0)) + h_2 \\ &= \text{conv}(W^0 \cdot h_1) + h_2 = \text{conv}(W^0 \cdot h) = \Xi_h \end{aligned}$$

To obtain the first (resp. second-last) equality, we have used that $h_2 \in \mathfrak{a}$ is a global fixed-point of $\text{Ad}(K_1)$ (resp. W^0). In the second line, the first equality follows from Lemma 3.6.3 (ii) and the second from the previous paragraph. The first equality in the third line is exactly the statement of Corollary 3.5.9 and the next two hold as φ_1' restricts to a W^0 -equivariant linear map $\mathfrak{a}^0 \rightarrow \mathfrak{a}_1$. Hence, the proof is finished in the case $n = 1$.

If $n > 1$, the induction hypothesis may be formulated as follows: Let $J \subsetneq I$ be any proper subset, let \mathcal{D}_J denote the J -restriction of \mathcal{D} and consider the \mathcal{D}_J -representation of $W_J = W(\mathbb{A}_J)$ on \mathfrak{a} . The Kac-Moody algebra of type \mathcal{D}_J over \mathbb{K} can be identified with the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ generated by the entire Cartan subalgebra \mathfrak{u}_0 and the subfamilies $(e_j)_{j \in J}, (f_j)_{j \in J}$ of the Chevalley generators. As remarked in Notation 3.4.13, its Iwasawa decomposition has the form $\mathfrak{g}_J = \mathfrak{k}_J \oplus \mathfrak{a} \oplus \mathfrak{u}_{J,+}$, where the three subalgebras are the respective intersections of \mathfrak{g}_J with \mathfrak{k} , \mathfrak{a} and \mathfrak{u}_+ , and the Iwasawa projection onto the \mathfrak{a} -component is simply the restriction $\pi_{\mathfrak{a}}|_{\mathfrak{g}_J}$. Since the Kac-Moody group of type \mathcal{D}_J over \mathbb{K} can be identified with $G_J = K_J A U_{J,+}$, the induction hypothesis states that whenever $h' \in \mathfrak{a}$ is almost virtually \mathcal{D}_J -singular, it holds that

$$\pi_{\mathfrak{a}}(\text{Ad}(K_J)h') \supset \text{conv}(W_J \cdot h'). \quad (5.4.3)$$

In particular, this can be applied to the virtually \mathcal{D} -singular point $h' = h$. More generally, let $w \in W$ be right J -reduced and denote by $\tilde{w} \in \tilde{W} \subset K$ the canonical representative of w in the extended Weyl group. We have seen in Section 2.5 and 3.2 that $\text{Ad}(\tilde{w}) = w^* \in \text{Aut}(\mathfrak{g})$ maps \mathfrak{u}_{α} to $\mathfrak{u}_{w(\alpha)}$ for all $\alpha \in \Phi^{\infty}$ and $\text{Ad}(\tilde{w})|_{\mathfrak{a}}$ coincides with the action of w defined by the \mathcal{D} -representation. The subalgebra $\mathfrak{g}_{w(J)} := \text{Ad}(\tilde{w})\mathfrak{g}_J$ is isomorphic to \mathfrak{g}_J and inherits an Iwasawa decomposition

$$\mathfrak{g}_{w(J)} = \mathfrak{k}_{w(J)} \oplus \mathfrak{a} \oplus \mathfrak{u}_{w(J),+},$$

where $\mathfrak{k}_{w(J)} := \text{Ad}(\tilde{w})\mathfrak{k}_J$ and $\mathfrak{u}_{w(J),+} := \text{Ad}(\tilde{w})\mathfrak{u}_{J,+}$. Here, we have $\mathfrak{k}_{w(J)} \subset \mathfrak{k}$ by Lemma 3.6.3 (iv) and $\mathfrak{u}_{w(J),+}$ is the direct sum of all root spaces $\mathfrak{u}_{w(\alpha)}$ with $\alpha \in \Phi_{J,+}^{\infty}$, which is contained in \mathfrak{u}_+ because the assumption that w is right J -reduced ensures that $w(\alpha) \in \Phi_+^{\infty}$ for all $\alpha \in \Phi_{J,+}^{\infty}$. Consequently, the Iwasawa decomposition of $\mathfrak{g}_{w(J)}$ is the restriction of that of \mathfrak{g} , so $\mathfrak{g}_{w(J)}$ contains the Iwasawa components of all its elements, the projection onto the \mathfrak{a} -component is simply the restriction $\pi_{\mathfrak{a}}|_{\mathfrak{g}_{w(J)}}$ and $\pi_{\mathfrak{a}}(\text{Ad}(\tilde{w})y) = \text{Ad}(\tilde{w})\pi_{\mathfrak{a}}(y) = w(\pi_{\mathfrak{a}}(y))$ holds for all $y \in \mathfrak{g}_J$. Hence, whenever $h' \in \mathfrak{a}$ has the property that $w^{-1}(h')$ is almost virtually \mathcal{D}_J -singular (e.g. if $h' = w(h)$), we arrive at the following generalization of (5.4.3):

$$\begin{aligned} \pi_{\mathfrak{a}}(\text{Ad}(K_{w(J)})h') &= \pi_{\mathfrak{a}}(\text{Ad}(\tilde{w}K_J\tilde{w}^{-1})h') = \pi_{\mathfrak{a}}(\text{Ad}(\tilde{w})\text{Ad}(K_J)\text{Ad}(\tilde{w}^{-1})h') \\ &= \pi_{\mathfrak{a}}(\text{Ad}(\tilde{w})\text{Ad}(K_J)w^{-1}(h')) = w(\pi_{\mathfrak{a}}(\text{Ad}(K_J)w^{-1}(h'))) \\ &\supset w(\text{conv}(W_J \cdot w^{-1}(h'))) = \text{conv}(wW_Jw^{-1} \cdot h') \end{aligned} \quad (5.4.4)$$

We now prove $\Xi_h \subset \pi_h$ using the same strategy as in Section 5.2.2. Let $i \in I = I_{\text{vis}}$ and $t \in T_h^{(i)}$ be arbitrary. It suffices to show that $P^{(i)}(t) \subset \pi_h$. To this end, suppose first that F' is a face of the essential part of $P^{(i)}(t)$ and let F denote the unique minimal face of Ξ_h with $F' = F \cap E^{(i)}(t)$. As explained in the non-linear case, it satisfies $\dim(F) \leq \dim(F') + 1 < n$, so its type $J := \tau(F)$ is a proper visible subset of I and there exists a right J -reduced element $w \in W$ such that

$$F = w(F_J) = w(\text{conv}(W_J \cdot h)) = \text{conv}(wW_Jw^{-1} \cdot w(h)).$$

Applying (5.4.4) to the subgroup $G_{w(J)} \subset G$ and the point $w(h) \in \mathfrak{a}$ yields

$$F' \subset F = \text{conv}(wW_J w^{-1} \cdot w(h)) \subset \pi_{\mathfrak{a}}(\text{Ad}(K_{w(J)})w(h)) = \pi_{\mathfrak{a}}(\text{Ad}(\tilde{w}K_J)h) \subset \pi_h,$$

where $\tilde{w} \in \tilde{W} \subset K$ is the canonical representative of w in the extended Weyl group. Since F' was arbitrary, we have shown that $P_{\text{ess}}^{(i)}(t) \subset \pi_h$, but the next step again requires the more precise inclusion $F' \subset \pi_{\mathfrak{a}}(\text{Ad}(K_{w(J)})w(h))$.

Let $h'' \in P^{(i)}(t)$ now be arbitrary. Proposition 4.3.55 asserts that there exists some $h' \in P_{\text{ess}}^{(i)}(t)$ with $h'' \in \text{conv}(W^{(i)} \cdot h')$. Let F' denote the smallest face of the essential part containing h' and F the minimal face of Ξ_h with $F' = F \cap E^{(i)}(t)$. By the previous argument, its type $J := \tau(F)$ is a proper visible subset of I and there is a right J -reduced element $w \in W$ such that $F = w(F_J)$ and $h' \in \pi_{\mathfrak{a}}(\text{Ad}(K_{w(J)})w(h))$. Thus, we find $k_1 \in K_{w(J)}$ and $x' \in \mathfrak{k}_{w(J)}$, $z' \in \mathfrak{u}_{w(J),+}$ with

$$\text{Ad}(k_1)w(h) = x' + h' + z'.$$

The same argument as in Section 5.2.2 shows that $h' \in F' \subset P_{\text{ess}}^{(i)}(t)$ is virtually $\mathcal{D}^{(i)}$ -singular. Consequently, we may apply (5.4.3) to the subgroup $G^{(i)} \subset G$ and the point $h' \in \mathfrak{a}$ to conclude that $h'' \in \pi_{\mathfrak{a}}(\text{Ad}(K^{(i)})h')$. Hence, there exist $k_2 \in K^{(i)}$ and $x'' \in \mathfrak{k}^{(i)}$, $z'' \in \mathfrak{u}_+^{(i)}$ such that

$$\text{Ad}(k_2)h' = x'' + h'' + z''.$$

Putting the previous two equations together leads to

$$\text{Ad}(k_2 k_1)w(h) = \text{Ad}(k_2)(x' + h' + z') = (\text{Ad}(k_2)x' + x'') + h'' + (\text{Ad}(k_2)z' + z''), \quad (5.4.5)$$

where $\text{Ad}(k_2)x' \in \mathfrak{k}$ by Lemma 3.6.3 (iv). Similar to the non-linear case, the crucial step towards finishing the proof is to show that $\text{Ad}(k_2)z' \in \mathfrak{u}_+$ and we establish a stronger statement.

Claim. If $g \in G^{(i)}$, then $\text{Ad}(g)\mathfrak{u}_{w(J),+} \subset \mathfrak{u}_+$.

Proof of Claim. As $\mathfrak{u}_{w(J),+}$ is the direct sum of all root spaces $\mathfrak{u}_{w(\alpha)}$ with $\alpha \in \Phi_{J,+}^{\infty}$, it suffices to prove that $\text{Ad}(g)$ maps every such root space into \mathfrak{u}_+ . Let $\alpha \in \Phi_{J,+}^{\infty}$ be arbitrary and $\beta := w(\alpha)$. Although it is possible that β is an imaginary root, the same argument as in the proof of the claim in Section 5.2.2 (which only involved the simple roots $(\alpha_j)_{j \in J}$) implies that $\beta \in \Phi_+^{\infty}$ and $i \in I(\beta)$. Now $G^{(i)}$ is generated by the torus T and all root groups U_{γ} with $\gamma \in \Phi^{(i)}$. By (3.2.3) and (3.2.4), the adjoint action of T preserves \mathfrak{u}_{β} and

$$\text{Ad}(U_{\gamma})\mathfrak{u}_{\beta} \subset \bigoplus_{l \in \mathbb{N}_0} \mathfrak{u}_{\beta + l\gamma}.$$

Since every root is either positive or negative and $i \in I(\beta)$ but $i \notin I(\gamma)$, we find that $\beta + l\gamma$ is either not a root or positive, which implies that $\text{Ad}(U_{\gamma})\mathfrak{u}_{\beta} \subset \mathfrak{u}_+$. Whenever the latter case occurs, we again have $i \in I(\beta + l\gamma)$, so if we act on such a root space with another root group in $G^{(i)}$, the same argument applies. Hence, we also cannot leave \mathfrak{u}_+ by iterating this step. Upon writing $g \in G^{(i)}$ as a product of elements in T and the root groups $(U_{\gamma})_{\gamma \in \Phi^{(i)}}$, it follows by induction on the number of factors that $\text{Ad}(g)\mathfrak{u}_{\beta} \subset \mathfrak{u}_+$, as claimed.

This claim implies that the sum on the right-hand side of (5.4.5) coincides with the Iwasawa decomposition of $\text{Ad}(k_2 k_1)w(h) \in \mathfrak{g}$ once the terms to the left (resp. right) of h'' are combined, so we deduce that

$$h'' = \pi_{\mathfrak{a}}(\text{Ad}(k_2 k_1)w(h)) \in \pi_{\mathfrak{a}}(\text{Ad}(K)w(h)) = \pi_{w(h)} = \pi_h.$$

Since the point $h'' \in P^{(i)}(t)$ was arbitrary, we have shown that $P^{(i)}(t) \subset \pi_h$ holds for all $t \in T_h^{(i)}$, which concludes the proof of Theorem 5.3.1 in the case where \mathcal{D} is free, cofree and cotorsion-free.

Remark 5.4.3. The proof of the claim above applies almost verbatim to establish a more general statement. Let $J \subset I$ be arbitrary and define

$$\mathfrak{u}_{J,+}^\perp := \bigoplus_{\alpha \in \Phi_+^\infty \setminus \Phi_{J,+}^\infty} \mathfrak{u}_\alpha \quad \text{and} \quad \mathfrak{p}_J := \mathfrak{g}_J \oplus \mathfrak{u}_{J,+}^\perp = \mathfrak{u}_0 \oplus \bigoplus_{\alpha \in \Phi_+^\infty} \mathfrak{u}_\alpha \oplus \bigoplus_{\alpha \in \Phi_{J,-}^\infty} \mathfrak{u}_\alpha.$$

Parallel to the definition of parabolic subgroups of G , it might be natural to call \mathfrak{p}_J the standard (positive) parabolic subalgebra of \mathfrak{g} of type J with Levi factor \mathfrak{g}_J and nilpotent radical $\mathfrak{u}_{J,+}^\perp$. Indeed, these notions are standard in the case where \mathfrak{g} is a finite-dimensional, complex, semisimple Lie algebra. Here, \mathfrak{g}_J is a reductive subalgebra and $\mathfrak{u}_{J,+}^\perp$ is the maximal nilpotent ideal of \mathfrak{p}_J . In the context of Kac-Moody algebras, they have also been introduced in [Kum02, Definition 1.2.2]. The Kac-Moody group G_J now acts on \mathfrak{g}_J via the adjoint representation and as a subgroup of G , it also acts on \mathfrak{g} . A simple modification of the argument in the previous proof shows that $\mathfrak{u}_{J,+}^\perp$ is invariant under the adjoint action of G_J .

For proving the claim in Section 5.2.2, such an additional step was not necessary because there, we made use of a reference to general Kac-Moody theory which asserted that the Levi factor G_J normalizes the unipotent radical $U_{J,+}^\perp$ of the parabolic subgroup $P_J \subset G$. On the Lie algebra level, the author was not aware of a similar result, but the argument essentially expresses the fact that $\mathfrak{u}_{J,+}^\perp$ is an ideal in \mathfrak{p}_J .

5.4.3 Step 3: Generalization to arbitrary root data

Finally, we generalize Theorem 5.3.1 to arbitrary Kac-Moody root data by using the machinery of morphisms and extensions. The arguments below are almost identical to the non-linear case and we freely apply the notation and results of Section 5.2.3.

Claim 1. Theorem 5.3.1 holds whenever \mathcal{D} is free.

Proof of Claim. Let \mathcal{D} be a free Kac-Moody root datum and $\tilde{\mathcal{D}}$ its cofree extension, which is free, cofree and cotorsion-free. The canonical projection $\tilde{\Lambda}^\vee \rightarrow \Lambda^\vee$ defines a morphism $\tilde{\mathcal{D}} \rightarrow \mathcal{D}$ and induces surjective homomorphisms $p : \tilde{G} \rightarrow G$ and $p' : \tilde{\mathfrak{g}} \rightarrow \mathfrak{g}$. The former satisfies $p(\tilde{K}) = K$ and $p(\tilde{A}) = A$, so $p'(\tilde{\mathfrak{a}}) = \mathfrak{a}$ follows from (3.6.7). If $h \in \mathfrak{a}$ is any almost virtually \mathcal{D} -singular element and $\tilde{h} \in \tilde{\mathfrak{a}}$ is chosen such that $p'(\tilde{h}) = h$, then \tilde{h} is almost virtually $\tilde{\mathcal{D}}$ -singular. Thus, we deduce from Theorem 5.3.1 for the free, cofree and cotorsion-free case that

$$\pi_{\tilde{\mathfrak{a}}}(\text{Ad}(\tilde{K})\tilde{h}) = \Xi_{\tilde{h}}.$$

Together with Lemma 3.6.3 (iii) and (3.6.5), this leads to the desired equality:

$$\begin{aligned} \pi_h &= \pi_{\mathfrak{a}}(\text{Ad}(K)h) = \pi_{\mathfrak{a}}(\text{Ad}(p(\tilde{K}))p'(\tilde{h})) = \pi_{\mathfrak{a}}(p'(\text{Ad}(\tilde{K})\tilde{h})) \\ &= p'(\pi_{\tilde{\mathfrak{a}}}(\text{Ad}(\tilde{K})\tilde{h})) = p'(\Xi_{\tilde{h}}) = \Xi_h \end{aligned}$$

Claim 2. Theorem 5.3.1 holds for all Kac-Moody root data.

Proof of Claim. Let \mathcal{D} be an arbitrary Kac-Moody root datum and $\hat{\mathcal{D}}$ its free extension. The canonical inclusion $\Lambda^\vee \rightarrow \hat{\Lambda}^\vee$ defines a morphism $\mathcal{D} \rightarrow \hat{\mathcal{D}}$ and induces injective homomorphisms $i : G \rightarrow \hat{G}$ and $i' : \mathfrak{g} \rightarrow \hat{\mathfrak{g}}$ with $i(K) \subset \hat{K}$ and $i'(\mathfrak{a}) \subset \hat{\mathfrak{a}}$. If $h \in \mathfrak{a}$ is any almost virtually \mathcal{D} -singular element, then $\hat{h} := i'(h) \in \hat{\mathfrak{a}}$ is almost virtually $\hat{\mathcal{D}}$ -singular and we conclude from Theorem 5.3.1 for the free case that

$$\pi_{\hat{\mathfrak{a}}}(\text{Ad}(i(K))\hat{h}) = \pi_{\hat{\mathfrak{a}}}(\text{Ad}(\hat{K})\hat{h}) = \Xi_{\hat{h}}.$$

Here, the first equality is justified by the fact that $\hat{K} = i(K)M_0$, where $M_0 = \hat{K} \cap T_0 \subset \hat{T}$ fixes $\hat{h} \in \hat{\mathfrak{a}} \subset \hat{\mathfrak{u}}_0$ under the adjoint action. Hence, Lemma 3.6.3 (iii) and (3.6.5) yield the following:

$$\begin{aligned} i'(\pi_h) &= i'(\pi_{\mathfrak{a}}(\text{Ad}(K)h)) = \pi_{\hat{\mathfrak{a}}}(i'(\text{Ad}(K)h)) = \pi_{\hat{\mathfrak{a}}}(\text{Ad}(i(K))i'(h)) \\ &= \pi_{\hat{\mathfrak{a}}}(\text{Ad}(i(K))\hat{h}) = \Xi_{\hat{h}} = i'(\Xi_h) \end{aligned}$$

As i' is injective, it follows that $\pi_h = \Xi_h$ and the proof of Theorem 5.3.1 is complete. \square

5.5 Comparison with similar results

Having established analogues of Kostant's Convexity Theorems for the class of (split, minimal) real or complex Kac-Moody groups, we conclude this chapter by comparing our linear result with an earlier convexity theorem for Kac-Moody algebras and investigate how our two theorems differ from Kostant's when specialized to the spherical case.

5.5.1 The Convexity Theorem of Kac & Peterson

The problem of generalizing Kostant's Linear Convexity Theorem to the Kac-Moody setting has already been considered much earlier by V.G. Kac and D.H. Peterson. In [KP84, Theorem 2 b)], they proved the following result which we have translated into the terminology of this thesis:

Kac-Peterson Convexity Theorem. *Let \mathbb{A} be a symmetrizable GCM with Weyl group W and \mathfrak{g} the Kac-Moody algebra of type $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ over \mathbb{C} with Cartan subalgebra \mathfrak{h} . Denote by $\mathfrak{a} := \Lambda_{\text{Kac}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ the canonical real form of \mathfrak{h} , by $X_+ \subset \mathfrak{a}$ the Tits cone of the $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ -representation of W and by $\pi_{\mathfrak{a}} : \mathfrak{g} \rightarrow \mathfrak{a}$ the natural projection. Moreover, let G be the Kac-Moody group of type $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ over \mathbb{C} with compact form K . For every $h \in X_+$, it holds that*

$$\{\pi_{\mathfrak{a}}(\text{Ad}(k)h) \mid k \in K\} = \text{conv}(W \cdot h). \quad (5.5.1)$$

Remark 5.5.1. The terminology in [KP84] is very different from ours, so we briefly need to justify why our formulation of the Kac-Peterson Convexity Theorem is equivalent to theirs.

- (i) Kac & Peterson do not define the algebra \mathfrak{g} using a root datum but via a \mathbb{C} -realization in the sense of Definition 2.3.4. Apart from a slightly different sign convention, their definition coincides with ours in the case of the root datum $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ and the field \mathbb{C} (cf. Remark 2.3.2). Another sign difference occurs in their construction of the compact involution of \mathfrak{g} which cancels with the previous one and is therefore equivalent to ours. Also, their definition of \mathfrak{a} (which is denoted by $\mathfrak{h}_{\mathbb{R}}$ in [KP84]) is the same as ours.
- (ii) Likewise, the group G is not defined via a root datum either but instead via the following representation-theoretic construction: Choose fundamental weights $\omega_1, \dots, \omega_n \in \mathfrak{a}^*$ and set $P_+ := \bigoplus_{i \in I} \mathbb{N}_0 \omega_i \subset \mathfrak{a}^* \subset \mathfrak{h}^*$. Recall that if $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is not invertible, these functionals are not uniquely determined. Since every $\lambda \in P_+$ is a dominant integral weight, the complex irreducible highest-weight \mathfrak{g} -module $L(\lambda)$ is integrable and the same is true for the direct sum of these modules. The authors consider the integrable representation (V, ρ) of \mathfrak{g} on the complex vector space $V := \bigoplus_{\lambda \in P_+} L(\lambda)$ and define G as the subgroup of $\text{GL}(V)$ generated by the set of operators $\{\exp(\rho(ze_i)), \exp(\rho(zf_i)) \mid i \in I, z \in \mathbb{C}\}$. It is a slightly non-trivial matter to verify that this coincides with the Kac-Moody group of type $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ over \mathbb{C} and we briefly propose the following argument:

Let G_{Kac} (resp. G_{sc}) denote the complex Kac-Moody group of type $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ (resp. $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$). Since $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ is free, cofree and cotorsion-free, the fundamental weights can be chosen to be contained in $\Lambda_{\text{Kac}} \subset \mathfrak{h}^*$, so that every $\lambda \in P_+$ becomes a proper dominant integral weight and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ lifts to a representation $\hat{\rho} : G_{\text{Kac}} \rightarrow \text{GL}(V)$ which may be restricted to the subgroup $G_{\text{sc}} \subset G_{\text{Kac}}$. Moreover, as the torus $T_{\text{sc}} = \Lambda_{\text{sc}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{C}^*$ of G_{sc} is generated by the set $\{r^{\alpha_i^{\vee}} \mid i \in I, r \in \mathbb{C}^*\}$, it follows from (KMG3) that G_{sc} is generated by the positive and negative simple root groups, so (3.2.5) implies that $\hat{\rho}(G_{\text{sc}}) = G$. On the one hand, it is clear from the definition that U_+ acts trivially on V . On the other hand, for every $i \in I$, the action of U_{-i} is non-trivial on the submodule $L(\omega_i) \subset V$ because $\rho(f_i)$ does not stabilize the highest-weight space $L(\omega_i)_{\omega_i}$. Thus, it is a consequence of the ‘‘Recognition Theorem’’ [Mar18, Theorem 7.71] that $\ker(\hat{\rho}|_{G_{\text{sc}}}) \subset T_{\text{sc}}$. In addition, the action of T_{sc} on V is faithful since if $t \in T_{\text{sc}} \setminus \{1\}$, the definition of T_{sc} shows that there exists some $i \in I$ with $t(\omega_i) \neq 1$, which implies that t acts non-trivially on the submodule $L(\omega_i) \subset V$. Hence, $\hat{\rho}$ induces an isomorphism $G_{\text{sc}} \cong G$ and under this identification, the compact involution of G defined in [KP84] is equivalent to ours, so the construction of the group K there coincides with the compact form of G_{sc} in our sense.

- (iii) The Weyl group W is also defined in a different way, but the authors assert that it acts faithfully on \mathfrak{a} and give explicit formulas for the actions of the generators s_1, \dots, s_n that coincide with the corresponding formulas in our definition of the $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ -representation of W . Thus, their notions of X_+ and $\text{conv}(W \cdot h)$ (which are denoted by X and $[W(h)]$ in [KP84]) are equivalent to ours.
- (iv) Instead of the projection $\pi_{\mathfrak{a}} : \mathfrak{g} \rightarrow \mathfrak{a}$ onto the real form \mathfrak{a} , the authors use the projection $p : \mathfrak{g} \rightarrow \mathfrak{h}$ onto the complex Cartan subalgebra $\mathfrak{h} = \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C}$, but these maps coincide on the set $\text{Ad}(K)h \subset \mathfrak{g}$. In fact, if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the eigenspace decomposition of the compact involution of \mathfrak{g} , then $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$ with $\mathfrak{a} \subset \mathfrak{p}$ and $i\mathfrak{a} \subset \mathfrak{k}$, so $\pi_{\mathfrak{a}}$ and p are identical on \mathfrak{p} . By Lemma 3.6.3 (iv), these eigenspaces are $\text{Ad}(K)$ -invariant, so $h \in \mathfrak{a}$ implies that $\text{Ad}(K)h \subset \mathfrak{p}$.

Apart from these differences in the definitions, the Kac-Peterson Convexity Theorem is very similar to Theorem 5.3.1. On the one hand, it is weaker in the sense that it neither incorporates the flexibility of Kac-Moody root data nor treats the real case. On the other hand, it is stronger as it only requires $h \in \mathfrak{a}$ to be contained in the Tits cone X_+ and does not make any assumptions on its stabilizer. In particular, the result applies to essentially singular elements as well which is not the case for Theorem 5.3.1. The proof in [KP84] is very different from ours. It is based on transferring concepts from symplectic geometry to highest-weight modules of Kac-Moody algebras. The authors introduce an analogue of the moment map in that situation and make use of the Atiyah-Guillemin-Sternberg Convexity Theorem. The details of the argument, however, are rather difficult to follow and the precise way in which that theorem is applied is not entirely clear to the author of this thesis. It is therefore an interesting but possibly hard problem to extract from their proof why the Kac-Peterson Convexity Theorem holds for essentially singular elements as well and whether Theorem 5.3.1 can also be proved by methods of symplectic geometry and thereby extended to the essentially singular case.

5.5.2 Kostant's Convexity Theorems for reductive algebraic \mathbb{R} -groups

Although the main focus of Theorem 5.1.1 and 5.3.1 clearly lies on the general Kac-Moody setting, they are equally valid if the underlying GCM is spherical. In this situation, they reduce to special cases of Kostant's original convexity theorems. More precisely, the non-linear convexity theorem for Kac-Moody groups leads to the following result:

Using the same notation as in Theorem 5.1.1, we now additionally assume that \mathbb{A} is spherical. Recall from Example 3.1.6 (i) that in this case, the Kac-Moody group $G = G_{\mathcal{D}}(\mathbb{K})$ over any field \mathbb{K} can be identified with the group $\mathcal{G}(\mathbb{K})$ of \mathbb{K} -points of a connected reductive algebraic group \mathcal{G} that is defined and split over \mathbb{K} . In particular, $G \cong \mathcal{G}(\mathbb{K})$ can be viewed as a Zariski-closed subgroup of $\text{GL}_N(\mathbb{K})$ for some $N \in \mathbb{N}$. If we further suppose that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, we may also consider G as a subgroup of $\text{GL}_N(\mathbb{K})$ that is closed in the standard Euclidean topology of $\mathbb{K}^{N \times N}$ and hence as a real or complex Lie group. Consequently, Theorem 5.1.1 reduces to a convexity theorem that applies to those Lie groups that arise as groups of real or complex points of connected, \mathbb{R} -split, reductive algebraic \mathbb{R} -groups, such as $\text{SL}_{n+1}(\mathbb{K})$ or $\text{Sp}_{2n}(\mathbb{K})$. Moreover, the condition on $h \in \mathfrak{a}$ to be almost virtually singular is vacuous in this case since the Weyl group of a spherical GCM is finite. However, not all reductive algebraic groups arise from Kac-Moody theory in this way. In particular, non-split groups are not covered by Theorem 5.1.1. This is merely a consequence of our definition of Kac-Moody groups and our methods can be adjusted to prove another variant of Kostant's Non-Linear Convexity Theorem in the setting of algebraic groups.

Theorem 5.5.2 (Kostant's Non-Linear Convexity Theorem for reductive algebraic \mathbb{R} -groups). *Let \mathcal{G} be a connected reductive algebraic \mathbb{R} -group, $\mathcal{T} \subset \mathcal{G}$ a maximal \mathbb{R} -split torus, $\Phi = \Phi(\mathcal{G}, \mathcal{T})$ the root system of \mathcal{G} relative to \mathcal{T} and W the Weyl group of Φ . For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, denote by $G := \mathcal{G}(\mathbb{K})$ the group of \mathbb{K} -points of \mathcal{G} , let $G = KAU_+$ be an Iwasawa decomposition of G and $\Pi_A : G \rightarrow A$, $g = kau \mapsto a$, the Iwasawa projection onto the A -component. Let \mathfrak{a} denote the Lie algebra of A . For every $h \in \mathfrak{a}$ with $a := \exp(h) \in A$, it holds that*

$$\{\Pi_A(ak) \mid k \in K\} = \exp(\text{conv}(W \cdot h)). \quad (5.5.2)$$

The major task in proving this result is to introduce all the relevant terminology. Once this is achieved, its proof is nothing but a minor modification of that of Theorem 5.1.1. In particular, the discussion of this thesis does not yet justify why the notions in the statement of Theorem 5.5.2 are well-defined. On the one hand, most of them are well-known from the structure theory of connected reductive algebraic groups over arbitrary fields. We will not develop this subject in detail and instead refer the reader to [Bor91, Chapter V, §18-21] and [Spr09, Chapter 15 and 16] for all relevant definitions and proofs. On the other hand, some of them are simple Lie-theoretic properties of groups of real or complex points of reductive algebraic \mathbb{R} -groups that are proved in [Bor91, Chapter V, §24.6] and [PR94, Section 3.2]. We freely use these references and only briefly introduce the required notation which will be very similar to the Kac-Moody setting. Occasionally, we also use suggestive notation that should be clear from the context. In the following discussion, we view all algebraic groups as affine algebraic varieties over \mathbb{C} in the classical sense of algebraic geometry and identify them with their groups of \mathbb{C} -points.

Let $d \in \mathbb{N}_0$ denote the \mathbb{R} -rank of \mathcal{G} , $n \in \mathbb{N}_0$ the semisimple \mathbb{R} -rank and set $I := \{1, \dots, n\}$. The chosen maximal \mathbb{R} -split torus $\mathcal{T} \subset \mathcal{G}$ has dimension d and acts diagonally on the Lie algebra of \mathcal{G} via the adjoint representation, which gives rise to a finite set of roots $\Phi = \Phi(\mathcal{G}, \mathcal{T}) \subset X(\mathcal{T}) \cong \mathbb{Z}^d$ that defines a root system in the n -dimensional subspace of $\mathfrak{a}^* := X(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ that it spans. We choose a set $\Phi_0 = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$ of simple roots and write $\Phi_+ \subset \Phi$ for the resulting set of positive roots. The Weyl group $W = W(\mathcal{G}, \mathcal{T})$ associated to this root system is a finite group that is generated by the set $S = \{s_1, \dots, s_n\}$ of reflections at the simple roots and (W, S) is a Coxeter system. By duality, it also acts on $\mathfrak{a} := (\mathfrak{a}^*)^*$ which can be identified with the Lie algebra of the real Lie group $\mathcal{T}(\mathbb{R}) \cong (\mathbb{R}^*)^d$.

Let $\mathcal{N}(\mathcal{T})$ (resp. $\mathcal{Z}(\mathcal{T})$) be the normalizer (resp. centralizer) of \mathcal{T} in \mathcal{G} . The quotient $\mathcal{N}(\mathcal{T})/\mathcal{Z}(\mathcal{T})$ is a finite group that acts faithfully on \mathcal{T} by conjugation and on $X(\mathcal{T})$ by the corresponding dual action. By extension of scalars, it acts on \mathfrak{a} and \mathfrak{a}^* via the adjoint and coadjoint representation. The action on \mathfrak{a}^* preserves Φ and induces an isomorphism $\mathcal{N}(\mathcal{T})/\mathcal{Z}(\mathcal{T}) \cong W$.³ For every $w \in W$, we choose a pre-image $\tilde{w} \in \mathcal{N}(\mathcal{T})$ under this isomorphism and denote by \tilde{W} the set of all such elements. Both $\mathcal{N}(\mathcal{T})$ and $\mathcal{Z}(\mathcal{T})$ are defined over \mathbb{R} and these representatives can be chosen to be contained in $N(\mathcal{T}) := \mathcal{N}(\mathcal{T})(\mathbb{R})$. This set will serve as a substitute for the extended Weyl group which is a priori not defined in this context.

For every root $\alpha \in \Phi$, there exists a connected unipotent \mathbb{R} -subgroup $\mathcal{U}_\alpha \subset \mathcal{G}$ that is normalized by $\mathcal{Z}(\mathcal{T})$. Given any subset $J \subset I$, we denote by \mathcal{T}_J the identity component of the subtorus $\bigcap_{j \in J} \ker(\alpha_j) \subset \mathcal{T}$ in the Zariski topology and by $\mathcal{G}_J := \mathcal{Z}(\mathcal{T}_J)$ its centralizer in \mathcal{G} , which is a connected reductive \mathbb{R} -subgroup of \mathcal{G} whose \mathbb{R} -rank (resp. semisimple \mathbb{R} -rank) is d (resp. $|J|$). The maximal \mathbb{R} -split torus \mathcal{T} is contained in \mathcal{G}_J , the corresponding root system $\Phi(\mathcal{G}_J, \mathcal{T}) \subset \mathfrak{a}^*$ can be identified with the subset $\Phi_J \subset \Phi$ spanned by $(\alpha_j)_{j \in J}$ and its Weyl group with the standard subgroup $W_J \subset W$. In addition, $\mathcal{G}_J = \mathcal{Z}(\mathcal{T}_J)$ is generated by $\mathcal{Z}(\mathcal{T})$ and the family $(\mathcal{U}_\alpha)_{\alpha \in \Phi_J}$. Let $\mathcal{U}_{J,+}$ (resp. $\mathcal{U}_{J,+}^\perp$) be the subgroup of \mathcal{G} generated by all \mathcal{U}_α with $\alpha \in \Phi_{J,+}$ (resp. $\alpha \in \Phi_{J,+}^\perp$), then $\mathcal{U}_{J,+}$ is contained in \mathcal{G}_J while $\mathcal{U}_{J,+}^\perp$ is normalized by \mathcal{G}_J and intersects it trivially. Hence, the subgroup $\mathcal{P}_J \subset \mathcal{G}$ generated by these two groups has the structure of a semidirect product $\mathcal{P}_J \cong \mathcal{G}_J \ltimes \mathcal{U}_{J,+}^\perp$. We also write $\mathcal{U}_+ := \mathcal{U}_{I,+}$.

All subgroups of \mathcal{G} that we have introduced are defined over \mathbb{R} and we denote their groups of \mathbb{K} -points by the respective Roman letters.⁴ In analogy with the definitions in Section 3.1, we call $U_\alpha = \mathcal{U}_\alpha(\mathbb{K})$ the root group associated to $\alpha \in \Phi$ and refer to $P_J = \mathcal{P}_J(\mathbb{K})$ as the standard parabolic subgroup of G of type J with Levi factor $G_J = \mathcal{G}_J(\mathbb{K})$ and unipotent radical $U_{J,+}^\perp = \mathcal{U}_{J,+}^\perp(\mathbb{K})$. It follows from the Bruhat decomposition that $G = \mathcal{G}(\mathbb{K})$ is generated by $Z(\mathcal{T}) = \mathcal{Z}(\mathcal{T})(\mathbb{K})$ and the root groups $(U_\alpha)_{\alpha \in \Phi}$. The obvious analogues of this statement are true for the parabolic subgroups, their Levi factors and their unipotent radicals. If \mathcal{G} is \mathbb{R} -split, then \mathcal{T} is a maximal torus of \mathcal{G} and $\mathcal{Z}(\mathcal{T}) = \mathcal{T}$. In this case, our setup is almost identical to the Kac-Moody setting.

³In fact, the group W is usually defined as $\mathcal{N}(\mathcal{T})/\mathcal{Z}(\mathcal{T})$ which is shown to be finite. This is used to prove that there exists a W -invariant inner product on \mathfrak{a}^* and that Φ is a root system whose Weyl group is isomorphic to that quotient. From the perspective of Kac-Moody theory, however, the converse approach might appear more natural.

⁴If $\mathbb{K} = \mathbb{C}$, this notation could be omitted since we identify an algebraic group with its group of \mathbb{C} -points, but it is convenient to treat the real and complex case simultaneously.

The only noteworthy differences that can arise if \mathcal{G} is not \mathbb{R} -split are the following:

- a) The root system Φ need not be reduced, i.e. it is possible that there exists some root $\alpha \in \Phi$ such that 2α is also a root in Φ .
- b) The root groups $(U_\alpha)_{\alpha \in \Phi}$ are not necessarily all isomorphic to the additive group of \mathbb{K} .

These details will not be relevant to our argument. We now turn to the Lie-theoretic aspects of Theorem 5.5.2 which were already briefly mentioned in Remark 3.4.2. The Iwasawa decomposition $G = KAU_+$ is constructed in [PR94, Section 3.2]. After possibly conjugating $G \subset \mathrm{GL}_N(\mathbb{K})$ by an element of $\mathrm{GL}_N(\mathbb{R})$, it is stable under the involution $g \mapsto (g^{-1})^*$ and K is defined as its fixed-point set in G or, equivalently, the intersection of G with the orthogonal or unitary group, which is a maximal compact subgroup of G with respect to the Lie group topology that meets every connected component and the elements of \tilde{W} can be chosen to be contained in the identity component of K . Moreover, $A \cong (\mathbb{R}_{>0})^d$ is defined as the identity component of $\mathcal{T}(\mathbb{R}) \cong (\mathbb{R}^*)^d$ with respect to its Lie group topology and $U_+ = \mathcal{U}_+(\mathbb{K})$. With a suitable choice of \mathcal{T} (resp. Φ_+), it can be ensured that A (resp. U_+) is the intersection of G with the group of diagonal matrices with positive real entries (resp. the group of upper-triangular matrices with 1's on the diagonal). Finally, A is normalized by $N(\mathcal{T})$, so the Weyl group acts on \mathfrak{a} and A and the exponential map of matrices defines a W -equivariant isomorphism $\exp : \mathfrak{a} \rightarrow A$. Having introduced all these notions, the proof of Theorem 5.5.2 consists of a simple adaptation of the arguments in Section 5.2 and we content ourselves with a rather brief summary of the main ideas.

Proof sketch. Since a reductive group is the product of its derived subgroup and its radical, which is contained in its center, we may assume that \mathcal{G} is semisimple, i.e. $d = n$. If \mathcal{G} is \mathbb{R} -anisotropic, i.e. $n = 0$, then $G = K$ and $A = U_+ = \{1\}$, so that the statement is trivial. If \mathcal{G} is \mathbb{R} -isotropic, i.e. $n > 0$, we define $\Pi_a := \Pi_A(aK) \subset A$ and $\Xi_h := \mathrm{conv}(W \cdot h) \subset \mathfrak{a}$. As these sets do not depend on the W -translate of $h \in \mathfrak{a} = X_+$, we may w.l.o.g. suppose that $h \in \overline{C}$.

- (i) **Step 1:** Let $k \in K$ be arbitrary and write $a' := \Pi_A(ak) \in \Pi_a$. In order to show that $a' \in \exp(\Xi_h)$, it suffices to prove that $h' := \log(a') \in \mathfrak{a}$ satisfies the following inequalities:

$$\langle \omega_i, w(h') \rangle \leq \langle \omega_i, h \rangle \quad \text{for all } w \in W \text{ and } i \in I$$

Here, the functionals $\omega_1, \dots, \omega_n \in \mathfrak{a}^*$ denote the fundamental weights of the root system Φ . In fact, the W -action on \mathfrak{a} is determined by $s_i(y) = y - \langle \alpha_i, y \rangle \alpha_i^\vee$ for all $i \in I$ and $y \in \mathfrak{a}$, where $\Phi_0^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{a}$ is the set of simple coroots of Φ , so this characterization of Ξ_h follows from the same results as in Section 4.3 and the assumption that $d = n$. Showing that Ξ_h is closed can be omitted in this case since W is finite and Ξ_h is a convex polytope. To prove these inequalities, we apply the same reasoning as in Section 5.2.1 to suitable representations of $G = \mathcal{G}(\mathbb{K})$. In contrast to the Kac-Moody setting, however, where the condition of cotorsion-freeness on the underlying root datum allowed us to consider the highest-weight modules associated to the fundamental weights, these representations have to be constructed in a different way which is inspired by [Hum75, Section 31.4].

For every $i \in I$, the standard parabolic subgroup $\mathcal{P}^{(i)}$ of type $I^{(i)}$ is a Zariski-closed subgroup of \mathcal{G} , so [Bor91, Chapter II, Theorem 5.1] asserts that there exists a faithful representation $\hat{\rho}_i : \mathcal{G} \rightarrow \mathrm{GL}(V)$ on a finite-dimensional complex vector space V and a one-dimensional subspace $L \subset V$ whose stabilizer is exactly $\mathcal{P}^{(i)}$. Because $\mathcal{U}_+ \subset \mathcal{P}^{(i)}$ fixes this line, the submodule $V' \subset V$ generated by a non-zero vector in L is a highest-weight representation of \mathcal{G} . The action of the Weyl group on $X(\mathcal{T}) \subset \mathfrak{a}^*$ permutes the weights of V' in the sense that $\tilde{w}(V'_\mu) = V'_{w(\mu)}$ holds for all $w \in W$ and $\mu \in \mathrm{wt}(V')$. The highest weight $\lambda \in \mathrm{wt}(V')$ is fixed by the subgroup $W^{(i)} \subset W$ as the corresponding set of representatives $\tilde{W}^{(i)} \subset \mathcal{G}$ can be chosen to be contained in $\mathcal{G}^{(i)} \subset \mathcal{P}^{(i)}$ which stabilizes the highest-weight space $V'_\lambda = L$. This implies that $\lambda = m_i \omega_i$ for some $m_i \in \mathbb{N}$. If $\mathbb{K} = \mathbb{C}$, the arguments in Section 5.2.1 now apply to the representation of $G = \mathcal{G}(\mathbb{C})$ on V' upon replacing ω_i by $m_i \omega_i$. If $\mathbb{K} = \mathbb{R}$, then $\mathcal{P}^{(i)}$ is defined over \mathbb{R} and the same is true for V and L , so $\hat{\rho}_i$ can be restricted to a real representation of $G = \mathcal{G}(\mathbb{R})$ on an \mathbb{R} -form of V' with identical properties. In both situations, the positive-definite, contravariant, Hermitian (resp. symmetric bilinear) form on $L(\omega_i)$ has to be replaced by a K -invariant inner product on (a real form of) V' with the property that distinct weight spaces are orthogonal, which exists by compactness of K .

- (ii) **Step 2:** The proof of the inclusion $\exp(\Xi_h) \subset \Pi_a$ proceeds by induction on n , but the base case is a bit simpler than in the Kac-Moody setting. If $n = 1$, then $\Phi = \{\alpha_1, -\alpha_1\}$, $W = \{1, s_1\} \cong \mathbb{Z}_2$ and $\dim(\mathfrak{a}) = 1$. Since the representative $\tilde{s}_1 \in \tilde{W}$ of $s_1 \in W$ can be chosen to be contained in the identity component of K w.r.t. the Lie group topology of G , there exists a continuous curve $c : [0, 1] \rightarrow K$ such that $c(0) = 1$ and $c(1) = \tilde{s}_1$. Moreover, the Iwasawa decomposition $K \times A \times U_+ \rightarrow G$ and the exponential map $\exp : \mathfrak{a} \rightarrow A$ are homeomorphisms, which implies that $t \mapsto \log(\Pi_A(ac(t)))$ is a continuous curve in \mathfrak{a} that connects h with $s_1(h)$. As $\dim(\mathfrak{a}) = 1$, it runs over the closed line segment $[h, s_1(h)] = \Xi_h$, so Π_a contains the exponential of that line segment, as required.
- If $n > 1$, the argument works verbatim as in Section 5.2.2. Let $J \subset I$ be arbitrary, $w \in W$ right J -reduced and consider the subgroup $G_{w(J)} \subset G$ generated by $Z(\mathcal{T})$ and all $U_{w(\alpha)}$ with $\alpha \in \Phi_J$. As conjugation in \mathcal{G} by $\tilde{w} \in \tilde{W} \subset K$ permutes the root groups in the sense that $\tilde{w}\mathcal{U}_\alpha\tilde{w}^{-1} = \mathcal{U}_{w(\alpha)}$ holds for all $\alpha \in \Phi$, it follows that $G_{w(J)} = \tilde{w}G_J\tilde{w}^{-1} = (\tilde{w}\mathcal{G}_J\tilde{w}^{-1})(\mathbb{K})$ is the group of \mathbb{K} -points of a connected reductive algebraic \mathbb{R} -group whose semisimple \mathbb{R} -rank is equal to $|J|$. Furthermore, $\mathcal{T} = \tilde{w}\mathcal{T}\tilde{w}^{-1}$ is a maximal \mathbb{R} -split torus of $\tilde{w}\mathcal{G}_J\tilde{w}^{-1}$ and the Weyl group of the root system $\Phi(\tilde{w}\mathcal{G}_J\tilde{w}^{-1}, \mathcal{T})$ can be identified with the parabolic subgroup $wW_Jw^{-1} \subset W$. In addition, $G_{w(J)}$ has an Iwasawa decomposition $G_{w(J)} = K_{w(J)}AU_{w(J),+}$, where $K_{w(J)} = \tilde{w}K_J\tilde{w}^{-1} = K \cap G_{w(J)}$ and $U_{w(J),+} = \tilde{w}U_{J,+}\tilde{w}^{-1} = U_+ \cap G_{w(J)}$, which is simply the restriction of the Iwasawa decomposition of G to $G_{w(J)}$. Hence, the induction hypothesis can be formulated exactly as in Section 5.2.2, where the condition of being almost virtually singular can be omitted since W is finite. From this point onwards, the arguments there apply without change in the present situation as well because all results of Section 4.3 that were used in that step are equally valid here.
- (iii) **Step 3** of the proof of Theorem 5.1.1 has no analogue in the current situation. Indeed, the simple roots and coroots of Φ are known to be linearly independent, which is the counterpart to freeness and cofreeness of Kac-Moody root data. The assumption of cotorsion-freeness, which was imposed in the Kac-Moody setting in order to lift highest-weight representations to the group level, is analogous to requiring \mathcal{G} to be simply connected. This has been avoided by the alternative construction of appropriate representations in Step 1. \square

Except for the terminology of algebraic geometry instead of Lie theory, Theorem 5.5.2 is almost identical to the original version of Kostant's Non-Linear Convexity Theorem. Kostant's result is formulated for connected semisimple Lie groups and ours for groups of real or complex points of connected reductive algebraic \mathbb{R} -groups, but the statement is the same in both cases. However, it should be observed that the two notions of connectedness are not equivalent. It refers to the Lie group topology in Kostant's theorem and to the Zariski topology in ours. If \mathcal{G} is a connected algebraic \mathbb{R} -group, then $\mathcal{G}(\mathbb{C})$ is also connected in its Lie group topology, while $\mathcal{G}(\mathbb{R})$ has finitely many connected components but is not necessarily connected as a Lie group. On the one hand, our result is thus more general than Kostant's in the sense that it also deals with some Lie groups that are not connected and only requires them to be reductive instead of semisimple. For instance, Theorem 5.5.2 applies to $\mathrm{GL}_N(\mathbb{R})$ for every $N \in \mathbb{N}$ which is neither connected nor semisimple as a Lie group, but GL_N is a connected reductive algebraic \mathbb{R} -group. On the other hand, not all connected semisimple Lie groups arise as groups of real or complex points of algebraic groups. One notable example is the universal cover of $\mathrm{SL}_2(\mathbb{R})$ which cannot be realized as a matrix group. Such groups are covered by Kostant's theorem but not ours.

Likewise, Theorem 5.3.1 reduces to a special case of Kostant's Linear Convexity Theorem if \mathbb{A} is spherical. While this variant only applies to a rather small family of groups, it can be generalized exactly as in the non-linear setting. We conclude this chapter by stating the resulting theorem. Its proof consists of a straightforward modification of that of Theorem 5.3.1 and will be omitted.

Theorem 5.5.3 (Kostant's Linear Convexity Theorem for reductive algebraic \mathbb{R} -groups). *With the assumptions and notation of Theorem 5.5.2, let \mathfrak{g} be the Lie algebra of G , $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$ the Iwasawa decomposition of \mathfrak{g} corresponding to $G = KAU_+$ and $\pi_a : \mathfrak{g} \rightarrow \mathfrak{a}$ the Iwasawa projection onto the \mathfrak{a} -component. For every $h \in \mathfrak{a}$, it holds that*

$$\{\pi_a(\mathrm{Ad}(k)h) \mid k \in K\} = \mathrm{conv}(W \cdot h). \quad (5.5.3)$$

Chapter 6

Applications

Having successfully established a natural analogue of Kostant’s Non-Linear Convexity Theorem for the classes of split real and complex Kac-Moody groups, we now use this result in order to investigate their algebraic structure. Along the way, we collect several properties of the Iwasawa projections that fit nicely into our discussion. This is the content of Section 6.1, where we describe an algorithm that (to a certain extent) makes it possible to describe the multiplication map in a Kac-Moody group in Iwasawa coordinates and to decide whether certain subsets are actually sub(semi-)groups. In Section 6.2, we apply the convexity theorem to establish the (non-)existence of various decompositions of Kac-Moody groups and prove Theorem B and C. Finally, we put these results in a somewhat “geometric” perspective in Section 6.3, which is motivated by the historical introduction, where we have explained how Kostant’s Non-Linear Convexity Theorem for semisimple Lie groups can be rephrased as a geometric statement about Riemannian symmetric spaces of non-compact type and how decompositions of semisimple Lie groups admit geometric interpretations in that framework. In order to extend this to the Kac-Moody setting, we present the construction of *Kac-Moody symmetric spaces* developed in [FHHK20] and reformulate our results in that context. We close this chapter by proving Theorem D, which will be an immediate consequence of Theorem C in the language of Kac-Moody symmetric spaces.

The notation will follow the previous chapters. Unless stated otherwise, we denote by $\mathbb{A} \in \mathbb{Z}^{n \times n}$ an arbitrary GCM, by $\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ a Kac-Moody root datum of rank $d \in \mathbb{N}$ and we employ the familiar terminology for the \mathcal{D} -representation of $W = W(\mathbb{A})$ on $\mathfrak{a} = \Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. We suppose that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and write $G = G_{\mathcal{D}}(\mathbb{K})$ for the Kac-Moody group of type \mathcal{D} over \mathbb{K} with Iwasawa decomposition $G = KAU_+$ and Iwasawa projections Π_K , Π_A and Π_{U_+} . In order to apply Theorem 5.1.1, we will occasionally have to assume that \mathbb{A} is symmetrizable, but we mention this explicitly when necessary.

6.1 The Iwasawa algorithm

Kostant’s Non-Linear Convexity Theorem is rooted in the question how the multiplication map $G \times G \rightarrow G$ can be described in terms of Iwasawa coordinates, but Theorem 5.1.1 provides a very limited answer as it only determines the A -components of the restricted map $\{a\} \times K \rightarrow G$ for certain elements $a \in A$ as a set instead of point-wise. Our first goal is to find an algorithm that describes more explicitly how the Iwasawa coordinates in G behave with respect to the group multiplication and reduces this problem to (2×2) -matrix computations as much as possible. This section heavily relies on the computational results about $G^0 = \mathrm{SL}_2(\mathbb{K})$ established in Section 3.5 and we freely use the notation introduced there. We begin with a straightforward generalization of Corollary 3.5.2 which will be useful in several instances.

Lemma 6.1.1. *Let $a \in A$ be arbitrary and set $h := \log(a) \in \mathfrak{a}$. For every $\alpha \in \Phi_+$, the image of the map $K_\alpha \rightarrow A$, $k \mapsto \Pi_A(ak)$, is the exponential of the closed line segment in \mathfrak{a} between h and $s_\alpha(h)$ or, equivalently, the set $\{ar^{h_\alpha} \mid r \in [1, a(c_\alpha)^{-1}]\} \subset A$.*

This result is a special case of Theorem 5.1.1, but it can be proved in a much simpler way. In fact, the following argument is almost identical to the base case of the induction in Section 5.2.2.

Proof. By Lemma 3.4.16, there exist $a_1, a_2 \in A$ with $a = a_1 a_2$, $a_1 \in A_\alpha$ and $a_2(c_\alpha) = 1$, so that a_2 commutes with $G_\alpha = \varphi_\alpha(G^0)$. Equivalently, on the Lie algebra level, we have $h = h_1 + h_2$, where $h_1 := \log(a_1) \in \mathfrak{a}_\alpha$ and $h_2 := \log(a_2) \in \mathfrak{a}$ satisfies $\langle c_\alpha, h_2 \rangle = 0$, so that h_2 is fixed by $s_\alpha \in W$. Since the group homomorphism $\varphi_\alpha : G^0 \rightarrow G$ restricts to a bijection $A^0 \rightarrow A_\alpha$, there exists a unique $a^0 \in A^0$ with $\varphi_\alpha(a^0) = a_1$. By (3.4.7) and (3.4.9), the Lie algebra homomorphism $\varphi'_\alpha : \mathfrak{g}^0 \rightarrow \mathfrak{g}$ satisfies $\varphi'_\alpha(h^0) = h_1$ for $h^0 := \log(a^0) \in \mathfrak{a}^0$ and the following holds for all $k^0 \in K^0$:

$$\Pi_A(a\varphi_\alpha(k^0)) = \Pi_A(a_1\varphi_\alpha(k^0))a_2 = \Pi_A(\varphi_\alpha(a^0k^0))a_2 = \varphi_\alpha(\Pi_{A^0}(a^0k^0))a_2$$

By Corollary 3.5.2, the image of the map $k^0 \mapsto \Pi_{A^0}(a^0k^0)$ is the exponential of the closed line segment $[h^0, -h^0] \subset \mathfrak{a}^0$ and we have $\varphi'_\alpha(h^0) = h_1$ and $\varphi'_\alpha(-h^0) = -h_1 = s_\alpha(h_1)$. The image of this set under φ_α is therefore the exponential of $[h_1, s_\alpha(h_1)] \subset \mathfrak{a}_\alpha$. As $s_\alpha(h) = -h_1 + h_2$, multiplying this set from the right by $a_2 = \exp(h_2)$ results in the exponential of $[h, s_\alpha(h)] \subset \mathfrak{a}$. Since $K_\alpha = \varphi_\alpha(K^0)$, the proof of the first assertion is complete and the second follows from Lemma 3.4.14 because $[h, s_\alpha(h)] = \{h + th_\alpha \mid t \in [0, -\langle c_\alpha, h \rangle]\}$. \square

We will frequently make use of the following “standard” expressions for elements of K and U_+ , which will allow us to apply inductive arguments and reduce many computations in the abstract Kac-Moody group G to explicit (2×2) -matrix calculations in $G^0 = \mathrm{SL}_2(\mathbb{K})$.

Notation 6.1.2. Since $K = MK'$, where K' is the subgroup of K generated by the K^0 -images $(K_i)_{i \in I}$ (cf. Corollary 3.4.11), and U_+ is generated by the positive root groups $(U_\alpha)_{\alpha \in \Phi_+}$, every element $k \in K$ and $u \in U_+$ can be written in one of the following ways:

$$k = k_0 k_1 \dots k_q \text{ for some } q \in \mathbb{N}_0, \text{ where } k_0 \in M \text{ and for every } p \in \{1, \dots, q\}, \quad (6.1.1)$$

the factor k_p is an element of a K^0 -image K_{i_p} for some $i_p \in I$.

$$u = u_1 \dots u_s \text{ for some } s \in \mathbb{N}_0, \text{ where for every } r \in \{1, \dots, s\}, \text{ the factor } u_r \text{ is} \quad (6.1.2)$$

an element of a positive root group U_{β_r} for some $\beta_r \in \Phi_+$.

In both cases, we may assume that $q, s \in \mathbb{N}_0$ are minimal among all such product decompositions of k and u . In particular, this ensures that no two consecutive factors of k (resp. u) are contained in the same K^0 -image (resp. root group). Moreover, it implies that $q = 0$ if and only if $k \in M$ and $s = 0$ if and only if $u = 1$. We call (6.1.1) (resp. (6.1.2)) a *minimal expression* of k (resp. u).

Suppose we are given two elements $g_1, g_2 \in G$ in Iwasawa form $g_1 = k_1 a_1 u_1$ and $g_2 = k_2 a_2 u_2$ and we want to compute the Iwasawa components of the product $g_1 g_2 \in G$. This can be achieved in three steps:

1. Compute the Iwasawa components of the subproduct $u_1 k_2$.
2. Set $k' := \Pi_K(u_1 k_2)$ and compute the Iwasawa components of $a_1 k'$.
3. Use the fact that A normalizes U_+ to bring the resulting expression into Iwasawa form.

Carrying out the first two steps is a non-trivial matter since the defining relations of G are rather hard to control. However, to a certain extent, it is possible to apply “elementary moves” in order to reduce them to explicit matrix computations inside the G^0 -images of G . The following procedure, which will be called the *Iwasawa algorithm*, makes these statements more precise.

In order to bring a product of the form $auk \in G$ for some elements $k \in K$, $a \in A$ and $u \in U_+$ into Iwasawa form, we may choose minimal expressions $k = k_0 k_1 \dots k_q$ and $u = u_1 \dots u_s$ as in (6.1.1) and (6.1.2), iteratively multiply the factors of u by those of k and compute the Iwasawa components after each step. The first product is $u_s k_0$, where $k_0 \in M \subset T$ normalizes U_{β_s} , which can be brought into Iwasawa form by writing $u_s k_0 = k_0 (k_0^{-1} u_s k_0)$, so we have $\Pi_K(u_s k_0) = k_0$, $\Pi_A(u_s k_0) = 1$ and $\Pi_{U_+}(u_s k_0) = k_0^{-1} u_s k_0 \in U_{\beta_s}$. More precisely, there exists a unique $z \in \mathbb{K}$ with $u_s = x_{\beta_s}(z)$ and $k_0^{-1} u_s k_0 = x_{\beta_s}(k_0(c_{\beta_s})^{-1} z)$. Such an operation will be called an *elementary UK-move of the first type*. Repeating it for all factors of u leads to the product

$$auk = ak_0 \hat{u}_1 \dots \hat{u}_s k_1 \dots k_q,$$

where $\hat{u}_r := k_0^{-1} u_r k_0 \in U_{\beta_r}$ for every $r \in \{1, \dots, s\}$. If $q > 0$, we next consider the subproduct $\hat{u}_s k_1$ with $k_1 \in K_{i_1}$. There are two possible cases, which will be called *elementary UK-moves of the second and third type*:

- a) If $\beta_s \neq \alpha_{i_1}$, then \hat{u}_s lies in the unipotent radical of the standard parabolic subgroup $P_{\{i_1\}}$ of type $\{i_1\}$ while k_1 is contained in its Levi factor. Hence, we have $\hat{u}_s k_1 = k_1 (k_1^{-1} \hat{u}_s k_1)$, where the product in parentheses lies in the unipotent radical of $P_{\{i_1\}}$ and therefore in U_+ . In this case, we also obtain $\Pi_K(\hat{u}_s k_1) = k_1$, $\Pi_A(\hat{u}_s k_1) = 1$ and $\Pi_{U_+}(\hat{u}_s k_1) = k_1^{-1} \hat{u}_s k_1$. The difference to a UK-move of the first type is that the U_+ -component need not be contained in U_{β_s} and may become very complicated due to the commutator relations (3.1.5).
- b) If $\beta_s = \alpha_{i_1}$, then \hat{u}_s and k_1 are both contained in the G^0 -image $G_{i_1} = \varphi_{i_1}(G^0)$. Hence, there exist $k^0 \in K^0$ and $u^0 \in U_+^0$ which are mapped to k_1 and \hat{u}_s by φ_{i_1} . The latter is uniquely determined, while the former is unique if φ_{i_1} is injective and unique up to sign if $\ker(\varphi_{i_1}) = \{\mathbb{I}_2, -\mathbb{I}_2\} \subset K^0$. Since φ_{i_1} is compatible with the Iwasawa decompositions $G^0 = K^0 A^0 U_+^0$ and $G = K A U_+$, the Iwasawa components of $\hat{u}_s k_1$ are simply the images of those of $u^0 k^0 \in G^0$ under φ_{i_1} and they can be computed explicitly via Proposition 3.5.3. More precisely, using the notation of (3.5.2), there exist unique $(z_1, z_2) \in B$ and $z \in \mathbb{K}$ such that $k^0 = k(z_1, z_2)$ and $u^0 = u(z)$ and (3.5.9) yields the following:

$$\begin{aligned}
\Pi_K(\hat{u}_s k_1) &= \varphi_{i_1}(\Pi_{K^0}(u(z)k(z_1, z_2))) = \varphi_{i_1}(k(\kappa_U(z, z_1, z_2))) \in K_{i_1} \\
\Pi_A(\hat{u}_s k_1) &= \varphi_{i_1}(\Pi_{A^0}(u(z)k(z_1, z_2))) = \varphi_{i_1}(a(\xi_U(z, z_1, z_2))) \\
&= \xi_U(z, z_1, z_2)^{h_{i_1}} \in A_{i_1} \\
\Pi_{U_+}(\hat{u}_s k_1) &= \varphi_{i_1}(\Pi_{U_+^0}(u(z)k(z_1, z_2))) = \varphi_{i_1}(u(\eta_U(z, z_1, z_2))) \\
&= x_{i_1}(\eta_U(z, z_1, z_2)) \in U_{i_1}
\end{aligned} \tag{6.1.3}$$

If φ_{i_1} is not injective, we could have replaced k_0 by $-k_0$ and thus (z_1, z_2) by $(-z_1, -z_2)$. The functions ξ_U and η_U are unaffected by this change and $\kappa_U(z, -z_1, -z_2) = -\kappa_U(z, z_1, z_2)$. This assertion merely mirrors the fact that $-\mathbb{I}_2$ is contained in K^0 and central in G^0 . After applying φ_{i_1} , the minus sign disappears, so the three components are independent of the choice of k^0 , as they should.

Performing one of these UK-moves results in the expression

$$auk = ak_0 \hat{u}_1 \dots \hat{u}_{s-1} k'_1 a'_s \hat{u}'_s k_2 \dots k_q,$$

where $k'_1 a'_s \hat{u}'_s$ is the Iwasawa form of $\hat{u}_s k_1$. For a UK-move of the second type, we have $k'_1 = k_1$ and $a'_s = 1$, but \hat{u}'_s may become a very complicated element of U_+ . For the third type, we have $k'_1 \in K_{i_1}$, $a'_s \in A_{i_1}$ and $\hat{u}'_s \in U_{i_1} = U_{\beta_s}$, but k'_1 and \hat{u}'_s might differ from k_1 and \hat{u}_s and a'_s may be non-trivial. In either case, it follows that $k'_1 \in K_{i_1}$, so if $s > 1$, we may perform another UK-move of the second or third type to write the product $\hat{u}_{s-1} k'_1$ in Iwasawa form as $k''_1 a''_{s-1} \hat{u}''_{s-1}$ for some $k''_1 \in K_{i_1}$, $a''_{s-1} \in A_{i_1}$ and $\hat{u}''_{s-1} \in U_+$, which leads to

$$auk = ak_0 \hat{u}_1 \dots \hat{u}_{s-2} k''_1 a''_{s-1} \hat{u}''_{s-1} a'_s \hat{u}'_s k_2 \dots k_q.$$

Since $A \subset T$ normalizes U_+ , the conjugate $\hat{u}'_{s-1} := a'^{-1}_{s-1} \hat{u}''_{s-1} a'_s$ is contained in U_+ . Note that it can be computed explicitly by choosing a minimal expression for \hat{u}''_{s-1} and using the fact that conjugation by a'^{-1}_{s-1} acts on each root group by multiplication with a positive real number. Thus, we may replace $\hat{u}''_{s-1} a'_s$ by $a'_s \hat{u}'_{s-1}$, which will be called an *elementary UA-move*. Upon setting $a'_{s-1} := a''_{s-1} a'_s \in A_{i_1}$, this move transforms the previous product into

$$auk = ak_0 \hat{u}_1 \dots \hat{u}_{s-2} k''_1 a'_{s-1} \hat{u}'_{s-1} \hat{u}'_s k_2 \dots k_q.$$

By iterating this process, it follows that, after a finite number of UK- and UA-moves, the initial product can be brought into the form

$$auk = ak_0 l'_1 b'_1 v'_1 k_2 \dots k_q,$$

where $l'_1 \in K_{i_1}$, $b'_1 \in A$ is the product of all the A -components created by the UK -moves, and $v'_1 \in U_+$ is the product of all the U_+ -components arising from the UK - and UA -moves. Since UK -moves of the second type and UA -moves simply replace the previous elements by conjugates and do not yield any “new” factors, a non-trivial A -component can only be created if a UK -move of the third type occurs and in this case, the result is contained in A_{i_1} , so the same holds for b'_1 , i.e. $b'_1 = (r'_1)^{h_{i_1}}$ for some $r'_1 \in \mathbb{R}_{>0}$. This assertion is also true if no such move has occurred and the A -component is trivial by taking $r'_1 = 1$. The U_+ -component v'_1 , however, can be very complicated and a minimal expression of v'_1 may contain many more factors than that of u .

In order to move k_0 and l'_1 further to the left, we consider the subproduct $ak_0l'_1$ and apply the following two operations, which will be called *elementary AK-moves of the first and second type*:

- a) Since $k_0 \in M \subset T$ commutes with $a \in A \subset T$, we have $ak_0 = k_0a$, so that $\Pi_K(ak_0) = k_0$, $\Pi_A(ak_0) = a$ and $\Pi_{U_+}(ak_0) = 1$.
- b) The previous move results in $ak_0l'_1 = k_0al'_1$ and we look at the subproduct al'_1 . As $l'_1 \in K_{i_1}$, Lemma 6.1.1 implies that $\Pi_A(al'_1) = ab'_1$ with $b'_1 = (r'_1)^{h_{i_1}} \in A_{i_1}$ for some $r'_1 \in [1, a(c_{i_1})^{-1}]$ which depends on a and l'_1 . More precisely, the exact Iwasawa components of al'_1 can be computed using Proposition 3.5.1. In fact, if we proceed as in the proof of Lemma 6.1.1 and write $a = a_1a_2$ with $a_1, a_2 \in A$ such that $a_1 \in A_{i_1}$ and a_2 commutes with G_{i_1} , then we can choose $a^0 \in A^0$ and $l^0 \in K^0$ which are mapped to a_1 and l'_1 by φ_{i_1} . The former is always unique and the latter is unique up to sign if φ_{i_1} is not injective. Furthermore, there exist unique $r \in \mathbb{R}_{>0}$ and $(y_1, y_2) \in B$ such that $a^0 = a(r)$ and $l^0 = k(y_1, y_2)$, so (3.5.6) yields the following:

$$\begin{aligned} \Pi_K(al'_1) &= \varphi_{i_1}(\Pi_{K^0}(a(r)k(y_1, y_2))) = \varphi_{i_1}(k(\kappa_A(r, y_1, y_2))) \in K_{i_1} \\ \Pi_A(al'_1) &= \varphi_{i_1}(\Pi_{A^0}(a(r)k(y_1, y_2)))a_2 = \varphi_{i_1}(a(\xi_A(r, y_1, y_2)))a_2 \\ &= \xi_A(r, y_1, y_2)^{h_{i_1}}a_2 = a \cdot (r^{-1}\xi_A(r, y_1, y_2))^{h_{i_1}} \in aA_{i_1} \\ \Pi_{U_+}(al'_1) &= \varphi_{i_1}(\Pi_{U_+^0}(a(r)k(y_1, y_2))) = \varphi_{i_1}(u(\eta_A(r, y_1, y_2))) \\ &= x_{i_1}(\eta_A(r, y_1, y_2)) \in U_{i_1} \end{aligned} \tag{6.1.4}$$

As in the case of a UK -move of the third type, the components do not depend on the choice of l^0 if φ_{i_1} is not injective.

After these two AK -moves and a UA -move that brings b'_1 to the left of $v''_1 := \Pi_{U_+}(al'_1)$, we arrive at the product

$$auk = k_0l_1ab_1v_1k_2 \dots k_q,$$

where $l_1 := \Pi_K(al'_1) \in K_{i_1}$, $b_1 := b'_1b''_1 = r_1^{h_{i_1}} \in A_{i_1}$ for $r_1 := r'_1r''_1 \in \mathbb{R}_{>0}$, $v_1 := (b'^{-1}_1v''_1b'_1)v'_1 \in U_+$. If $q = 1$, the result is now in Iwasawa form. If $q > 1$, we iterate the procedure by choosing a minimal expression of v_1 and applying a finite number of UK -moves of the second and third type to the subproduct v_1k_2 and then some UA -moves to order the appearing factors, which leads to an expression of the form

$$auk = k_0l_1ab_1l'_2b'_2v'_2k_3 \dots k_q,$$

where $l'_2 \in K_{i_2}$, $b'_2 \in A_{i_2}$ and $v'_2 \in U_+$ are the Iwasawa components of v_1k_2 . By an AK -move of the second type applied to the subproduct $ab_1l'_2$ and a UA -move that brings b'_2 to the left of $v''_2 := \Pi_{U_+}(ab_1l'_2)$, this product becomes

$$auk = k_0l_1l_2ab_1b_2v_2k_3 \dots k_q,$$

where $l_2 := \Pi_K(ab_1l'_2) \in K_{i_2}$, $b_2 := b'_2(ab_1)^{-1}\Pi_A(ab_1l'_2) = r_2^{h_{i_2}} \in A_{i_2}$ for some $r_2 \in \mathbb{R}_{>0}$ and $v_2 := (b'^{-1}_2v''_2b'_2)v'_2 \in U_+$. Iterating this process q times leads to the desired Iwasawa decomposition

$$auk = k_0l_1 \dots l_qab_1 \dots b_qv_q,$$

where $l_p \in K_{i_p}$ and $b_p = r_p^{h_{i_p}} \in A_{i_p}$ with $r_p \in \mathbb{R}_{>0}$ for all $p \in \{1, \dots, q\}$. However, describing v_q without further assumptions on k , a and u is hopeless because the U_+ -component is very difficult

to control as soon as an elementary UK -move of the second type has occurred in the process, which is almost always the case. In fact, since no two consecutive factors of k (resp. u) lie in the same K^0 -image (resp. root group), the only three configurations that do not involve a UK -move of the second type are the cases $q = 0$ or $s = 0$ (i.e. $k \in M$ or $u = 1$) or $q = s = 1$ with $\beta_1 = \alpha_{i_1}$. In particular, if $k \notin M$ and u is not contained in a simple root group, then UK -moves of the second type are always necessary.

Returning to the situation and notation before this algorithm, the three steps to compute the Iwasawa components of a product $g_1 g_2 \in G$ involve the following operations:

1. Bring the subproduct $u_1 k_2$ into Iwasawa form $u_1 k_2 = k' a' u'$ by finding minimal expressions of u_1 and k_2 and applying a finite number of elementary moves. The resulting product is $g_1 g_2 = k_1 a_1 k' a' u' a_2 u_2$.
2. Apply an AK -move to the subproduct $a_1 k'$ to write it in Iwasawa form as $a_1 k' = k'' a'' u''$, which leads to $g_1 g_2 = k_1 k'' a'' u'' a' u' a_2 u_2$.
3. Use a UA -move to bring $u'' a' u' a_2$ into Iwasawa form $u'' a' u' a_2 = a''' u'''$, where $a''' = a' a_2$ and $u''' = a_2^{-1}((a'^{-1} u'' a') u') a_2$. The final result $g_1 g_2 = k_1 k'' a'' a''' u''' u_2$ is in Iwasawa form upon combining the first, middle and last two factors, respectively.

Finally, it is worth noting that the Iwasawa algorithm is “continuous” in the following sense: If we choose a basis $\{v_1, \dots, v_d\}$ of \mathfrak{a} and minimal expressions of $k_1, k_2 \in K$ and $u_1, u_2 \in U_+$, we can parametrize the Iwasawa coordinates of the factors as follows:

$$\begin{aligned} k_1 &= l_0 \varphi_{i_1}(y_1^{(1)}, y_2^{(1)}) \dots \varphi_{i_p}(y_1^{(p)}, y_2^{(p)}) & k_2 &= m_0 \varphi_{j_1}(z_1^{(1)}, z_2^{(1)}) \dots \varphi_{j_q}(z_1^{(q)}, z_2^{(q)}) \\ a_1 &= \prod_{j=1}^d p_j^{v_j} & a_2 &= \prod_{j=1}^d q_j^{v_j} \\ u_1 &= x_{\beta_1}(y_1) \dots x_{\beta_s}(y_s) & u_2 &= x_{\gamma_1}(z_1) \dots x_{\gamma_t}(z_t) \end{aligned}$$

Here, it is understood that $l_0, m_0 \in M$ and $(y_1^{(j)}, y_2^{(j)}), (z_1^{(j)}, z_2^{(j)}) \in B$, $p_j, q_j \in \mathbb{R}_{>0}$ and $y_j, z_j \in \mathbb{K}$ whenever j is a natural number for which the respective elements are defined. By inspecting the Iwasawa algorithm, we conclude that the Iwasawa coordinates of $g_1 g_2$ have the following shape:

$$\begin{aligned} \Pi_K(g_1 g_2) &= \Pi_K(g_1) m_0 \varphi_{j_1}(\hat{z}_1^{(1)}, \hat{z}_2^{(1)}) \dots \varphi_{j_q}(\hat{z}_1^{(q)}, \hat{z}_2^{(q)}) \\ \Pi_A(g_1 g_2) &= \Pi_A(g_1) r_1^{h_{j_1}} \dots r_q^{h_{j_q}} \Pi_A(g_2) = \prod_{j=1}^d t_j^{v_j} \\ \Pi_{U_+}(g_1 g_2) &= x_{\delta_1}(\hat{y}_1) \dots x_{\delta_u}(\hat{y}_u) \Pi_{U_+}(g_2) \end{aligned} \tag{6.1.5}$$

Moreover, the values of $(\hat{z}_1^{(j)}, \hat{z}_2^{(j)}) \in B$, $r_j, t_j \in \mathbb{R}_{>0}$ and $\hat{y}_j \in \mathbb{K}$ depend continuously on all input parameters. To justify this, it suffices to observe that the output parameters of every elementary move are continuous with respect to the input. This is obvious for an AK -move of the first type. For an AK -move of the second type or a UK -move of the third type, it holds in view of (6.1.3) and (6.1.4) and continuity of the functions $\kappa_A, \xi_A, \eta_A, \kappa_U, \xi_U, \eta_U$. For a UK -move of the second type, it follows from the commutator relations (3.1.5). Finally, a UK -move of the first type or a UA -move acts on every root group by multiplication with a scalar that depends continuously on the conjugating element in M_0 or A , so the assertion is true in these cases as well. In particular, we arrive at the following useful consequence:

Corollary 6.1.3. *The map $\mathfrak{a} \rightarrow \mathfrak{a}$, $h \mapsto \log(\Pi_A(\exp(h)k))$, is continuous for every $k \in K$. \square*

Remark 6.1.4. It might appear that our proof of Corollary 6.1.3 could be simplified considerably by endowing G with an appropriate group topology (e.g. the Kac-Peterson topology) and showing that the three maps involved are continuous. For the exponential map and the logarithm, this is probably simple and in a less general setting, it is proved in [FHHK20, Proposition 3.11]. However, in order to show that the Iwasawa projection onto the A -component is continuous, it would be

natural to establish the stronger statement that the Iwasawa decomposition is a homeomorphism. Although the author is not aware whether this assertion is true without any assumptions on the underlying Kac-Moody root datum, special cases have been proved in [FHHK20, Theorem 3.31] and [KV24, Theorem 3.20] by similar methods that rely on non-trivial topological considerations.

We will apply the Iwasawa algorithm in various ways to obtain more detailed information on the Iwasawa projections of G . Before, we record a very simple property.

Lemma 6.1.5. *The map $K \rightarrow K$, $k \mapsto \Pi_K(gk)$, is injective for every $g \in G$.*

Proof. Upon replacing $G^0 = K^0 A^0 U_+^0$ by $G = K A U_+$, the argument for injectivity in the proof of Corollary 3.5.5 applies verbatim to an arbitrary Kac-Moody group. \square

Our first application of the Iwasawa algorithm will be to show that the map in the statement of Lemma 6.1.5 is also surjective. To achieve this, we consider the cases $g = a \in A$ and $g = u \in U_+$ separately and reduce them to the SL_2 -case that we have already dealt with in Corollary 3.5.5. First, we establish the following “local” surjectivity result. Afterwards, we will derive the desired “global” analogue.

Lemma 6.1.6. *Let $a \in A$ and $u \in U_+$ be arbitrary. The maps $k \mapsto \Pi_K(ak)$ and $k \mapsto \Pi_K(uk)$ define bijections $K_i \rightarrow K_i$ for every $i \in I$.*

Proof. It is a consequence of the Iwasawa algorithm that $k \mapsto \Pi_K(ak)$ and $k \mapsto \Pi_K(uk)$ restrict to maps $K_i \rightarrow K_i$ (cf. the first line of (6.1.5)) which are injective by Lemma 6.1.5, so it only remains to prove surjectivity. First, we consider the A -case. Here, Lemma 3.4.16 implies that there exist $a_1, a_2 \in A$ such that $a = a_1 a_2$, $a_1 = r^{h_i} \in A_i$ for some $r \in \mathbb{R}_{>0}$ and a_2 commutes with $G_i = \varphi_i(G^0)$. If a^0 denotes the unique element of A^0 with $\varphi_i(a^0) = a_1$, then the following diagram commutes by (3.4.7):

$$\begin{array}{ccc} K^0 & \xrightarrow{k^0 \mapsto \Pi_{K^0}(a^0 k^0)} & K^0 \\ \varphi_i \downarrow & & \downarrow \varphi_i \\ K_i & \xrightarrow{k \mapsto \Pi_K(a_1 k)} & K_i \end{array}$$

By Corollary 3.5.5, the horizontal map on top of this diagram is bijective while $\varphi_i : G^0 \rightarrow G_i$ is either bijective or its kernel is $\{\mathbb{I}_2, -\mathbb{I}_2\}$. Since $-\mathbb{I}_2$ is contained in K^0 and central in G^0 , we have $\Pi_{K^0}(a^0(-k^0)) = -\Pi_{K^0}(a^0 k^0)$ for all $k^0 \in K^0$. In either case, it follows that the horizontal map on the bottom is also bijective and the same is then true for $k \mapsto \Pi_K(a_1 k) a_2 = \Pi_K(ak)$.

Next, we investigate the U_+ -case. If $u = 1$, the statement is trivial. Otherwise, we choose a minimal expression $u = u_1 \dots u_s$ as in (6.1.2) and argue by induction on s . If $s = 1$ and $\beta_1 = \alpha_i$, then $u \in U_i \subset G_i$ and the same argument as in the A -case shows that $k \mapsto \Pi_K(uk)$ defines a bijection $K_i \rightarrow K_i$. If $\beta_1 \neq \alpha_i$, then u lies in the unipotent radical of the standard parabolic subgroup $P_{\{i\}}$ of type $\{i\}$ while $K_i \subset G_{\{i\}}$ is contained in its Levi factor. Thus, for every $k \in K_i$, we have $k^{-1}uk \in U_+$ and $\Pi_K(uk) = \Pi_K(k(k^{-1}uk)) = k$, so $k \mapsto \Pi_K(uk)$ restricts to the identity map on K_i which is trivially bijective.

Suppose now that $s > 1$, set $v := u_2 \dots u_s \in U_+$ and let $k' \in K_i$ be arbitrary. By the induction hypothesis, there exist $k_1, k_2 \in K_i$ such that $\Pi_K(u_1 k_1) = k'$ and $\Pi_K(v k_2) = k_1$, which implies

$$\Pi_K(uk_2) = \Pi_K(u_1 v k_2) = \Pi_K(u_1 \Pi_K(v k_2)) = \Pi_K(u_1 k_1) = k'.$$

Here, the second equality is justified by the fact that A normalizes U_+ . Hence, the required map is surjective and the proof is complete. \square

Proposition 6.1.7. *The map $k \mapsto \Pi_K(gk)$ defines a bijection $K \rightarrow K$ for every $g \in G$.*

Proof. By Lemma 6.1.5, it remains to prove surjectivity. To this end, let $k' \in K$ be arbitrary and choose a minimal expression $k' = k'_0 k'_1 \dots k'_q$ as in (6.1.1). We are going to show by induction on q that there exists an element $k \in K$ such that $\Pi_K(gk) = k'$.

First, we consider the special case $g = a \in A$. If $q = 0$, then $k' = k'_0 \in M \subset T$ commutes with $a \in A \subset T$, so we may simply take $k := k'$. Now assume that $q > 0$ and set $l' := k'_0 k'_1 \dots k'_{q-1}$. By the induction hypothesis, there exists $l \in K$ such that $\Pi_K(al) = l'$, i.e. $al = l'a'u'$ for some $a' \in A$ and $u' \in U_+$. As $k'_q \in K_{i_q}$, it follows from Lemma 6.1.6 that there exist $l_q, l'_q \in K_{i_q}$ with $\Pi_K(a'l'_q) = k'_q$ and $\Pi_K(u'l_q) = l'_q$. The element $k := ll_q \in K$ has the required property:

$$\Pi_K(ak) = \Pi_K(all_q) = l'\Pi_K(a'u'l_q) = l'\Pi_K(a'\Pi_K(u'l_q)) = l'\Pi_K(a'l'_q) = l'k'_q = k'$$

The third equality holds since A normalizes U_+ , so the proof is finished if $g = a \in A$. Likewise, if $g = u \in U_+$, the case $q = 0$ follows from the fact that $M \subset T$ normalizes U_+ and the induction step works verbatim upon replacing a by u . Finally, let $g \in G$ be arbitrary and write it in Iwasawa form as $g = k_0 a_0 u_0$. By the previous two cases, there exist $k, k'' \in K$ such that $\Pi_K(u_0 k) = k''$ and $\Pi_K(a_0 k'') = k_0^{-1} k'$, which leads to the desired result:

$$\Pi_K(gk) = k_0 \Pi_K(a_0 u_0 k) = k_0 \Pi_K(a_0 \Pi_K(u_0 k)) = k_0 \Pi_K(a_0 k'') = k' \quad \square$$

On the one hand, Proposition 6.1.7 implies that $\Pi_K(gK) = K$ holds for every $g \in G$. For the A -projection, on the other hand, we already know from Theorem 5.1.1 that $\Pi_A(gK)$ may well be a proper subset of A . Our next goal is to decide whether certain naturally occurring subsets of G are in fact sub(semi-)groups. To this end, it is useful to establish two more results that similarly describe the images of subsets of G under an Iwasawa projection. We require a bit of notation: Let $J \subset I$ be arbitrary and set $\mathfrak{a}'_J := \sum_{j \in J} \mathbb{R} h_j \subset \mathfrak{a}$. This definition reduces to that in (4.3.3) if \mathcal{D} is cofree, which is not assumed at the moment, and $\mathfrak{a}'_I = \mathfrak{a}'$ has been introduced in (2.6.2). We further define $A'_J := \exp(\mathfrak{a}'_J)$, which coincides with the subgroup of A generated by the set $\{r^{h_j} \mid j \in J, r \in \mathbb{R}_{>0}\}$, and write $A' := A'_I$.

Proposition 6.1.8. $\Pi_A(U_+ K) = A'$

Proof. It follows from the Iwasawa algorithm that $\Pi_A(U_+ K) \subset A'$ (cf. the second line of (6.1.5)), so it remains to show the converse inclusion. Given any subset $J \subset I$, we consider the Iwasawa decomposition $G_J = K_J A U_{J,+}$ of the Levi factor G_J of the standard parabolic subgroup $P_J \subset G$ of type J . We are going to prove by induction on $|J|$ that $A'_J \subset \Pi_A(U_{J,+} K_J)$. Once this is established, taking $J = I$ will conclude the proof.

If $|J| = 1$, then $J = \{i\}$ for some $i \in I$ and the homomorphism $\varphi_i : G^0 \rightarrow G$ takes values in $G_i \subset G_{\{i\}}$. As φ_i is compatible with the Iwasawa decompositions $G^0 = K^0 A^0 U_+^0$ and $G = K A U_+$, Corollary 3.5.7 implies that

$$\Pi_A(U_{\{i\},+} K_{\{i\}}) \supset \Pi_A(U_i K_i) = \Pi_A(\varphi_i(U_+^0 K^0)) = \varphi_i(\Pi_{A^0}(U_+^0 K^0)) = \varphi_i(A^0) = A_i = A'_{\{i\}}.$$

Suppose now that $|J| > 1$, choose any $j \in J$ and set $J' := J \setminus \{j\}$. Let $a \in A'_J$ be arbitrary, then there exist $b \in A'_{J'}$ and $r \in \mathbb{R}_{>0}$ with $a = br^{h_j}$. By the induction hypothesis, we find $u_1 \in U_{J',+}$ and $k_1 \in K_{J'}$ such that $\Pi_A(u_1 k_1) = b$. Hence, the Iwasawa decomposition of $u_1 k_1 \in G_{J'}$ is

$$u_1 k_1 = l_1 b v_1 \quad \text{for some } l_1 \in K_{J'} \text{ and } v_1 \in U_{J',+}.$$

Write $q_1 := b(c_j) \in \mathbb{R}_{>0}$ and $q := \max\{1, q_1^{-1}\}$. By Lemma 6.1.1, the set $\Pi_A(bK_j) \subset A'_J$ is equal to $\{bp^{h_j} \mid p \in [1, q_1^{-1}]\}$. Moreover, by Lemma 3.5.6, there exists some $\gamma' \in [0, 2\pi)$ such that the image of the map $\mathbb{K} \rightarrow \mathbb{R}_{>0}$, $z \mapsto \xi_U(z, \cos(\gamma'), \sin(\gamma'))$, contains the interval $[rq^{-1}, \infty) \subset \mathbb{R}_{>0}$ and Proposition 3.5.3 then implies that $\{a(s) \mid s \in [rq^{-1}, \infty)\} \subset \Pi_{A^0}(U_+^0 R_{\gamma'})$. We now define $k_{\gamma'} := \varphi_j(R_{\gamma'}) \in K_j$ and apply φ_j to this inclusion, which shows that

$$\{s^{h_j} \mid s \in [rq^{-1}, \infty)\} \subset \Pi_A(U_j k_{\gamma'}).$$

Furthermore, using Lemma 6.1.6 in the real case, we find $\gamma \in [0, 2\pi)$ such that $k_\gamma := \varphi_j(R_\gamma) \in K_j$ satisfies $\Pi_K(bk_\gamma) = k_{\gamma'}$. More precisely, the Iwasawa decomposition of $bk_\gamma \in G_{\{j\}}$ has the form

$$bk_\gamma = k_{\gamma'} b q_2^{h_j} v \quad \text{for some } q_2 \in [1, q_1^{-1}] \text{ and } v \in U_j.$$

As $q_2 \leq q$ implies that $rq_2^{-1} \in [rq^{-1}, \infty)$, there is some $z \in \mathbb{K}$ with $\xi_U(z, \cos(\gamma'), \sin(\gamma')) = rq_2^{-1}$. Thus, if we set $u_2 := x_j(z) = \varphi_j(u(z)) \in U_j$, then

$$u_2 k_{\gamma'} = l_2 (rq_2^{-1})^{h_j} v_2 \quad \text{for some } l_2 \in K_j \text{ and } v_2 \in U_j.$$

All the elements introduced so far are contained in G_J and satisfy the following relations:

- $u_2 \in U_j \subset U_{J',+}^\perp$ lies in the unipotent radical of the parabolic subgroup $P_{J'}$ of type J' , whereas $l_1 \in K_{J'} \subset G_{J'}$ is contained in its Levi factor. Hence, it holds that

$$u'_2 := l_1 u_2 l_1^{-1} \in U_{J',+}^\perp \cap G_J \subset U_{J,+}.$$

- $v_1 \in U_{J',+} \subset U_{\{j\},+}^\perp$ lies in the unipotent radical of the parabolic subgroup $P_{\{j\}}$ of type $\{j\}$, whereas $k_\gamma^{-1} \in K_j \subset K_{\{j\}} \subset G_{\{j\}}$ is contained in its Levi factor. It follows that

$$v'_1 := k_\gamma^{-1} v_1 k_\gamma \in U_{\{j\},+}^\perp \cap G_J \subset U_{J,+}.$$

- $v_2 \in U_j$, whereas $(bq_2^{h_j})^{-1} \in A'_J \subset A$ normalizes every root group, which implies that

$$v'_2 := (bq_2^{h_j})^{-1} v_2 (bq_2^{h_j}) \in U_j \subset U_{J,+}.$$

We now put all the pieces together. Upon setting $u := u'_2 u_1 \in U_{J,+}$ and $k := k_1 k_\gamma \in K_J$, we may perform the following computation, where the brackets merely serve the purpose of illustrating the order in which we apply the previous considerations:

$$\begin{aligned} uk &= u'_2 (u_1 k_1) k_\gamma = (u'_2 l_1) b (v_1 k_\gamma) = l_1 u_2 (b k_\gamma) v'_1 \\ &= l_1 (u_2 k_{\gamma'}) b q_2^{h_j} v v'_1 = l_1 l_2 (rq_2^{-1})^{h_j} v_2 (b q_2^{h_j}) v v'_1 \\ &= (l_1 l_2) (b r^{h_j}) (v'_2 v v'_1) = (l_1 l_2) a (v'_2 v v'_1) \in K_J a U_{J,+} \end{aligned}$$

We conclude that $a = \Pi_A(uk) \in \Pi_A(U_{J,+} K_J)$. Since $a \in A'_J$ was arbitrary, the induction step is complete and the proof is finished. \square

Proposition 6.1.9. $\Pi_A(A'K) = A'$

Proof. On the one hand, we have $1 \in K$ and thus $A' \subset \Pi_A(A'K)$. On the other hand, it follows from the Iwasawa algorithm that $\Pi_A(A'K) \subset A'$ (cf. the second line of (6.1.5)). \square

We now combine the previous two propositions with Theorem 5.1.1 to prove the anticipated result on sub(semi-)groups of G . As a motivation, recall that the positive and negative Tits cones and their topological interiors are convex cones in \mathfrak{a} , so their exponential images $A_\pm := \exp(X_\pm)$ and $A_\pm^\circ := \exp(X_\pm^\circ)$ are subsemigroups of A , but unless the GCM \mathbb{A} is spherical, they are proper subsets of A and not subgroups in general. Note that if \mathcal{D} is not admissible, then $X_\pm^\circ = A_\pm^\circ = \emptyset$.

Theorem 6.1.10.

- (i) $KA'U_+$ is a subgroup of G .
- (ii) KA_+U_+ is a subsemigroup of G if and only if \mathbb{A} is spherical. In this case, it coincides with the entire group G .
- (iii) If \mathcal{D} is admissible and \mathbb{A} is symmetrizable, then $KA_+^\circ U_+$ is a subsemigroup of G if and only if every indecomposable component of \mathbb{A} is of finite or affine type. If \mathbb{A} is spherical, it coincides with G . Otherwise, it is not a subgroup of G .

Proof. Throughout this proof, we use Notation 4.1.1 to denote the indecomposable components of \mathbb{A} and all related objects.

- (i) It is clear that $1 \in KA'U_+$. Let $g_1 = k_1 a_1 u_1$ and $g_2 = k_2 a_2 u_2$ be two elements of $KA'U_+$ in Iwasawa form. Proposition 6.1.8 and 6.1.9 show that $u_1 k_2 = l_1 b_1 v_1$ and $a_1 l_1 = l_2 b_2 v_2$ for some $l_1, l_2 \in K$, $b_1, b_2 \in A'$ and $v_1, v_2 \in U_+$. Since A' normalizes U_+ , it follows that

$$\begin{aligned} g_1 g_2 &= k_1 a_1 (u_1 k_2) a_2 u_2 = k_1 (a_1 l_1) b_1 v_1 a_2 u_2 = k_1 l_2 b_2 v_2 b_1 v_1 a_2 u_2 \\ &= (k_1 l_2) (b_2 b_1 a_2) (a_2^{-1} b_1^{-1} v_2 b_1 v_1 a_2 u_2) \in KA'U_+. \end{aligned} \tag{6.1.6}$$

Hence, $KA'U_+$ is closed under multiplication. Likewise, let $g = kau$ be any element of $KA'U_+$ in Iwasawa form, then $g^{-1} = u^{-1}a^{-1}k^{-1}$. Proposition 6.1.8 and 6.1.9 imply that $a^{-1}k^{-1} = lbv$ and $u^{-1}l = l'b'v'$ for some $l, l' \in K$, $b, b' \in A'$ and $v, v' \in U_+$, which leads to

$$g^{-1} = u^{-1}(a^{-1}k^{-1}) = (u^{-1}l)bv = l'b'v'bv = l'b'b(b^{-1}v'bv) \in KA'U_+.$$

Thus, $KA'U_+$ is closed under inversion as well and therefore a subgroup of G .

- (ii) If \mathbb{A} is spherical, then $X_+ = \mathfrak{a}$ and $A_+ = A$, so the statement is trivially true. Otherwise, we have $U_+K \subset (KA_+U_+)^2$ because $1 \in A_+$ and $\Pi_A(U_+K) = A'$ by Proposition 6.1.8. However, we claim that \mathfrak{a}' is not contained in X_+ if \mathbb{A} is not spherical. This will show that U_+K cannot be contained in KA_+U_+ , so that KA_+U_+ is not closed under multiplication. The following argument is a minor reformulation of the proof of Proposition 4.4.1, where we worked with free and cofree Kac-Moody root data, and we use the same notation.

If \mathbb{A} is not spherical, we may w.l.o.g. assume that the first indecomposable component \mathbb{A}_1 is not of finite type. Let $i \in I_1$ be arbitrary. If \mathbb{A}_1 is of affine type and $\delta \in \Phi_+^{\text{im}}(\mathbb{A}_1)$ is its positive imaginary root of minimal height, the same argument as in that proof shows that the \mathcal{D}_1 -Tits cone is the set of all $h \in \mathfrak{a}$ that satisfy $\langle c_\delta, h \rangle > 0$ or $\langle c_j, h \rangle = 0$ for all $j \in I_1$. Because $\langle c_\delta, h_i \rangle = \langle \delta, \alpha_i^\vee \rangle = 0$ but $\langle c_i, h_i \rangle = 2$, it follows that $h_i \in \mathfrak{a}'$ is not contained in $X_{1,+} \supset X_+$. Similarly, if \mathbb{A}_1 is of indefinite type, there exists a positive imaginary root $\gamma \in \Phi_+^{\text{im}}(\mathbb{A}_1)$ with $\langle \gamma, \alpha_j^\vee \rangle < 0$ for all $j \in I_1$. Since c_γ is non-negative on the \mathcal{D}_1 -Tits cone but $\langle c_\gamma, h_i \rangle = \langle \gamma, \alpha_i^\vee \rangle < 0$, we again conclude that $h_i \in \mathfrak{a}'$ does not lie in $X_{1,+} \supset X_+$.

- (iii) Suppose first that some indecomposable component of \mathbb{A} is of indefinite type, then we may w.l.o.g. assume that \mathbb{A}_1 has this property and that $I_1 = \{1, \dots, p\}$ for some $p \in \{2, \dots, n\}$. Since \mathbb{A}_1^T is also of indefinite type, there exists some $x = (x_1, \dots, x_p)^T \in \mathbb{R}^p$ such that $x > 0$ and $\mathbb{A}_1^T x < 0$. Hence, upon replacing x by a sufficiently large scalar multiple, we may ensure that the vector $h' := \sum_{j \in I_1} x_j h_j \in \mathfrak{a}'$ satisfies $\langle c_i, h' \rangle \leq -1$ for all $i \in I_1$. If we set $a' := \exp(h') \in A'$, then Proposition 6.1.8 implies that there exist $u \in U_+$ and $k \in K$ such that $\Pi_A(uk) = a'$, i.e. $uk = k'a'u'$ for some $k' \in K$ and $u' \in U_+$.

By Corollary 6.1.3, the map $f : \mathfrak{a} \rightarrow \mathfrak{a}$, $h \mapsto \log(\Pi_A(\exp(h)k'))$, is continuous with respect to the standard Euclidean topology on $\mathfrak{a} \cong \mathbb{R}^d$ and we claim that it preserves $X_+^\circ \neq \emptyset$. In fact, if $h \in X_+^\circ$, then h is virtually singular and $\Xi_h \subset X_+^\circ$. Since \mathbb{A} is assumed to be symmetrizable, we may apply Theorem 5.1.1 to conclude that $f(h) \in \Xi_h \subset X_+^\circ$, so the claim is proved. As the origin lies in the closure of X_+° and $f(0) = 0$, there exists a point $h_1 \in X_+^\circ$ that is sufficiently close to 0 such that $h_2 := f(h_1) \in X_+^\circ$ satisfies $\langle c_i, h_2 \rangle < \frac{1}{2}$ for all $i \in I$. We now define $a_1 := \exp(h_1)$ and $a_2 := \exp(h_2)$, so that $a_1, a_2 \in A_+^\circ$ and $\Pi_A(a_1 k') = a_2$, i.e. $a_1 k' = k'' a_2 u''$ for some $k'' \in K$ and $u'' \in U_+$. If $KA_+^\circ U_+$ was a subsemigroup, then the product $a_1 u k a_2$ would be contained in that set. However, we have

$$a_1(uk)a_2 = (a_1 k')a'u'a_2 = k''a_2 u''a'u'a_2 = k''(a_2^2 a')(a_2^{-1}a'^{-1}u''a'u'a_2),$$

where the product in the last parentheses lies in U_+ which is normalized by A . It follows that $a := \Pi_A(a_1 u k a_2) = a_2^2 a'$, where $h := \log(a) = 2h_2 + h'$ satisfies $\langle c_i, h \rangle < 0$ for all $i \in I_1$, so h lies in $-C_1$ which is disjoint from $X_{1,+} \supset X_+ \supset X_+^\circ$ by Corollary 2.6.7. Thus, $a \notin A_+^\circ$ and $KA_+^\circ U_+$ is not a subsemigroup in this case.

Conversely, suppose that every indecomposable component of \mathbb{A} is of finite or affine type. We may w.l.o.g. assume that the components are ordered such that $\mathbb{A}_1, \dots, \mathbb{A}_s$ are of affine type and $\mathbb{A}_{s+1}, \dots, \mathbb{A}_m$ are of finite type for some $s \in \{0, \dots, m\}$. As in the proof of (ii), given any $r \in \{1, \dots, s\}$, let $\delta_r \in \Phi_+^{\text{im}}(\mathbb{A}_r)$ be the positive imaginary root of \mathbb{A}_r of minimal height. The \mathcal{D}_r -Tits cone $X_{r,+}$ is the set of all $h \in \mathfrak{a}$ with $\langle c_{\delta_r}, h \rangle > 0$ or $\langle c_i, h \rangle = 0$ for every $i \in I_r$ and its topological interior $X_{r,+}^\circ$ only contains those points for which the former condition holds. Since $\mathcal{D} = \bigcup_{r=1}^m \mathcal{D}_r$ and $X_{r,+}^\circ = X_{r,+} = \mathfrak{a}$ for $r \in \{s+1, \dots, m\}$, it follows that

$$X_+^\circ = \bigcap_{r=1}^m X_{r,+}^\circ = \bigcap_{r=1}^s \{h \in \mathfrak{a} : \langle c_{\delta_r}, h \rangle > 0\}.$$

We now repeat the computation in (6.1.6) with $g_1, g_2 \in KA_+^\circ U_+$, so that $\Pi_A(g_1 g_2) = b_2 b_1 a_2$, where this time, we have $a_2 \in A_+^\circ$ by assumption, $b_2 = \Pi_A(a_1 l_1) \in \exp(\Xi_{\log(a_1)}) \subset A_+^\circ$ by Theorem 5.1.1 and $b_1 = \Pi_A(u_1 k_2) \in A'$ by Proposition 6.1.8. In addition, given any $r \in \{1, \dots, s\}$, we have $\langle c_{\delta_r}, h_i \rangle = \langle \delta_r, \alpha_i^\vee \rangle = 0$ for every $i \in I$, so the functional $c_{\delta_r} \in \mathfrak{a}^*$ vanishes identically on the subspace $\mathfrak{a}' \subset \mathfrak{a}$. Hence, it is zero on $\log(b_1)$ but strictly positive on $\log(a_2)$ and $\log(b_2)$, which implies that $\log(b_2 b_1 a_2) \in X_+^\circ$ and thus $g_1 g_2 \in KA_+^\circ U_+$. Consequently, $KA_+^\circ U_+$ is a subsemigroup in this case. If \mathbb{A} is spherical, it coincides with the entire group G . If \mathbb{A} contains at least one component of affine type and $h \in X_+^\circ$, then $-h \notin X_+^\circ$ by Corollary 2.6.7, which shows that $KA_+^\circ U_+$ is not closed under inversion. \square

While the most interesting assertion of Theorem 6.1.10 is surely the third, which is also the only one whose proof makes use of our convexity theorem, the first implies that if A' is a proper subgroup of A , then $KA'U_+$ is a proper subgroup of G containing K , so in this case, K is not maximal as an abstract subgroup of G . However, it is a surprising result of Lie theory that if G is a connected reductive Lie group and K the fixed-point set of a Cartan involution of G , then the subsemigroups of G containing K can be completely classified. In particular, if G is simple, then K is a maximal subgroup of G (cf. [Nee94, Proposition 1.17 and Corollary 1.18]). The previous result suggests that our convexity theorem might be useful to study the same question in the Kac-Moody setting: If $G = G_{\mathcal{D}}(\mathbb{K})$ is a real or complex Kac-Moody group, is it possible to classify the subsemigroups of G containing its compact form K ? In particular, is K a maximal subgroup of G ? For this statement to be true, it is certainly necessary that the underlying GCM \mathbb{A} is indecomposable, that $A' = A$ and that the center $Z = \{t \in T : t(c_i) = 1 \ \forall i \in I\}$ of G is already contained in $K \cap T = M$. As we will explain at the end of Section 6.3, this question is closely related to the long-standing open problem of simplicity of real and complex Kac-Moody groups. It appears that Kostant's Non-Linear Convexity Theorem and the Iwasawa algorithm are promising tools to investigate this in more detail. At the time of writing, the author has obtained some partial results on subsemigroups of G containing K , but the statements are rather weak and technical and not worthy of reproduction. Pursuing this problem further is without doubt an exciting and important topic for future research.

6.2 Decompositions of Kac-Moody groups

Another application of Kostant's Non-Linear Convexity Theorem for Kac-Moody groups concerns the question of whether certain decompositions of semisimple Lie groups continue to hold in the Kac-Moody setting. As a motivation, we recall some famous facts from Lie theory.

Theorem 6.2.1. *Let G be a connected semisimple Lie group and let $G = KA U_+$ be an Iwasawa decomposition of G , where $K \subset G$ is the fixed-point set of a Cartan involution $\Theta : G \rightarrow G$. Define the set $\Omega := \{g\Theta(g^{-1}) \mid g \in G\} \subset G$. Then G can be decomposed in the following ways:*

- (i) (“Cartan decomposition”) $G = K\Omega$
- (ii) (“ KAK -decomposition”) $G = KAK$
- (iii) (“horospherical decomposition”) $G = KU_+ K$

Proof. All assertions are well-known results of classical Lie theory. In fact, the existence of the Cartan decomposition $G = K\Omega$ is usually proved before the Iwasawa decomposition $G = KA U_+$. In [Kna02, Theorem 6.31], it is shown that $G = K \exp(\mathfrak{p})$, where \mathfrak{p} is the (-1) -eigenspace of the differential of Θ , but it is not difficult to see that $\Omega = \exp(\mathfrak{p})$. The KAK -decomposition follows from this and $\mathfrak{p} = \bigcup_{k \in K} \text{Ad}(k)\mathfrak{a}$, where \mathfrak{a} is the maximal abelian subspace of \mathfrak{p} with $A = \exp(\mathfrak{a})$ (cf. [Kna02, Theorem 6.51]), and the horospherical decomposition is a consequence of Kostant's Non-Linear Convexity Theorem for semisimple Lie groups (cf. [Kos73, Theorem 5.1]). \square

Remark 6.2.2. The first two decompositions in Theorem 6.2.1 apply more generally to suitable notions of “reductive” Lie groups. For example, if \mathcal{G} is a connected reductive algebraic \mathbb{R} -group, recall from Section 5.5.2 that $G := \mathcal{G}(\mathbb{K})$ can be identified with a closed subgroup of $\text{GL}_N(\mathbb{K})$ for some $N \in \mathbb{N}$ that is stable under the map $\Theta : G \rightarrow G, g \mapsto (g^{-1})^*$, but it need not be connected

or semisimple as a Lie group. In this situation, the Cartan (or “polar”) decomposition $G = K\Omega$ is established in [PR94, Proposition 3.10 and 3.11] or [Kna02, Proposition 1.143]. Together with the fact that $K \subset \mathrm{U}(N)$ is compact, this result implies that (G_e, K_e, θ, B) is a reductive Lie group in the sense of [Kna02, Chapter VII, Section 2], where G_e is the identity component of G , $K_e := G_e \cap K$ is a maximal compact subgroup of G_e , $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ is the map $X \mapsto -X^*$ and B is the bilinear form on \mathfrak{g} defined by $B(X, Y) := \mathrm{Re}(\mathrm{Tr}(XY))$. Hence, [Kna02, Theorem 7.39] shows that $G_e = K_e A K_e$ and if \mathcal{G} is semisimple, then $G_e = K_e U_+ K_e$ holds by [Kos73, Theorem 5.1]. Using that K intersects every connected component of G , this even implies $G = KAK$ and if \mathcal{G} is semisimple, then $G = KU_+ K$.

If $G = \mathrm{GL}_N(\mathbb{K})$, then Ω is the set of positive-definite matrices, so the Cartan decomposition $G = K\Omega$ reduces to the assertion that every invertible matrix can be written as the product of an orthogonal (or unitary) and a positive-definite matrix, which is well-known from linear algebra. The KAK -decomposition is less obvious from this perspective. The horospherical decomposition, however, is not satisfied in this example because $|\det(g)| = 1$ for all $g \in KU_+ K$. It does apply to the semisimple subgroup $\mathrm{SL}_N(\mathbb{K})$, where K is the special orthogonal (or unitary) group.

Since the Iwasawa decomposition continues to exist for real and complex Kac-Moody groups, it is natural to ask whether the same is true for the other three decompositions in Theorem 6.2.1. This question was studied by M. Horn who proved the result below in a slightly different setup (cf. [Hor17]). We return to the usual notation from the beginning of this chapter and denote by \mathcal{D} a Kac-Moody root datum of rank $d \in \mathbb{N}$ associated to a GCM $\mathbb{A} \in \mathbb{Z}^{n \times n}$, by $G = G_{\mathcal{D}}(\mathbb{K})$ the real or complex Kac-Moody group of type \mathcal{D} and by $G = KAU_+$ an Iwasawa decomposition of G , where K is the fixed-point set of the compact involution $\Theta : G \rightarrow G$.

Theorem 6.2.3. *If \mathbb{A} is not spherical, the following assertions hold:*

- (i) *G does not have a Cartan decomposition: If $\Omega := \{g\Theta(g^{-1}) \mid g \in G\}$, then $G \neq K\Omega$.*
- (ii) *G does not have a KAK -decomposition: $G \neq KAK$. However, it satisfies*

$$G = \bigcup_{N=1}^{\infty} (KAK)^N. \quad (6.2.1)$$

- (iii) *If G is generated by the root groups $(U_{\alpha})_{\alpha \in \Phi}$, then $G = (KU_+ K)^{n+1}$.*

Before turning to the proof, we remark that the assumption in the third statement is necessary. In general, G is generated by the torus T and the root groups $(U_{\alpha})_{\alpha \in \Phi}$, but in view of (KMG3), the subgroup $G' \subset G$ generated by the root groups alone only contains the subtorus $T' \subset T$ generated by the set $\{r^{h_i} \mid i \in I, r \in \mathbb{K}^*\}$. Similarly, G' only contains the subgroup $A' = \exp(\mathfrak{a}')$ of $A = \exp(\mathfrak{a})$ generated by $\{r^{h_i} \mid i \in I, r \in \mathbb{R}_{>0}\}$ which coincides with A if and only if \mathfrak{a} is spanned by $(h_i)_{i \in I}$. This is always satisfied if \mathcal{D} is coadjoint, but it cannot hold if $d > n$. In the remaining cases, the issue is more subtle and not relevant to our discussion. Since $K = MK'$, where $K' \subset G'$ is the subgroup generated by the K^0 -images $(K_i)_{i \in I}$ and $M \subset T$ normalizes G' , it follows that $(KU_+ K)^N \subset MG'$ for all $N \in \mathbb{N}$ and MG' is a proper subgroup of G if $A' \subsetneq A$. The condition that G is generated by the root groups should be thought of as an analogue of semisimplicity, which is not defined in the Kac-Moody setting.

Proof. The first two assertions are proved in [Hor17, Proposition 6.3], where the terminology is different from ours and the notion of Kac-Moody root data is not built into the definitions. By inspecting the arguments, however, it is apparent that they only require the Kac-Moody group to give rise to an RGD system and this is true for $G = G_{\mathcal{D}}(\mathbb{K})$, irrespective of the underlying root datum (cf. Remark 3.1.5), so the results also hold in our setting. The proof of (6.2.1) is contained in [Hor17, Lemma 6.4], but the author always supposes his Kac-Moody groups to be generated by the root groups. Hence, his argument only applies to our group G' , but we have $G = TG'$ and $T = MA \subset KAK$ is contained in the union on the right-hand of (6.2.1) side as well. The third statement is shown in [Hor17, Lemma 6.6], which again applies to G' , but the condition that $G = G'$ has to be generated by the root groups cannot be removed in this case because if $A' \subsetneq A$, then $A \setminus A'$ is not contained in $(KU_+ K)^N \subset MG'$ for any $N \in \mathbb{N}$. \square

Remark 6.2.4. Recall from the historical introduction that if G is a connected semisimple Lie group, the KAK -decomposition enabled B. Kostant to describe the set $\Pi_A(gK)$ for any $g \in G$ as the exponential image of the convex hull of a Weyl group orbit. In the Kac-Moody setting, this decomposition is not available, so Theorem 5.1.1 cannot be generalized in the same way.

The statement of Theorem 6.2.3 (iii) is not optimal and does not decide whether the horospherical decomposition $G = KU_+K$ might also hold for Kac-Moody groups (cf. [Hor17, Question 6.8]). In most situations, however, our convexity theorem yields a negative answer and also provides a slightly more precise result in the spherical case. The following is the first part of Theorem B:

Theorem 6.2.5. *If \mathcal{D} is admissible and $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is symmetrizable, then G has a horospherical decomposition $G = KU_+K$ if and only if \mathbb{A} is spherical and $d = n$.*

Proof. The decomposition $G = KU_+K$ is equivalent to the statement that for every $g \in G$, there exists some $k \in K$ such that $gk \in KU_+$ or, equivalently, $\Pi_A(gk) = 1$. We are going to show that, unless \mathbb{A} is spherical and $d = n$, we can choose $g = a \in A$ for which this equality is not possible. In view of Theorem 5.1.1, it suffices to prove that there is an almost virtually singular element $h \in \mathfrak{a}$ with $0 \notin \Xi_h$.

Assume first that \mathbb{A} is not spherical. Since \mathcal{D} is admissible, there exists a regular element $h \in C$. In particular, h lies in the topological interior of the Tits cone, which implies that $\Xi_h \subset X_+^\circ$. If the origin was contained in $\Xi_h \subset X_+^\circ$, its stabilizer under the W -action on \mathfrak{a} would be finite, but this is the entire group W , so $0 \in \Xi_h$ is only possible if $|W| < \infty$, contradicting the assumption that \mathbb{A} is not spherical.

Second, suppose that \mathbb{A} is spherical and $d \neq n$. Because \mathcal{D} is free and cofree as \mathbb{A} is invertible, its rank is at least n , so the assumption states that $d > n$ and the subspace $\mathfrak{a}_0 := \bigcap_{i \in I} \ker(c_i) \subset \mathfrak{a}$ has dimension $d - n > 0$. If we choose some $h \in \mathfrak{a}_0 \setminus \{0\}$, then h is a global fixed-point of W and $\Xi_h = \{h\}$ does not contain the origin, as required.

Finally, if \mathbb{A} is spherical and $d = n$, then Example 3.1.6 (i) shows that $G = \mathcal{G}(\mathbb{K})$ coincides with the group of \mathbb{K} -points of a connected semisimple algebraic \mathbb{R} -group \mathcal{G} , so it has a horospherical decomposition by Remark 6.2.2. \square

The condition that \mathcal{D} is admissible in Theorem 6.2.5 is satisfied in many situations. For example, it is guaranteed to hold if no indecomposable component of \mathbb{A} is of affine type (cf. Lemma 2.6.2). However, the affine case is an important part of Kac-Moody theory and the result does not apply to some noteworthy examples. If $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is an indecomposable GCM of affine type, there exist Kac-Moody root data associated to \mathbb{A} that are not admissible. The first is the simply connected root datum $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ whose rank is $n+1$. In this case, the family $(c_i)_{i \in I}$ is linearly dependent and the space $\mathfrak{a}_0 := \bigcap_{i \in I} \ker(c_i) \subset \mathfrak{a}$ is one-dimensional. In fact, we have $\overline{C} = X_+ = \mathfrak{a}_0$. Hence, the same argument as in the first spherical case above proves that $G \neq KU_+K$. Second, recall from Example 2.2.5 that the rank of \mathcal{D} may even be equal to n , so that \mathcal{D} is neither free nor cofree and $\overline{C} = X_+ = \{0\}$. In this situation, the next result shows that a horospherical decomposition does indeed exist, which establishes the second part of Theorem B.

Theorem 6.2.6. *If $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is an indecomposable GCM of affine type and $d = n$, then G has a horospherical decomposition $G = KU_+K$.*

In accordance with Convention 2.1.2, we have replaced the integer n in Theorem B by $n+1$ in the statement of Theorem 6.2.6 and we will replace every index i by a superscript $^\circ$ above the respective symbol in its proof.

Proof. Let $\mathring{\mathcal{D}}$ denote the \mathring{I} -restriction of \mathcal{D} and let $\mathring{P} \cong \mathring{G} \ltimes \mathring{U}_+^\perp$ be the parabolic subgroup of G of type \mathring{I} . Its Levi factor has the Iwasawa decomposition $\mathring{G} = \mathring{K} \mathring{A} \mathring{U}_+$. Since \mathbb{A} is spherical and $d = n = |\mathring{I}|$, Theorem 6.2.5 implies that $\mathring{G} = \mathring{K} \mathring{U}_+ \mathring{K}$. Given any $\alpha \in \Phi_+$, the root group $U_\alpha \subset G$ is either contained in \mathring{U}_+ or \mathring{U}_+^\perp . In the former case, it lies in the Levi factor $\mathring{G} \subset \mathring{P}$. In the latter, it is contained in the unipotent radical $\mathring{U}_+^\perp \subset \mathring{P}$, so that $g_0 U_\alpha g_0^{-1} \subset \mathring{U}_+^\perp \subset U_+$ for all $g_0 \in \mathring{G}$. As U_+ is generated by the positive root groups, it follows that $U_+ = \mathring{U}_+ \mathring{U}_+^\perp$ and $G = K \mathring{A} \mathring{U}_+ \mathring{U}_+^\perp$.

Thus, if $g \in G$ is arbitrary, there exist $k \in K$, $a \in A$, $u_1 \in \mathring{U}_+$ and $u_2 \in \mathring{U}_+^\perp$ with $g = kau_1u_2$. Since $au_1 \in \mathring{G}$, we also find $l_1, l_2 \in \mathring{K}$ and $v \in \mathring{U}_+$ such that $au_1 = l_1vl_2$. This implies

$$g = kau_1u_2 = kl_1vl_2u_2 = (kl_1)(vl_2u_2l_2^{-1})l_2 \in KU_+K$$

because $l_2 \in \mathring{K} \subset \mathring{G}$ lies in the Levi factor of \mathring{P} and $u_2 \in \mathring{U}_+^\perp$ in its unipotent radical. \square

Remark 6.2.7. (i) The proof of Theorem 6.2.6 did not use the assumption that \mathbb{A} is of affine type very explicitly. Instead, it exploited that \mathbb{A} is not invertible, which is necessary for the existence of Kac-Moody root data associated to \mathbb{A} whose rank is smaller than $|I|$, and that there exists some $i \in I$ such that the submatrix $\mathbb{A}^{(i)}$ is spherical, which allowed us to use Theorem 6.2.5 in a positive way. However, these two properties already imply that \mathbb{A} is of affine type (cf. Lemma 2.1.3). In particular, if \mathbb{A} is of indefinite type where \mathcal{D} is admissible, it is not possible to establish the horospherical decomposition by an argument that is similar to the previous proof, so there is no contradiction to Theorem 6.2.5.

(ii) The situation in Theorem 6.2.6 is the simplest case where \mathcal{D} is not admissible. In general, given an arbitrary GCM \mathbb{A} , it may happen that \mathbb{A} has several affine components or both affine and non-affine components. By investigating how the corresponding components of \mathcal{D} “overlap”, it is probably possible to give more precise conditions on \mathcal{D} for the existence of the horospherical decomposition, but we will not pursue this any further.

Related to the non-existence of the KAK -decomposition for Kac-Moody groups, the authors of [FHHK20] studied the question of whether (6.2.1) also holds if A is replaced by $A_+^\circ = \exp(X_+^\circ)$. Their interest in this problem is rooted in a property of “Kac-Moody symmetric spaces” that will be explained in the next section, where we have to pass to a restricted setting for the results of [FHHK20] to be valid. Nevertheless, the question itself can be investigated from a purely algebraic perspective without this geometric context and our convexity theorem provides a negative answer in the non-spherical case. The following result is the content of Theorem C:

Theorem 6.2.8. *If \mathcal{D} is admissible and \mathbb{A} is symmetrizable, then exactly one of the following conditions holds:*

- (i) *If \mathbb{A} is spherical, then $KA_+^\circ K = G$.*
- (ii) *If \mathbb{A} is not spherical, then $(KA_+^\circ K)^N \subset KA_+^\circ U_+ \subsetneq G$ for all $N \in \mathbb{N}$.*

Proof. If \mathbb{A} is spherical, then $A_+^\circ = A$ and $G = KAK$ holds by Remark 6.2.2 because $G = \mathcal{G}(\mathbb{K})$ can be identified with the group of \mathbb{K} -points of a connected reductive algebraic \mathbb{R} -group \mathcal{G} . If \mathbb{A} is not spherical, then $\emptyset \neq X_+^\circ \subsetneq \mathfrak{a}$ and $KA_+^\circ U_+$ is a non-empty proper subset of $G = KAU_+$ since $\exp : \mathfrak{a} \rightarrow A$ is bijective. In this case, we prove (ii) by induction on N .

Suppose first that $N = 1$ and consider an arbitrary element of the form $kal \in KA_+^\circ K \subset G$ with $k, l \in K$ and $a \in A_+^\circ$. Then $h := \log(a) \in X_+^\circ \subset \mathfrak{a}$ is virtually singular, so that $\Pi_a = \exp(\Xi_h)$ holds by Theorem 5.1.1. Thus, if we write $kal = k'a'u'$ in Iwasawa form and set $h' := \log(a') \in \mathfrak{a}$, we conclude that $h' \in \Xi_h \subset X_+^\circ$ and $a' \in A_+^\circ$, so that $kal = k'a'u' \in KA_+^\circ U_+$.

Assume now that $N > 1$ and let $g = \prod_{r=1}^N k_r a_r l_r \in (KA_+^\circ K)^N \subset G$ be arbitrary, where $k_r, l_r \in K$ and $a_r \in A_+^\circ$ for all $r \in \{1, \dots, N\}$. We set $g' := k_1 a_1 l_1$ and $g'' := \prod_{r=2}^N k_r a_r l_r$, so that $g = g'g''$. The induction hypothesis implies that $g'' = k''a''u'' \in KA_+^\circ U_+$ for some $k'' \in K$, $a'' \in A_+^\circ$ and $u'' \in U_+$ and hence

$$g = g'g'' = k_1 a_1 l_1 g'' = k_1 a_1 l_1 k'' a'' u''.$$

The case $N = 1$ proves that the subproduct $k_1 a_1 l_1 k'' \in KA_+^\circ K$ is contained in $KA_+^\circ U_+$, so its Iwasawa form is $k_1 a_1 l_1 k'' = k'a'u'$ for some $k' \in K$, $a' \in A_+^\circ$ and $u' \in U_+$. As A normalizes U_+ , the following expression yields the Iwasawa decomposition of g :

$$g = g'g'' = k_1 a_1 l_1 k'' a'' u'' = k'a'u'a''u'' = k'(a'a'')(a''^{-1}u'a''u'')$$

Since A_+° is a subsemigroup of A , we conclude that $a'a'' \in A_+^\circ$ and $g \in KA_+^\circ U_+$. \square

At this point, we have proved most of our main results. The final goal of this thesis is to interpret Theorem C as a statement about certain “geometric objects” associated to Kac-Moody groups.

6.3 Kac-Moody symmetric spaces

Although real and complex Kac-Moody groups are natural generalizations of semisimple Lie groups from an algebraic point of view, they do not carry any differentiable structure, which makes it difficult to extend one very important aspect of Lie theory to that setting. Every semisimple Lie group gives rise to a *Riemannian symmetric space of non-compact type*, which is a complete contractible Riemannian manifold of non-positive sectional curvature on which the Lie group acts by isometries, and this geometric object is an essential tool to study the Lie group itself. For a general introduction to Riemannian geometry, we refer to [Lee18]. A thorough treatment of Riemannian symmetric spaces can be found in [Hel78] and particular emphasis on the non-compact type is laid in [Ebe96, Chapter 2]. For Kac-Moody groups, it is not immediately obvious how to construct a reasonable analogue of a symmetric space due to the lack of a manifold structure. Nevertheless, the authors of [FHHK20] have developed an approach that circumvents this obstacle. It is based on the following concept (cf. [FHHK20, Definition 2.2]):

Definition 6.3.1. Let \mathcal{X} be a set and $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ a map.

- (i) The pair (\mathcal{X}, μ) is a *reflection space* if the following three properties hold for all $x, y, z \in \mathcal{X}$:
 - (RS1) $\mu(x, x) = x$
 - (RS2) $\mu(x, \mu(x, y)) = y$
 - (RS3) $\mu(x, \mu(y, z)) = \mu(\mu(x, y), \mu(x, z))$
- (ii) A reflection space (\mathcal{X}, μ) is called a *symmetric space* if the following additional condition is also fulfilled for all $x, y \in \mathcal{X}$:
 - (RS4) $\mu(x, y) = y$ implies $y = x$.
- (iii) A *morphism* between symmetric spaces (\mathcal{X}, μ) and (\mathcal{X}', μ') is a map $\varphi : \mathcal{X} \rightarrow \mathcal{X}'$ that satisfies

$$\varphi(\mu(x, y)) = \mu'(\varphi(x), \varphi(y)) \quad \text{for all } x, y \in \mathcal{X}. \quad (6.3.1)$$

If φ is bijective, then $\varphi^{-1} : \mathcal{X}' \rightarrow \mathcal{X}$ is also a morphism and φ is called an *isomorphism*. In this case, if $(\mathcal{X}, \mu) = (\mathcal{X}', \mu')$, then φ is an *automorphism* of (\mathcal{X}, μ) . The group of all automorphisms of (\mathcal{X}, μ) will be denoted by $\text{Aut}(\mathcal{X}, \mu)$.

Remark 6.3.2. (i) This definition of a symmetric space is motivated by the approach towards Riemannian symmetric spaces developed by O. Loos (cf. [Loo69]). In fact, the definition of a symmetric space in [Loo69, Chapter II, §1] is almost identical to Definition 6.3.1. The sole differences are that Loos requires \mathcal{X} to be a smooth manifold and μ to be a smooth map and only imposes that (RS4) holds if y is contained in a sufficiently small neighbourhood of x . He shows that his definition is equivalent to that of a symmetric space in the language of Riemannian geometry and manages to reprove most results of the classical theory in his setting. For arbitrary $x \in \mathcal{X}$, the map $y \mapsto s_x(y) := \mu(x, y)$ should be thought of as the geodesic reflection at the point x . In fact, (RS1) states that x is a fixed-point of s_x , (RS2) means that s_x has order two and (RS3) that s_x is an automorphism of (\mathcal{X}, μ) , which are well-known properties of the geodesic reflections in Riemannian symmetric spaces. The axiom (RS4) asserts that x is the only fixed-point of s_x . This holds in Riemannian symmetric spaces of non-compact type but fails for the compact type, such as the sphere with its standard Riemannian metric. In general, it is only true if y lies in a sufficiently small neighbourhood of x , which explains Loos' variant of this condition. Hence, a symmetric space in the sense of Definition 6.3.1 should be viewed as a Riemannian symmetric space of non-compact type without any differentiable structure. The notion of an automorphism serves as a substitute for the concept of isometries.

- (ii) In [FHHK20, Definition 2.12], the authors also introduce “topological” symmetric spaces and begin to formulate analogues of classical geometric objects (e.g. geodesics) in that setting. This is motivated by their definition of real Kac-Moody groups which they always view as topological groups endowed with the Kac-Peterson topology, but all their results hold in a weaker form by omitting the topological aspects. As we have not introduced this topology formally and consider Kac-Moody groups only as abstract groups defined by a presentation, we restate the definitions and results of [FHHK20] without any topological notions.

Example 6.3.3. (i) Given any $l \in \mathbb{N}$, the l -dimensional *Euclidean symmetric space* \mathbb{E}^l is the symmetric space $(\mathbb{R}^l, \mu_{\text{eucl}})$ defined by:

$$\begin{aligned} \mu_{\text{eucl}} : \mathbb{R}^l \times \mathbb{R}^l &\rightarrow \mathbb{R}^l \\ (x, y) &\mapsto 2x - y \end{aligned} \quad (6.3.2)$$

For each $x \in \mathbb{R}^l$, the map $s_x : \mathbb{R}^l \rightarrow \mathbb{R}^l$, $y \mapsto 2x - y = x + (x - y)$, fixes x and reflects every point $y \neq x$ along the line spanned by x and y . Hence, s_x is indeed the geodesic reflection of \mathbb{R}^l with its usual flat Riemannian metric.

- (ii) One of the simplest ways to construct symmetric spaces in the sense of Definition 6.3.1 is very general and paves the way towards Kac-Moody symmetric spaces. For the moment, suppose that G is an arbitrary group and that $\Theta : G \rightarrow G$ is an involutive automorphism. Let $K \subset G$ denote the fixed-point set of Θ and define the following map:

$$\begin{aligned} \mu : G/K \times G/K &\rightarrow G/K \\ (gK, hK) &\mapsto g\Theta(g^{-1}h)K \end{aligned} \quad (6.3.3)$$

It is straightforward to verify that $(G/K, \mu)$ is a reflection space. It is symmetric, i.e. **(RS4)** is satisfied, if and only if $K \cap \{g\Theta(g^{-1}) \mid g \in G\} = \{1\}$ (cf. [FHHK20, Proposition 4.2]). Moreover, G acts transitively on G/K by automorphisms via the natural left-action and the stabilizer of the point $1 \cdot K \in G/K$ is precisely the subgroup K .

Remark 6.3.4. The motivation for this construction of abstract symmetric spaces comes from Riemannian geometry. If M is a Riemannian symmetric space, its isometry group $I(M)$ has a Lie group structure such that its natural action on M is smooth. The action of its identity component $G := I_e(M)$ is transitive, the stabilizer K of any chosen “base-point” $o \in M$ under the G -action is a compact subgroup of G and M is diffeomorphic to the quotient manifold G/K . Conjugation by the geodesic reflection $s_o \in I(M)$ defines an involutive automorphism $\Theta : G \rightarrow G$ and, under the identification $M \cong G/K$, the geodesic reflection s_x at a point $x \equiv gK$ corresponds to the map $hK \mapsto g\Theta(g^{-1}h)K$. Proofs of these assertions can be found in [Hel78, Chapter IV, §3].

We will apply the construction in Example 6.3.3 (ii) in the following way: Let \mathcal{D} be a Kac-Moody root datum, $G = G_{\mathcal{D}}(\mathbb{K})$ the real or complex Kac-Moody group of type \mathcal{D} , $\Theta : G \rightarrow G$ the compact involution of G and K its compact form. We obtain a reflection space $(G/K, \mu)$ defined by (6.3.3). In order to show that it is even symmetric, we define the set $\Omega := \{g\Theta(g^{-1}) \mid g \in G\}$ and need to prove the following:

Proposition 6.3.5. $K \cap \Omega = \{1\}$

This result is established in [FHHK20, Proposition 3.38] for a class of Kac-Moody groups that is less general than ours (cf. Remark 6.3.7). For the reader’s convenience, we therefore provide a self-contained proof, but our argument is almost identical to that reference.

Proof. Let $g \in G$ be such that $k := g\Theta(g^{-1}) \in K \cap \Omega$, then

$$k = \Theta(k) = \Theta(g)g^{-1} = (g\Theta(g^{-1}))^{-1} = k^{-1},$$

so the order of k is at most two. By the general theory of RGD systems, the pair (B_+, N) , where B_+ is the positive Borel subgroup of G and N the subgroup generated by the torus T and the extended Weyl group \tilde{W} , is a BN -pair for G (cf. [AB08, Proposition 8.54]). Since k generates a finite subgroup of G , it follows from [AB08, Corollary 12.68] that it lies in a parabolic subgroup of spherical type, i.e. there exists a spherical subset $J \subset I$ such that k is contained in the parabolic subgroup $P_J \subset G$. Now the Levi decomposition $P_J \cong G_J \ltimes U_{J,+}^\perp$ and the fact that Θ stabilizes T and maps U_α to $U_{-\alpha}$ for all $\alpha \in \Phi$ imply that $k = \Theta(k) \in \Theta(P_J) \cong G_J \ltimes U_{J,-}^\perp$, where $U_{J,-}^\perp$ is the subgroup of U_- generated by all U_α with $\alpha \in \Phi_- \setminus \Phi_{J,-}$. These groups satisfy $P_J \cap \Theta(P_J) = G_J$, so k is already contained in their common Levi factor G_J .

Since \mathbb{A}_J is spherical, it follows from Example 3.1.6 (i) that $G_J \cong G_{\mathcal{D}_J}(\mathbb{K})$ can be identified with the group of \mathbb{K} -points of a connected reductive algebraic \mathbb{R} -group and thus with a Zariski-closed subgroup of $\mathrm{GL}_N(\mathbb{K})$ for some $N \in \mathbb{N}$. Moreover, as mentioned in Remark 3.4.2, after possibly conjugating G_J by an element of $\mathrm{GL}_N(\mathbb{R})$, its compact involution coincides with the restriction of the map $y \mapsto (y^{-1})^*$ to $G_J \subset \mathrm{GL}_N(\mathbb{K})$. Hence, there exists an injective homomorphism $f : G_J \rightarrow \mathrm{GL}_N(\mathbb{K})$ such that $f(k) \in \mathrm{U}(N)$ is a unitary matrix. This also holds if $\mathbb{K} = \mathbb{R}$ in which case $f(k) \in \mathrm{O}(N) \subset \mathrm{U}(N)$. At the same time, $f(k) = f(g\Theta(g^{-1})) = f(g)f(g)^*$ is positive-definite, but this is only possible if $f(k) = \mathbb{I}_N$ is the identity matrix and therefore $k = 1$, as required. \square

Definition 6.3.6. Let \mathcal{D} be a Kac-Moody root datum, $G = G_{\mathcal{D}}(\mathbb{K})$ the Kac-Moody group of type \mathcal{D} over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\Theta : G \rightarrow G$ the compact involution of G and K its compact form. The symmetric space $(G/K, \mu)$ defined by (6.3.3) is called the *Kac-Moody symmetric space* of G .

The Kac-Moody group G acts transitively on its symmetric space $(G/K, \mu)$ by automorphisms via the natural left-action, which defines a homomorphism

$$\begin{aligned} G &\rightarrow \mathrm{Aut}(G/K, \mu) \\ g &\mapsto l_g := (hK \mapsto ghK) \end{aligned} \tag{6.3.4}$$

and it is simple to verify that its kernel is the normal subgroup

$$C_K(\Omega) := \{k \in K : kp = pk \ \forall p \in \Omega\} = \{g \in G : gp = p\Theta(g) \ \forall p \in \Omega\} \subset G. \tag{6.3.5}$$

Hence, the quotient $G_{\mathrm{eff}} := G/C_K(\Omega)$ acts effectively on G/K and can be viewed as a subgroup of $\mathrm{Aut}(G/K, \mu)$. The compact involution $\Theta : G \rightarrow G$ descends to an involutive automorphism of $(G/K, \mu)$ which coincides with the reflection $s_o \in \mathrm{Aut}(G/K, \mu)$ at the base point $o := 1 \cdot K \in G/K$. Given any $g \in G$, the reflection at the point $gK \in G/K$ is the conjugate $s_{gK} = l_g \circ s_o \circ l_{g^{-1}}$. Moreover, since s_o has order two and $s_o \circ l_g \circ s_o = l_{\Theta(g)}$, the group $\langle s_o \rangle \cong \mathbb{Z}_2$ normalizes G_{eff} . Thus, the *main group* of $(G/K, \mu)$, i.e. the subgroup of $\mathrm{Aut}(G/K, \mu)$ generated by all point reflections, is isomorphic to the semidirect product $G_{\mathrm{eff}} \rtimes \langle s_o \rangle$.

Remark 6.3.7. There are some minor differences between our notion of a Kac-Moody symmetric space and the one in [FHHK20, Definition 4.5]:

- (i) The Kac-Moody groups in [FHHK20] are defined in a different way as a certain amalgamated product that only depends on a previously fixed GCM \mathbb{A} . If \mathbb{A} is 2-spherical, the authors assert that this definition coincides with the “simply connected” Kac-Moody group over \mathbb{K} associated to \mathbb{A} , i.e. the group of type $\mathcal{D}_{\mathrm{sc}}^{\mathbb{A}}$. As a consequence, their Kac-Moody groups are already generated by the root groups and the torus does not have to be included separately. This implies that they are generated by the subset Ω , which is why their definition of G_{eff} is a bit simpler than ours (cf. [FHHK20, Proposition 3.39 and Lemma 4.7]).
- (ii) The authors of [FHHK20] almost exclusively work over the field of real numbers. Their construction has been extended to the complex case and the root datum $\mathcal{D}_{\mathrm{Kac}}^{\mathbb{A}}$ in [KV24], where most of their results remain true. Only the classification of automorphisms of Kac-Moody symmetric spaces in [FHHK20, Section 6] does not readily generalize to that setting. Moreover, they typically view Kac-Moody groups as topological groups endowed with the Kac-Peterson topology and the resulting symmetric spaces as topological symmetric spaces with respect to the quotient topology.

While it is very likely that a large part of the theory developed in [FHHK20] and [KV24] can be extended to arbitrary Kac-Moody root data, performing this generalization appears to be a difficult and lengthy task that is beyond the scope of this thesis. However, our primary goal in this section is to interpret Theorem C as a statement about Kac-Moody symmetric spaces and for this, we require the construction of the “causal boundary”, which is developed in [FHHK20, Section 7] and relies heavily on earlier parts. This is why we will soon restrict to the setting of that reference in order to freely use the results established there.

- (iii) Definition 6.3.6 is analogous to the notion of an “unreduced” Kac-Moody symmetric space in the sense of [FHHK20]. The authors also introduce a “reduced” space associated to \mathbb{A} that

is obtained from a central quotient of $G_{\mathcal{D}_{\text{sc}}^{\mathbb{A}}}(\mathbb{K})$ and has many similar properties. We have not included such a definition on purpose in order to keep the discussion relatively short and simplify the notation. If \mathbb{A} is invertible, the reduced and unreduced Kac-Moody symmetric space coincide. If \mathbb{A} is of affine type, however, there are some subtle differences between these spaces and several assertions of [FHHK20] only hold if \mathbb{A} is not of affine type. This is one of the advantages of the construction in [KV24], where the choice of the root datum $\mathcal{D}_{\text{Kac}}^{\mathbb{A}}$ makes it possible to extend these results to the affine case as well.

Although it is not apparent from the definition that the Kac-Moody symmetric space $(G/K, \mu)$ has any reasonable sort of “geometric structure”, the authors of [FHHK20] establish the existence of several natural analogues of familiar geometric objects in this setting. We will only be interested in the following notion, for which we have adjusted [FHHK20, Definition 2.18, 2.23 and 2.28] to our purposes.

Definition 6.3.8. Let (\mathcal{X}, μ) be a symmetric space.

- (i) A subset $F \subset \mathcal{X}$ is called a *Euclidean flat of rank $l \in \mathbb{N}$* , or simply an *l -flat*, if $\mu(x, y) \in F$ for all $x, y \in F$ and the symmetric space $(F, \mu|_{F \times F})$ is isomorphic to \mathbb{E}^l .
- (ii) A Euclidean flat of rank 1 is called a *geodesic*. If a point $x \in \mathcal{X}$ lies on a geodesic $\gamma \subset \mathcal{X}$ and $\varphi : \gamma \rightarrow \mathbb{R}$ is an isomorphism with $a := \varphi(x)$, the set $\varphi^{-1}([a, \infty)) \subset \gamma$ is a *geodesic ray* emanating from x . If $y \in \mathcal{X}$ also lies on γ and $b := \varphi(y)$, the set $\varphi^{-1}([a, b]) \subset \gamma$ is called a *geodesic segment* from x to y . Here, the case $x = y$ is allowed.
- (iii) We say that (\mathcal{X}, μ) is *geodesically connected* if for every pair of points $x, y \in \mathcal{X}$, there exists some $N \in \mathbb{N}$ and a finite sequence $x = x_0, x_1, \dots, x_{N-1}, x_N = y$ of points in \mathcal{X} such that x_{k-1} and x_k lie on a common geodesic for every $k \in \{1, \dots, N\}$.

In the case where $G = G_{\mathcal{D}}(\mathbb{K}) = KAU_+$ is a real or complex Kac-Moody group endowed with an Iwasawa decomposition, it is simple to construct many flats in the Kac-Moody symmetric space $(G/K, \mu)$ whose rank coincides with the rank $d \in \mathbb{N}$ of the underlying Kac-Moody root datum \mathcal{D} . To this end, we define the set

$$F_0 := A \cdot o = \{aK \mid a \in A\} \subset G/K. \quad (6.3.6)$$

The proof of the following proposition is a very simple formal computation based on (6.3.3) that is identical to [FHHK20, Proposition 5.2].

Proposition 6.3.9. *For every $g \in G$, the set $F := g(F_0) = gA \cdot o \subset G/K$ is a Euclidean flat of rank $d \in \mathbb{N}$ in $(G/K, \mu)$. More precisely, if $\mathfrak{a} \cong \mathbb{R}^d$ is endowed with the structure of a d -dimensional Euclidean symmetric space, the following map defines an isomorphism of symmetric spaces:*

$$\begin{aligned} \varphi : \mathfrak{a} &\rightarrow F \\ h &\mapsto g \exp(h)K \end{aligned} \quad (6.3.7)$$

Moreover, for every $g \in G$ and $h \in \mathfrak{a}$, the set $\{g \exp(th)K \mid t \in \mathbb{R}\} \subset G/K$ is a geodesic. □

A flat in $(G/K, \mu)$ of the form $g(F_0)$ for some $g \in G$ is called a *standard flat*. This notion allows us to prove an interesting geometric property of Kac-Moody symmetric spaces. The next result has already been established in [FHHK20, Corollary 5.12 and Proposition 5.14], but we can give a slightly different proof that is adapted to our setting and relates nicely to Section 6.2.

Proposition 6.3.10. *The Kac-Moody symmetric space $(G/K, \mu)$ is always geodesically connected. However, if \mathbb{A} is not spherical, there exist points in G/K that do not lie on a common geodesic.*

Proof. On the one hand, let $x, y \in G/K$ be arbitrary. Because G acts transitively on G/K , there exist some $g, h \in G$ such that $g \cdot o = x$ and $h \cdot o = g^{-1} \cdot y$. Moreover, Theorem 6.2.3 (ii) states that $G = \bigcup_{N=1}^{\infty} (KAK)^N$, so h can be written as $h = \prod_{r=1}^N p_r a_r q_r$ for some $N \in \mathbb{N}$ and $p_r, q_r \in K$, $a_r \in A$ for all $r \in \{1, \dots, N\}$. Consider the sequence of points $x_0, \dots, x_N \in G/K$ defined by

$$x_0 := g \cdot o = x \quad \text{and} \quad x_k := g \prod_{r=1}^k p_r a_r q_r \cdot o \quad \text{for all } k \in \{1, \dots, N\}.$$

Since K stabilizes the base point $o = 1 \cdot K$ and the A -orbit through o is the standard flat F_0 , the following holds for every $k \in \{1, \dots, N\}$:

$$\begin{aligned} x_{k-1} &= g \prod_{r=1}^{k-1} p_r a_r q_r \cdot o = g \prod_{r=1}^{k-1} p_r a_r q_r \cdot (p_k \cdot o) \in g \left(\prod_{r=1}^{k-1} p_r a_r q_r \right) p_k(F_0) \\ x_k &= g \prod_{r=1}^k p_r a_r q_r \cdot o = g \prod_{r=1}^{k-1} p_r a_r q_r \cdot (p_k a_k \cdot o) \in g \left(\prod_{r=1}^{k-1} p_r a_r q_r \right) p_k(F_0) \end{aligned}$$

Thus, the points x_{k-1} and x_k are contained in a common standard flat. Because every such flat is isomorphic to the Euclidean symmetric space \mathbb{E}^d , where any two points lie on a common geodesic, it follows that x_{k-1} and x_k also have this property. Since $x_0 = g \cdot o = x$ and $x_N = gh \cdot o = y$ were arbitrary, we conclude that $(G/K, \mu)$ is geodesically connected.

On the other hand, if \mathbb{A} is not spherical, then $G \neq K\Omega$ by Theorem 6.2.3 (i). We claim that if $g \in G \setminus K\Omega$, then the points $o = 1 \cdot K$ and $x := g^{-1} \cdot o = g^{-1}K$ do not lie on a common geodesic. Suppose to the contrary that there exists a geodesic $\gamma \subset \mathcal{X}$ containing them and let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an isomorphism with $a := \varphi(o)$ and $b := \varphi(x)$. The midpoint $c := \frac{1}{2}(a+b) \in [a, b]$ satisfies $\mu_{\text{eucl}}(c, a) = b$ and $\mu_{\text{eucl}}(c, b) = a$. As φ is an isomorphism, the point $y := \varphi^{-1}(c) \in \gamma$ has the property that $\mu(y, o) = x$ and $\mu(y, x) = o$. Let $h \in G$ be such that $y = hK$, then the first equality can be written as $g^{-1}K = \mu(hK, 1 \cdot K) = h\Theta(h^{-1})K$. This is equivalent to $k := gh\Theta(h^{-1}) \in K$ and hence to $g = k\Theta(h)h^{-1} = k\Theta(h)\Theta(\Theta(h)^{-1}) \in K\Omega$, a contradiction. \square

Remark 6.3.11. Conversely, if \mathbb{A} is spherical, then $G = KAK$ by Theorem 6.2.8, which implies that any two points of G/K lie in a common standard flat and can therefore be joined by a geodesic segment. This is a well-known property of Riemannian symmetric spaces of non-compact type, which are complete, simply connected Riemannian manifolds of non-positive sectional curvature. In fact, it follows from the *Cartan-Hadamard Theorem* that the geodesic is unique and minimizes the Riemannian distance between the points (cf. [Lee18, Theorem 12.8 and Proposition 12.9]).

Similarly, if we define a *horosphere* in $(G/K, \mu)$ to be any set of the form $gU_+ \cdot o$ for some $g \in G$, the various results on the (non-)existence of a horospherical decomposition of G in Section 6.2 have the following geometric analogues:

Proposition 6.3.12. *The Kac-Moody symmetric space $(G/K, \mu)$ has the following properties:*

- (i) *If G is generated by the root groups $(U_\alpha)_{\alpha \in \Phi}$, then for all $x, y \in G/K$, there exists a sequence $x = x_0, x_1, \dots, x_n, x_{n+1} = y$ of points in G/K such that x_{k-1} and x_k are contained in a common horosphere for every $k \in \{1, \dots, n+1\}$.*
- (ii) *If \mathcal{D} is admissible and $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is symmetrizable, then any two points in G/K are contained in a common horosphere if and only if \mathbb{A} is spherical and $d = n$.*
- (iii) *If $\mathbb{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$ is an indecomposable GCM of affine type and $d = n$, then any two points in G/K are contained in a common horosphere.*

Proof. Let $x, y \in G/K$ be arbitrary. By the same argument as in the proof of Proposition 6.3.10, there exist $g, h \in G$ with $g \cdot o = x$ and $h \cdot o = g^{-1} \cdot y$ and if $h \in (KU_+K)^N$ for some $N \in \mathbb{N}$, we can construct a sequence $x = x_0, x_1, \dots, x_{N-1}, x_N = y$ of points in G/K such that x_{k-1} and x_k lie on a common horosphere for every $k \in \{1, \dots, N\}$. If G is generated by the root groups, then $G = (KU_+K)^{n+1}$ by Theorem 6.2.3 (iii), so the proof of (i) is complete. Likewise, the assertion of (iii) follows from Theorem 6.2.6. In the situation of (ii), the condition that \mathbb{A} is spherical and $d = n$ is equivalent to $G = KU_+K$ by Theorem 6.2.5. If this holds, then any two points in G/K lie on a common horosphere. Conversely, if $G \neq KU_+K$ and $g \in G \setminus KU_+K$, we claim that the point $x := g \cdot o = gK$ does not lie on a horosphere containing $o = 1 \cdot K$. Suppose to the contrary that there is some $g' \in G$ such that the horosphere $g'U_+ \cdot o$ contains both o and x and let $g' = k'a'u'$ be the Iwasawa form of g' . By assumption, there exists some $u_1 \in U_+$ such that $o = g'u_1 \cdot o$, which is equivalent to $g'u_1 \in K$. This implies that $a' = 1$ and $u' = u_1^{-1}$ and therefore $g'U_+ \cdot o = k'U_+ \cdot o$. Since this horosphere also contains $x = g \cdot o$, we find $u_2 \in U_+$ with $g \cdot o = k'u_2 \cdot o$ and hence $u_2^{-1}k'^{-1}g \in K$, so that $g \in k'u_2K \subset KU_+K$, a contradiction. \square

Proposition 6.3.10 and 6.3.12 may appear somewhat artificial in the sense that the “geometric” results are merely restatements of algebraic properties of Kac-Moody groups and there does not seem to be an obvious way to deduce them directly from the theory of Kac-Moody symmetric spaces. In particular, it is not clear whether the notion of a horosphere is useful in this setting. The standard flats, however, do indeed play an important role in [FHHK20]. It is very likely that many of their results can be adjusted to our definitions, but in order to reproduce them without technical comments, we now impose the following restrictions:

Assumption 6.3.13. For the rest of Section 6.3, we suppose that $\mathbb{A} \in \mathbb{Z}^{n \times n}$ is an indecomposable, symmetrizable and invertible GCM. Moreover, we assume that $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is the simply connected Kac-Moody root datum associated to \mathbb{A} and that $\mathbb{K} = \mathbb{R}$.

Remark 6.3.14. These assumptions imply that our definition of the Kac-Moody symmetric space $(G/K, \mu)$ coincides with that of the “unreduced” symmetric space associated to \mathbb{A} in the sense of [FHHK20, Definition 4.5]. They also ensure that all results of [FHHK20] are true simultaneously, but most of the properties listed below hold under weaker conditions. In fact, the authors never suppose that \mathbb{A} is invertible, but many results of [FHHK20, Section 6 and 7] require that \mathbb{A} is not of affine type or only apply to the “reduced” Kac-Moody symmetric space associated to \mathbb{A} that we have not defined formally. Our assumption that \mathbb{A} is invertible guarantees that it is not of affine type and that the “reduced” symmetric space associated to \mathbb{A} coincides with the “unreduced” one. This allows us to reproduce their findings without having to introduce a lot of new notation.

Some of the main results of [FHHK20] about the Kac-Moody symmetric space $(G/K, \mu)$ that we have not already mentioned are the following:

- a) The standard flats of $(G/K, \mu)$ are in bijective correspondence with the G -conjugates of the torus $T \subset G$ and with the twin apartments in the twin building associated to the RGD system $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ (cf. [FHHK20, Corollary 5.6]).
- b) Every flat of $(G/K, \mu)$ is contained in a standard flat. In particular, since the rank of $\mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is equal to n , all maximal flats of $(G/K, \mu)$ are isomorphic to \mathbb{E}^n and G acts transitively on the set of maximal flats. Moreover, the stabilizer K of the base point o acts transitively on the set of maximal flats containing o (cf. [FHHK20, Theorem 5.17 and Corollary 5.18]). Consequently, given any point $x \in G/K$, the geodesics in $(G/K, \mu)$ containing x are precisely the sets of the form $\{g \exp(th)K \mid t \in \mathbb{R}\}$, where $g \in G$ is any element such that $x = gK$ and $h \in \mathfrak{a}$ is arbitrary. Analogues of these statements are classical results about Riemannian symmetric spaces of non-compact type (cf. [Ebe96, Proposition 2.10]).
- c) Based on the work of P.E. Caprace on homomorphisms of Kac-Moody groups (cf. [Cap09]), the authors classify automorphisms of $(G/K, \mu)$ and find a very precise description of its automorphism group. In particular, they establish that $\text{Aut}(G/K, \mu)$ contains a normal subgroup $\text{Aut}^+(G/K, \mu)$ of index 2 that satisfies $\text{Aut}(G/K, \mu) \cong \text{Aut}^+(G/K, \mu) \rtimes \langle s_o \rangle$ and $G_{\text{eff}} \subset \text{Aut}^+(G/K, \mu)$ (cf. [FHHK20, Theorem 6.12 and Remark 6.13]).
- d) The authors also describe the group of “local” automorphisms of $(G/K, \mu)$, i.e. the subgroup of $\text{Aut}(G/K, \mu)$ preserving a pair (x, F) , where x is a point in G/K and F a standard flat containing x . They define “regular” and “singular” points in F and a “geometric Weyl group” which acts on F and preserves these point sets. They show that, under an isomorphism $\mathfrak{a} \rightarrow F$ as in (6.3.7), these notions coincide with those of regular and singular points in \mathfrak{a} in the sense of Definition 4.1.2 and the “geometric” Weyl group is isomorphic to the “algebraic” Weyl group $W(\mathbb{A})$ of the GCM \mathbb{A} (cf. the discussion from 6.14 until the end of Section 6 in [FHHK20]). This is another natural generalization of a well-known correspondence between “geometric” Weyl chambers in maximal flats of Riemannian symmetric spaces of non-compact type and “algebraic” Weyl chambers in maximal abelian subspaces of semisimple Lie algebras (cf. [Ebe96, Section 2.9-2.12]).

Proposition 6.3.10 and 6.3.12 provide examples, where the (non-)existence of decompositions of Kac-Moody groups can be interpreted as “geometric” properties of Kac-Moody symmetric spaces. In order to reformulate Theorem C in this framework as well, we have to explain the construction of the “causal pre-order” on Kac-Moody symmetric spaces developed in [FHHK20, Section 7].

Since \mathbb{A} is invertible, the root datum $\mathcal{D} = \mathcal{D}_{\text{sc}}^{\mathbb{A}}$ is admissible, so the positive and negative Tits cone $X_{\pm} \subset \mathfrak{a}$ both have non-empty interior. If \mathbb{A} is spherical, then X_{+}° and X_{-}° both coincide with \mathfrak{a} , otherwise these sets are disjoint by Corollary 2.6.7. They are open convex cones in \mathfrak{a} which can be “exponentiated” to the standard flat $F_0 = A \cdot o \subset G/K$ by setting

$$C_o^{+,-}(F_0) := C_o^{+}(F_0) \cup C_o^{-}(F_0) \quad \text{where} \quad C_o^{\pm}(F_0) := A_{\pm}^{\circ} \cdot o. \quad (6.3.8)$$

The set $C_o^{+,-}(F_0)$ is a “double cone” in F_0 whose two “halves” $C_o^{\pm}(F_0)$ coincide if \mathbb{A} is spherical and are disjoint otherwise. If $\varphi \in \text{Aut}(G/K, \mu)$ is an automorphism that fixes o and maps F_0 to itself, then φ preserves this double cone. If it is even contained in the index-2 subgroup $\text{Aut}^{+}(G/K, \mu)$, it preserves both halves and otherwise, it exchanges $C_o^{+}(F_0)$ and $C_o^{-}(F_0)$. If F is an arbitrary maximal flat of $(G/K, \mu)$ containing o , there exists an element $k \in K$ such that $F = k(F_0)$ and we define a double cone in F by

$$C_o^{+,-}(F) := C_o^{+}(F) \cup C_o^{-}(F) \quad \text{where} \quad C_o^{\pm}(F) := kA_{\pm}^{\circ} \cdot o. \quad (6.3.9)$$

As the G -action on G/K takes values in $G_{\text{eff}} \subset \text{Aut}^{+}(G/K, \mu)$, these definitions are independent of the choice of k . From this, we construct a “double cone based at o ” as

$$C_o^{+,-} := C_o^{+} \cup C_o^{-} \quad \text{where} \quad C_o^{\pm} := \bigcup C_o^{\pm}(F). \quad (6.3.10)$$

Here, the union is taken over all maximal flats F of $(G/K, \mu)$ containing o . This double cone is invariant under all automorphisms of $(G/K, \mu)$ that map o to itself and each of its two halves is preserved by the automorphisms in $\text{Aut}^{+}(G/K, \mu)$ fixing o . Finally, if $x \in G/K$ is an arbitrary point and $g \in G$ is such that $x = gK$, we define a double cone based at x by

$$C_x^{+,-} := C_x^{+} \cup C_x^{-} \quad \text{where} \quad C_x^{\pm} := g(C_o^{\pm}). \quad (6.3.11)$$

This definition is now independent of the choice of g . The family $(C_x^{+,-})_{x \in G/K}$ is invariant under the automorphism group of $(G/K, \mu)$ and called the *canonical double cone field* on $(G/K, \mu)$. Its two halves $(C_x^{\pm})_{x \in G/K}$ are invariant under $\text{Aut}^{+}(G/K, \mu)$ and called the *positive* (resp. *negative*) *causal structure* on $(G/K, \mu)$.

- Definition 6.3.15.** (i) A map $\gamma : U \rightarrow G/K$, where $U \subset \mathbb{R}$ is allowed to be any interval, is called a *causal* (resp. *anti-causal*) *curve* if for every $t \in U$, there exists some $\varepsilon > 0$ such that $\gamma((t, t + \varepsilon)) \subset C_{\gamma(t)}^{+}$ (resp. $\gamma((t, t + \varepsilon)) \subset C_{\gamma(t)}^{-}$).¹ If the image of γ is also a geodesic, then γ is called a *causal* (resp. *anti-causal*) *geodesic* and we define causal (resp. anti-causal) geodesic rays and segments in the obvious ways.
- (ii) Given two points $x, y \in G/K$, we say that x *strictly causally precedes* y and write $x \prec y$ if there is some $N \in \mathbb{N}$ and a finite sequence $x = x_0, x_1, \dots, x_{N-1}, x_N = y$ of points in G/K such that for every $k \in \{1, \dots, N\}$, there exists a causal geodesic segment from x_{k-1} to x_k . We write $x \preceq y$ and say that x *causally precedes* y if $x = y$ or $x \prec y$. This defines a reflexive and transitive relation on G/K that is called the *causal pre-order*.
- (iii) For a point $x \in G/K$, the *causal future* and *causal past* of x are the two sets $(G/K)_x^{+}$ and $(G/K)_x^{-}$ defined by

$$(G/K)_x^{+} := \{y \in G/K : x \preceq y\} \quad \text{and} \quad (G/K)_x^{-} := \{y \in G/K : y \preceq x\}. \quad (6.3.12)$$

Every geodesic in $(G/K, \mu)$ is the image of a curve of the form $\mathbb{R} \rightarrow G/K, t \mapsto g \exp(th)K$, for some $g \in G$ and $h \in \mathfrak{a}$ and this curve is causal (resp. anti-causal) if and only if $h \in X_{+}^{\circ}$ (resp. $h \in X_{-}^{\circ}$). Moreover, the causal pre-order is invariant under every automorphism $\varphi \in \text{Aut}^{+}(G/K, \mu)$ in the sense that

$$x \preceq y \implies \varphi(x) \preceq \varphi(y) \quad \text{for all } x, y \in G/K. \quad (6.3.13)$$

In particular, it is invariant under the action of G .

¹If U has a maximum $t_0 \in \mathbb{R}$, it is tacitly assumed that γ can be extended to a larger interval $U' \supset U$ for which t_0 is not maximal.

Remark 6.3.16. The notions of Definition 6.3.15 are interesting from a combinatorial perspective: Let Δ^+ (resp. Δ^-) denote the set of all causal (resp. anti-causal) geodesic rays in $(G/K, \mu)$ and $\Delta_x^\pm \subset \Delta^\pm$ the respective subset of rays emanating from a given point $x \in G/K$. In [FHHK20, Proposition 7.26], the authors show that Δ_x^+ (resp. Δ_x^-) can be endowed with the structure of an “ideal polyhedral complex” whose poset of cells is isomorphic to the subcomplex of the positive (resp. negative) half of the twin building associated to the RGD system $(G, (U_\alpha)_{\alpha \in \Phi}, T)$ consisting of all simplices of spherical type. They further introduce a notion of “asymptotic” causal (resp. anti-causal) geodesic rays and prove that the quotient of Δ^+ (resp. Δ^-) by this equivalence relation has an ideal polyhedral structure with the same property (cf. [FHHK20, Corollary 7.41]). This generalizes a well-known construction of Lie theory: If G is a connected semisimple Lie group, $K \subset G$ the fixed-point set of a Cartan involution and G/K the associated Riemannian symmetric space of non-compact type, then geodesic rays in G/K are said to be asymptotic if their distance with respect to the Riemannian metric is bounded. This defines an equivalence relation on the set of unit-speed geodesic rays in G/K whose set of equivalence classes can be endowed with the structure of a simplicial complex that is isomorphic to the spherical building associated to G . More details about this construction can be found in [Ebe96, Section 3.6].

If \mathbb{A} is spherical, then $X_+^\circ = X_-^\circ = \mathfrak{a}$, so for every point $x \in G/K$ and every maximal flat F containing x , we have $C_x^+(F) = C_x^-(F) = F$. Since $G = KAK$, every other point in G/K lies in some maximal flat containing x , so it even follows that $C_x^+ = C_x^- = G/K$. Therefore, every curve in $(G/K, \mu)$ is both causal and anti-causal and the causal pre-order is trivial, i.e. it holds that $x \preceq y$ for all $x, y \in G/K$ or, equivalently, $(G/K)_x^+ = (G/K)_x^- = G/K$ for all $x \in G/K$. If \mathbb{A} is not spherical, then X_+° and X_-° are proper disjoint subsets of \mathfrak{a} , which suggests that the causal pre-order should be antisymmetric in this case and hence a partial order. In other words, $(G/K)_x^+ \cap (G/K)_x^- = \{x\}$ should hold for all $x \in G/K$. At the time of writing, the authors of [FHHK20] were not able to prove this assertion, but they arrived at a partial result that is based on the following observations:

Because G acts transitively on G/K and the causal pre-order is G -invariant, it is a partial order if and only if $(G/K)_o^+ \cap (G/K)_o^- = \{o\}$. To decide whether this is satisfied, let S_+ (resp. S_-) be the subsemigroup of G generated by K and A_+° (resp. A_-°) or, equivalently:

$$S_\pm := K \cup \bigcup_{N=1}^{\infty} (KA_\pm^\circ K)^N \quad (6.3.14)$$

Upon replacing A by A_\pm° in the first part of the proof of Proposition 6.3.10, the same argument can now be used to show that $(G/K)_o^\pm = S_\pm \cdot o$. Since S_+ and S_- contain K , these two subsemigroups can be equivalently described as

$$S_+ = \{g \in G : o \preceq g \cdot o\} \quad \text{and} \quad S_- = \{g \in G : g \cdot o \preceq o\}. \quad (6.3.15)$$

Hence, the causal pre-order is antisymmetric if and only if $S_+ \cap S_- = K$. Although it is not immediately apparent whether this equality holds, it can only fail for a trivial reason.

Proposition 6.3.17. *The causal pre-order on the Kac-Moody symmetric space $(G/K, \mu)$ satisfies exactly one of the following conditions:*

- (i) $S_+ = S_- = G$ and the causal pre-order is trivial.
- (ii) $S_+ \cap S_- = K$ and the causal pre-order is a partial order.

This result is already stated and proved in [FHHK20, Proposition 7.46], where the definitions of S_+ and S_- (resp. $(G/K)_o^+$ and $(G/K)_o^-$) appear to contain a minor error as they are not quite consistent with the assertions. For the reader’s convenience, we provide a self-contained and more explicit proof, but we follow the existing one very closely.

Proof. If the causal pre-order is not antisymmetric, i.e. if (ii) is not satisfied, there exists a point $x \in G/K$ such that $o \prec x \prec o$. Hence, we find $N_1 < N_2 \in \mathbb{N}$ and a finite sequence

$$o = x_0, x_1, \dots, x_{N_1-1}, x_{N_1} = x, x_{N_1+1}, \dots, x_{N_2-1}, x_{N_2} = x_0 = o$$

of points in G/K such that for every $k \in \{1, \dots, N_2\}$, there exists a causal geodesic segment from x_{k-1} to x_k . In particular, x_{N_2-1} lies in a maximal flat containing $x_{N_2} = o$, so there is some $l \in K$ with $y := l \cdot x_{N_2-1} \in F_0$ and a unique $h \in \mathfrak{a}$ with $y = \exp(-h)K$. It follows that y and $l \cdot o = o$ are both contained in F_0 , where they lie on a unique common geodesic which can be parametrized by the curve $\gamma : \mathbb{R} \rightarrow G/K$, $t \mapsto \exp(-th)K$. As there exists a causal geodesic segment from $l \cdot x_{N_2-1} = y = \gamma(1)$ to $l \cdot x_{N_2} = o = \gamma(0)$, this parametrization of the geodesic is anti-causal, which implies that $-h \in X_-^\circ$ and $h \in X_+^\circ$.

We now define $z := \gamma(\frac{1}{2}) = \exp(-\frac{h}{2})K \in F_0$ and consider the map $\varphi := s_z \circ s_o \in \text{Aut}(G/K, \mu)$. By the observations preceding Remark 6.3.7, the main group of $(G/K, \mu)$ is isomorphic to the semidirect product $G_{\text{eff}} \rtimes \langle s_o \rangle$ and the following computation holds:

$$\begin{aligned} \varphi &= s_z \circ s_o = (l_{\exp(-\frac{h}{2})} \circ s_o \circ l_{\exp(-\frac{h}{2})^{-1}}) \circ s_o = l_{\exp(-\frac{h}{2})} \circ (s_o \circ l_{\exp(\frac{h}{2})} \circ s_o) \\ &= l_{\exp(-\frac{h}{2})} \circ l_{\Theta(\exp(\frac{h}{2}))} = l_{\exp(-\frac{h}{2})} \circ l_{\exp(-\frac{h}{2})} = l_{\exp(-h)} \end{aligned}$$

Hence, φ is contained in $G_{\text{eff}} \subset \text{Aut}^+(G/K, \mu)$. It preserves the image of γ and acts on this set by $\varphi(\gamma(t)) = \gamma(t+1)$ for all $t \in \mathbb{R}$. As the causal pre-order is $\text{Aut}^+(G/K, \mu)$ -invariant and $o \prec y$ by transitivity, it follows that $\gamma(r) = \varphi^r(o) \prec \varphi^r(y) = \gamma(r+1)$ and thus $o \prec \gamma(r) = \exp(-rh)K$ for all $r \in \mathbb{N}$. We claim that this implies $F_0 \subset (G/K)_o^+$.

Let $x' \in F_0$ be arbitrary and let $h' \in \mathfrak{a}$ be the unique element with $x' = \exp(h')K$. Since $h \in X_+^\circ$, there exists some $w_1 \in W$ such that $w_1(h) \in \overline{C}$ and the set $J := \{i \in I : \langle \alpha_i, w_1(h) \rangle = 0\}$ is spherical. Thus, the positive Tits cone $X_{J,+}$ of the J -restriction of \mathcal{D} coincides with \mathfrak{a} , so there exists some $w_2 \in W_J$ such that $w := w_2 w_1$ satisfies $w(h') \in \overline{C_J}$, i.e. $\langle \alpha_j, w(h') \rangle \geq 0$ for all $j \in J$. As $\langle \alpha_i, w_1(h) \rangle > 0$ for all $i \in I \setminus J$, we may choose some sufficiently large $r \in \mathbb{N}$ to ensure that $v_r := r w_1(h) + w(h') \in \mathfrak{a}$ satisfies $\langle \alpha_i, v_r \rangle \geq 0$ for all $i \in I$, where equality is only possible if $i \in J$, so that v_r is contained in a spherical cell of \overline{C} and therefore in X_+° . Because w_2^{-1} stabilizes $w_1(h)$, we may rewrite h' as follows:

$$h' = w^{-1}w(h') = w^{-1}(v_r - r w_1(h)) = w^{-1}(v_r) - rh \in w^{-1}(X_+^\circ) - rh = X_+^\circ - rh$$

Hence, there exists a causal geodesic segment in F_0 from $\gamma(r) = \exp(-rh)K$ to $x' = \exp(h')K$, which implies that $o \prec \gamma(r) \prec x'$, so the claim is proved. Using that $A \cdot o = F_0 \subset (G/K)_o^+ = S_+ \cdot o$ and $K \subset S_+$, we now conclude that $A \subset S_+$. Therefore, S_+ contains $(KAK)^N$ for every $N \in \mathbb{N}$, so it coincides with G by Theorem 6.2.3 (ii). A similar argument shows that $S_- = G$. \square

If \mathbb{A} is spherical, we have already observed that the first case occurs because the causal structure is trivial, but we can also deduce this from the fact that $S_+ = S_- = KAK = G$ holds in this case. In order to prove that the causal pre-order is a partial order if \mathbb{A} is not spherical, it suffices to show that S_+ is a proper subset of G . In view of Assumption 6.3.13, this is exactly what we have achieved in Theorem 6.2.8 (ii), so we arrive at the following geometric result:

Theorem 6.3.18. *The causal pre-order on the Kac-Moody symmetric space $(G/K, \mu)$ satisfies exactly one of the following conditions:*

- (i) *If \mathbb{A} is spherical, then $S_+ = S_- = G$ and the causal pre-order is trivial.*
- (ii) *If \mathbb{A} is not spherical, then $S_+ \cap S_- = K$ and the causal pre-order is a partial order.* \square

Therefore, we have proved Theorem D, the last of our main results. It is likely that our version of Kostant's Non-Linear Convexity Theorem has many more applications to algebraic properties of Kac-Moody groups and that several other results that we have established in earlier sections have interesting geometric reformulations in the language of Kac-Moody symmetric spaces.

One noteworthy open question in this subject is whether the action of G on G/K is *primitive*. By definition, this holds if and only if there does not exist any non-trivial G -invariant equivalence relation on G/K . If such a relation exists, the stabilizer L of the equivalence class containing the base point o is a subgroup of G with $K \subsetneq L \subsetneq G$. Conversely, if such a subgroup exists, the G -translates of the L -orbit of o define a non-trivial G -invariant partition of G/K . Thus, G acts primitively on G/K if and only if K is a maximal subgroup of G .

If the action of G on G/K was primitive, the same would be true for the action of the main group $G_{\text{eff}} \rtimes \langle s_o \rangle$, so $(G/K, \mu)$ would be a *primitive symmetric space* in the sense of [Cap05]. The core result of [Cap05] states that the main group of a primitive symmetric space is either simple or a semidirect product of the form $H \rtimes \langle \sigma \rangle$, where H is a simple group and $\sigma : H \rightarrow H$ an outer involutive automorphism or $H = H' \times H'$ is the direct product of a non-abelian simple group H' with itself and $\sigma : H \rightarrow H, (h_1, h_2) \mapsto (h_2, h_1)$, is the canonical flip. Thus, if $(G/K, \mu)$ could be shown to be primitive, the classification of Caprace would shed light on the problem of simplicity of G_{eff} and hence on that of G itself.

The question whether (split, minimal) Kac-Moody groups over fields are simple is an important problem that is mostly open, but a few conditions are clearly necessary (cf. [Mar18, Section 9.1]):

- The subgroup $G' \subset G$ generated by the root groups $(U_\alpha)_{\alpha \in \Phi}$ is normal, so it should hold that $G' = G$. For example, this is always the case if \mathcal{D} is coadjoint, but never if $d > n$.
- Root groups belonging to different indecomposable components of \mathbb{A} commute. Therefore, the subgroups of G' generated by the root groups of the respective components are normal, so \mathbb{A} should be indecomposable.
- The center $Z' \subset G'$ may be non-trivial, so the question should be formulated for G'/Z' .

Over fields of characteristic zero, the problem of simplicity for G'/Z' is open (unless \mathbb{A} is of finite type where G' is a Chevalley group), but there do exist some partial results over finite fields. One notable further restriction is that \mathbb{A} should not be of affine type. This is because affine Kac-Moody groups are related to loop groups of the form $\hat{\mathcal{G}}(\mathbb{K}[t, t^{-1}])$, where $\hat{\mathcal{G}}$ is a Chevalley-Demazure group scheme, which admit non-trivial quotients induced by non-injective homomorphisms from $\mathbb{K}[t, t^{-1}]$ into other rings. Thus, it remains to study the indefinite type.

In the real and complex case, a major contribution to the question of simplicity would be to decide whether the compact form K is a maximal subgroup of G . As indicated at the end of Section 6.1, it seems that our convexity theorem might be a powerful tool to tackle this problem. Ideally, it would be desirable to classify all sub(semi-)groups of G containing K in a way that generalizes the known result for reductive Lie groups, but this is probably much more difficult. Investigating these questions in greater detail would go far beyond the scope of this thesis, but they definitely serve as interesting and promising topics for future research.

Chapter 7

Conclusion and Outlook

Although the principal goal of this thesis was merely the generalization of Kostant's Non-Linear Convexity Theorem to the class of split real or complex Kac-Moody groups, the pursuit of this objective naturally led to a vast variety of different questions and required tools from diverse areas of mathematics such as combinatorics, representation theory and convex geometry. Hence, not only the statement of the convexity theorem, which provides a useful tool to investigate algebraic properties of Kac-Moody groups, appears to be a valuable contribution to Kac-Moody theory, but also its proof may be of independent interest.

From the combinatorial perspective, the core of this thesis is the investigation of the convex hull $\Xi_h = \text{conv}(W \cdot h)$ of the Weyl group orbit of an element $h \in \mathfrak{a}$ under the \mathcal{D} -representation and the description of its face poset. While Ξ_h has a rather simple structure if h is regular, understanding the way in which faces degenerate in the singular case was the motivation to introduce coloured Coxeter diagrams and the notion of an exposing forest, which made it possible to describe the face poset of Ξ_h in the almost virtually singular case as well. To the best of the author's knowledge, this concept had not been introduced in the general theory of Coxeter groups earlier and may have further applications in this field.

For the proof of the convexity theorem, however, the combinatorial aspects of these results were not sufficient. Instead, it was crucial to understand the facial structure of Ξ_h also from a metric perspective in order to characterize this set by a family of weak linear inequalities. At that point, basic notions from convex geometry entered the discussion, but they had to be applied to convex hulls of infinite sets which are typically not studied in this way. Hence, if h is almost virtually singular, the combinatorial description of the face poset of Ξ_h is not implied by any general theory and the fact that Ξ_h is closed is non-trivial. Conversely, the result that Ξ_h is not closed if h is essentially singular is both surprising and a major obstruction to extending the convexity theorem beyond the almost virtually singular case.

While the investigation of Ξ_h was largely independent of the Kac-Moody context and often only required simple facts about Coxeter groups and arguments from linear algebra, the functionals appearing in the defining inequalities of Ξ_h could be viewed as the dominant integral weights of highest-weight representations of Kac-Moody groups. This observation provided the essential link between the “non-linear” group-theoretic property in the statement of the convexity theorem and the “linear” metric and combinatorial problems that were studied before and made it possible to prove one of the inclusions in Theorem A by a simple geometric argument.

The main ingredient in the proof of the converse inclusion was an inductive approach that reduced the problem to the SL_2 -case, where the result could be established by explicit matrix computations. This strategy was motivated by the author's attempt at the beginning of his research to find an alternative proof of Kostant's original convexity theorem for various semisimple matrix groups, such as $\text{SL}_n(\mathbb{K})$ or $\text{Sp}_{2n}(\mathbb{K})$, by reducing it to subgroups that are isomorphic to $\text{SL}_2(\mathbb{K})$. For every such matrix group $G = KAU_+$, the statement could be shown by an inductive argument, where the base case could be reduced to $\text{SL}_2(\mathbb{K})$ and the inductive step only required a sufficient condition for the conjugate of an upper-triangular matrix to remain upper-triangular in order to

compute the Iwasawa components of a product $ak_1k_2 \in G$ with $a \in A$ and $k_1, k_2 \in K$ by iteration. In general, such a condition is provided by the Levi decompositions of parabolic subgroups. In the inductive step, it was also necessary to consider subgroups of G that were only reductive instead of semisimple. These two observations are the reason why the class of real or complex points of connected reductive algebraic \mathbb{R} -groups is particularly tailored to satisfy a variant of Kostant's Non-Linear Convexity Theorem that, in contrast to its Lie-theoretic predecessor, does not require any analytic arguments. Therefore, the results of Section 5.5.2 about reductive algebraic \mathbb{R} -groups were the author's first steps towards the Kac-Moody case and greatly inspired the general proof. In such finite-dimensional examples, it was also possible to visualize the convex polytope Ξ_h and these images suggested that its defining hyperplanes were level sets of the fundamental weights of the finite root system associated to G .

Conversely, in the spherical case, Theorem A reduces to a close analogue of Kostant's original convexity theorem and its proof provides an alternative approach that exploits combinatorial properties of Lie groups instead of their analytic structure. In addition, the same arguments could be applied almost verbatim to generalize Kostant's Linear Convexity Theorem to the Kac-Moody realm as well, which suggests that both results are perhaps instances of a more general unifying convexity property.

Altogether, this thesis highlights that algebraic and combinatorial methods are extremely powerful tools in Kac-Moody theory and often the best substitute for the analytic arguments that are typical to Lie theory but not available in this generalized setting. Besides the results of Chapter 6, Kostant's Non-Linear Convexity Theorem is likely to have many more applications to algebraic properties of Kac-Moody groups and the considerations of this thesis may be relevant for other problems in this subject as well. As a complement to the principal objective, especially in Part I, the author has taken the liberty of providing explicit proofs for minor results about Kac-Moody groups & algebras that appear to be well-known to experts in the field, but for which an exact reference is perhaps difficult to find. Hence, some parts of this thesis might also serve as useful footnotes to the existing literature about Kac-Moody theory in general.

Despite having successfully generalized both of Kostant's Convexity Theorems to the Kac-Moody setting, the results of this thesis have some noteworthy shortcomings and the discussion leads to some natural open questions that might provide interesting topics for further research. The most obvious problem concerns the statements of the main theorems.

Question 7.1. To what extent is it possible to weaken the assumptions in Kostant's Convexity Theorems for Kac-Moody groups? In particular, can they be generalized to points in \mathfrak{a} that are not almost virtually singular?

On the one hand, the proofs of Theorem 5.1.1 and 5.3.1 crucially relied on two assumptions: First, the GCM \mathbb{A} was required to be symmetrizable. Second, the point $h \in \mathfrak{a}$ had to be almost virtually singular. On the other hand, the statements themselves do not need these restrictions, which raises the question whether they actually hold in greater generality by a different proof. The methods of this thesis are probably not suitable to study this problem because Ξ_h is not closed in the essentially singular case and its face poset appears to contain little to no information in the critical or external case. At the moment, however, this assertion is merely motivated by the examples in Section 4.4 where no general results have been established. Hence, the following task may nevertheless be of independent interest:

Problem 7.2. Investigate the structure of Ξ_h if $h \in \mathfrak{a}$ is not almost virtually singular in more detail. In particular, study the situation where the underlying GCM \mathbb{A} is of affine type and $h \in \mathfrak{a}$ is critical.

Although it is rather unlikely that results on this problem allow for any further generalizations of Theorem 5.1.1 and 5.3.1, it would be interesting to explain the phenomena of Section 4.4 in a uniform way and establish rigorous proofs. In the essentially singular case, Ξ_h is not closed and thus not determined by a family of weak inequalities, but perhaps it can be characterized by adding certain strict inequalities. If such a description exists, it might actually be possible to

extend the convexity theorems to this situation. Likewise, if A is of affine type and h is critical, then Ξ_h appears to be closed, but the defining hyperplanes are very different from the almost virtually singular case in the sense that they need not be level sets of dominant integral weights. A precise explanation of these phenomena might also lead to generalizations of the main results of this thesis and provide further applications to affine Kac-Moody groups.

Question 7.3. Is Kostant’s Non-Linear Convexity Theorem an efficient tool in order to classify sub(semi-)groups of a real or complex Kac-Moody group G that contain its compact form K ? Under which conditions is K a maximal subgroup of G ?

The findings of Chapter 6 show that Theorem A is occasionally useful to decide whether certain subsets of a Kac-Moody group are sub(semi-)groups. In particular, this works well if the subset contains the compact form $K \subset G$. It is likely that the full strength of this method has not been exploited yet and that further results on the structure of sub(semi-)groups of G containing K are within reach. For example, it was suggested to the author that the question whether KA_+K and $KA_+^\circ K$ are subsemigroups of G is relevant for applications of Kac-Moody groups to theoretical physics. Ideally, it would be desirable to obtain a complete classification of sub(semi-)groups of G containing K that generalizes the analogous result for reductive Lie groups. While this might be a rather challenging task, an interesting simpler question is to decide under which conditions the compact form K is maximal as an abstract subgroup of G , which is known to be the case for simple Lie groups. This is closely related to the important open problem of simplicity of real and complex Kac-Moody groups.

Question 7.4. Is it possible to formulate and prove analogues of Kostant’s Convexity Theorems for other types of Kac-Moody groups, e.g. non-split or maximal groups?

Throughout this thesis, every Kac-Moody group G was “split” in the sense that its torus T was isomorphic to $(\mathbb{K}^*)^d$ for some $d \in \mathbb{N}$ and “minimal” as the definition of G only involved root groups associated to real roots, whereas the imaginary roots were not incorporated on the group level. A theory of Galois descent for Kac-Moody groups, which allows for the definition of “non-split” groups, was developed by Rémy (cf. [Rém02]). Moreover, there exist various notions of “maximal” Kac-Moody groups that arise as certain completions of the minimal ones (cf. [Mar18, Chapter 8]). Both generalizations are significantly more involved than the split and minimal groups and have not yet been investigated by the author. In view of Kostant’s original results and their analogues for connected reductive algebraic \mathbb{R} -groups in Section 5.5.2, however, it is tempting to conjecture that suitable variants of Kostant’s Convexity Theorems should also hold for real and complex Kac-Moody groups that are not necessarily split.

Question 7.5. Does there exist a connection between Kostant’s Non-Linear Convexity Theorem for semisimple Lie groups, its analogue in the Kac-Moody setting and Schwer’s Convexity Theorem for affine buildings?

As discussed in the historical introduction, the original versions of Kostant’s Convexity Theorems for semisimple Lie groups were transferred in a multitude of ways to diverse areas of mathematics. This thesis contributes yet another such generalization to Kac-Moody theory. The strategy in the proof of Theorem 5.1.1 bears some resemblance to Kostant’s own proof in [Kos73, Theorem 4.1] and to Schwer’s proof of her convexity theorem for affine buildings in [Hit10]. All three approaches contain a representation-theoretic part and an inductive argument, although the details of these steps are very different in the three settings. The proof of Kostant’s Linear Convexity Theorem for Kac-Moody groups is almost identical to the non-linear case and follows this pattern as well. This observation suggests that all these results might have a common core and perhaps arise as different manifestations of a more abstract unifying convexity theorem. While investigating this relationship in detail appears to be a difficult task, any results in this direction would surely serve as major contributions to the theory of Lie and Kac-Moody groups over both Archimedean and non-Archimedean local fields and provide deeper insights into the question which structures are fundamental in these subjects and how they are linked with each other.

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List of Symbols

$\langle \cdot, \cdot \rangle$	dual pairing of a vector space or free \mathbb{Z} -module	xviii
\prec, \preceq	causal pre-order on $(G/K, \mu)$	168
\sim	equivalence relation on $\Sigma^*(W, S)$ defined in (4.3.19)	104
$] \alpha, \beta[_{\mathbb{N}}$	open interval between $\alpha, \beta \in \Phi$	48
A	$\exp(\mathfrak{a})$	58
A'	$\exp(\mathfrak{a}')$	155
$A_+, A_-, A_+^\circ, A_-^\circ$	$\exp(X_+), \exp(X_-), \exp(X_+^\circ), \exp(X_-^\circ)$	156
$\text{Aut}(\mathcal{X}, \mu)$	automorphism group of (\mathcal{X}, μ)	162
$\text{aff}(P)$	affine span of a convex set P	82
a_0, \dots, a_n	labels of the vertices in an affine Dynkin diagram	15
$a(r)$	$h(r, r^{-1})$ for $r \in \mathbb{R}_{>0}$	66
$\mathbb{A} = (a_{ij})_{i,j \in I}$	generalized Cartan matrix	13
$\mathring{\mathbb{A}} = (a_{ij})_{i,j \in \mathring{I}}$	reduced matrix of a GCM \mathbb{A} of affine type	15
\mathfrak{a}	$\Lambda^\vee \otimes_{\mathbb{Z}} \mathbb{R}$	35
\mathfrak{a}'	$\sum_{i \in I} \mathbb{R} h_i \subset \mathfrak{a}$	35
B	$\{(z_1, z_2) \in \mathbb{K}^2 : z_1 ^2 + z_2 ^2 = 1\}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$	66
B_+, B_-	positive/negative Borel subgroup of G	49
$\mathcal{B}, \mathcal{B}^*$	dual bases of \mathfrak{a} and \mathfrak{a}^*	45
\mathcal{B}_Φ	$\{e_\alpha \mid \alpha \in \Phi\}$	34
χ	$\sum_{i \in I_{\text{vis}}} \omega_i \in \mathfrak{a}^*$	112
C, \overline{C}	open/closed fundamental chamber	35
$C_K(\Omega)$	$\{k \in K : kp = pk \ \forall p \in \Omega\}$	164
$C(W_J), C_h(W_J)$	centralizer of W_J in W (resp. W_h)	90
$(C_x^+)_{x \in G/K}, (C_x^-)_{x \in G/K}$	positive/negative causal structure on $(G/K, \mu)$	168
c	character map $Q \rightarrow \Lambda$	22
c_α	$c(\alpha) \in \Lambda$	22
$\text{conv}(Y)$	convex hull of a set Y	84
$\Delta = \Delta(W, S)$	Coxeter diagram of (W, S)	4
Δ°	exposing forest for Δ	7
$\delta(i)$	minimal distance in Δ from $i \in I$ to a vertex in $I \setminus I^\bullet$	7
δ_{ij}	Kronecker delta	xviii
D_{2m}	finite dihedral group of order $2m$	4
D_∞	infinite dihedral group	4
D_J	cell of \overline{C} of type J	37
$\mathcal{D} = (I, \mathbb{A}, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$	Kac-Moody root datum	19
$\mathcal{D}_{\text{Kac}}^\mathbb{A}, \mathcal{D}_{\text{sc}}^\mathbb{A}, \mathcal{D}_{\text{min}}^\mathbb{A}$	Kac/simply connected/minimal adjoint root datum of \mathbb{A}	19
$\partial_{\text{rel}}(P)$	relative boundary of a convex set P	82
E_h	affine span of Ξ_h	88
$E^{(1)}, \dots, E^{(n)}$	maximal supporting hyperplanes of Ξ_h	89
$E^{(i)}(t)$	$\{y \in E_h : \langle \omega_i, y \rangle = t\}$	112
$E_+^{(i)}(t)$	$\{y \in E_h : \langle \omega_i, y \rangle \geq t\}$	112
$E_+(t)$	$\{y \in E_h : \langle \chi, y \rangle \geq t\}$	112

$e_1, f_1, \dots, e_n, f_n$	Chevalley generators of \mathfrak{g}	23
\exp	exponential map $\mathfrak{a} \rightarrow A$	58
\mathbb{E}^l	l -dimensional Euclidean symmetric space	163
F_0	$A \cdot o = \{aK \mid a \in A\} \subset G/K$	165
F_J	$\text{conv}(W_J \cdot h) \subset \Xi_h$	86
$\mathcal{F}(P)$	face poset of a convex set P	82
$\Gamma(\mathbb{A})$	Dynkin diagram of \mathbb{A}	14
$G = G_{\mathcal{D}}(\mathbb{K})$	Kac-Moody group of type \mathcal{D} over \mathbb{K}	49
G^0	$\text{SL}_2(\mathbb{K})$	57
G_α	G^0 -image: $\varphi_\alpha(G^0) = \langle U_\alpha, U_{-\alpha} \rangle \cong (\text{P})\text{SL}_2(\mathbb{K})$	57
G_{eff}	$G/C_K(\Omega)$	164
G_J	Levi factor of P_J	51
$(G/K, \mu)$	Kac-Moody symmetric space	164
$(G/K)_x^+, (G/K)_x^-$	causal future/past of $x \in G/K$	168
\mathcal{G}	connected reductive algebraic \mathbb{K} -group	52
$\mathfrak{g} = \mathfrak{g}_{\mathcal{D}}(\mathbb{K})$	Kac-Moody algebra of type \mathcal{D} over \mathbb{K}	23
\mathfrak{g}^0	$\mathfrak{sl}_2(\mathbb{K})$	57
\mathfrak{g}_α	$\varphi'_\alpha(\mathfrak{g}^0) = \mathbb{K}e_\alpha \oplus \mathbb{K}h_\alpha \oplus \mathbb{K}f_\alpha \cong \mathfrak{sl}_2(\mathbb{K})$	34
H	contravariant symmetric bilinear or Hermitian form on $L(\lambda)$	45
H_1, \dots, H_n	walls of \bar{C}	37
h	cocharacter map $Q^\vee \rightarrow \Lambda^\vee$	22
h_α	$h(\alpha^\vee) \in \Lambda^\vee$	22
$h(r, s)$	$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$ for $r, s \in \mathbb{K}^*$	56
$\text{ht}(\alpha)$	height of $\alpha \in Q$ or Q^\vee	22
$\mathfrak{h} = \mathfrak{h}_{\mathcal{D}}(\mathbb{K})$	Cartan subalgebra of \mathfrak{g}	23
ι_K^J	$[W_{K^*} : W_{K^* \cap (J \cup J^*)}]$	106
I	$\{1, \dots, n\}$	xviii
$\overset{\circ}{I}$	see Convention 2.1.2	15
I^\bullet	set of coloured vertices of Δ	6
$I_{\text{vis}}, I_{\text{inv}}$	visible/invisible vertices of I	6
$I_{\text{deg}}, I_{\text{nd}}$	degenerate/non-degenerate vertices of I	10
I_h	$\{i \in I : \langle \alpha_i, h \rangle = 0\}$	85
I_∞	union of non-spherical connected components of I	6
$I(\alpha)$	support of $\alpha \in Q$ or Q^\vee	22
$I(w)$	support of $w \in W$	4
\mathbb{I}_n	$(n \times n)$ -identity matrix	xviii
J^\perp	$\{i \in I : a_{ij} = 0 \ \forall j \in J\}$	90
J^*	$J^\perp \cap I_h$	90
κ_A, ξ_A, η_A	see (3.5.4) and (3.5.5)	66
κ_U, ξ_U, η_U	see (3.5.7) and (3.5.8)	68
K	compact form of G	58
K_α	K^0 -image: $\varphi_\alpha(K^0)$	59
$k(z_1, z_2)$	$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$ for $(z_1, z_2) \in B$	66
\mathbb{K}	field of characteristic zero	xviii
\mathbb{K}^*	$\mathbb{K} \setminus \{0\}$	xviii
$\mathbb{K}[t, t^{-1}]$	ring of Laurent polynomials with coefficients in \mathbb{K}	28
\mathfrak{k}	compact form of \mathfrak{g}	45
Λ, Λ^\vee	dual free \mathbb{Z} -modules	xviii
$L(\lambda)$	irreducible highest-weight module with highest weight $\lambda \in \mathfrak{h}^*$	31
\log	logarithm $A \rightarrow \mathfrak{a}$	58
$l(w)$	word length of $w \in W$ with respect to S	4
ℓ	$\text{rank}(\mathbb{A})$	14
$M = (m_{ij})_{i,j \in I}$	Coxeter matrix of (W, S)	3
$M \subset G$	complement of A in $T \cong M \times A$	58

$M(\lambda)$	Verma module with highest weight $\lambda \in \mathfrak{h}^*$	31
Ω	$\{g\Theta(g^{-1}) \mid g \in G\}$	163
ω	Chevalley involution of \mathfrak{g}	25
$\omega_1, \dots, \omega_n$	fundamental weights	45
o	$1 \cdot K \in G/K$	164
$\Phi_0 = \{\alpha_1, \dots, \alpha_n\}$	set of simple roots	22
$\Phi_0^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$	set of simple coroots	22
$\Phi^\infty, \Phi_+^\infty, \Phi_-^\infty$	set of all/positive/negative roots	24
Φ, Φ_+, Φ_-	set of all/positive/negative real roots	33
$\Phi^\vee, \Phi_+^\vee, \Phi_-^\vee$	set of all/positive/negative real coroots	33
$\Phi^{\text{im}}, \Phi_+^{\text{im}}, \Phi_-^{\text{im}}$	set of all/positive/negative imaginary roots	33
$\Phi_{J,+}^\perp$	$\{\alpha \in \Phi_+ : I(\alpha) \not\subset J\}$	34
Φ_w	$\{\alpha \in \Phi_+ : w(\alpha) \in \Phi_-\}$	33
Π_K, Π_A, Π_{U_+}	Iwasawa projections in $G = KAU_+$	62
Π_a	$\Pi_A(aK)$ for $a \in A$	130
φ_α	group homomorphism $G^0 \rightarrow G_\alpha \subset G$	57
φ'_α	Lie algebra homomorphism $\mathfrak{g}^0 \rightarrow \mathfrak{g}_\alpha \subset \mathfrak{g}$	34
$\pi_{\mathfrak{k}}, \pi_{\mathfrak{a}}, \pi_{\mathfrak{u}_+}$	Iwasawa projections in $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}_+$	60
π_h	$\pi_{\mathfrak{a}}(\text{Ad}(K)h)$ for $h \in \mathfrak{a}$	138
$\pi_J^l(w), \pi_J^r(w), \pi_{J,K}(w)$	elements of minimal length in $W_J w, wW_J, W_J wW_K$	5
ψ	order-preserving map $\Sigma^*(W, S) \rightarrow \mathcal{F}(\Xi_h), wW_J \mapsto w(F_J)$	90
$\tilde{\psi}$	poset isomorphism $\tilde{\Sigma}^*(W, S) \rightarrow \mathcal{F}(\Xi_h)$ induced by ψ	104
$P^{(i)}(t)$	$\Xi_h \cap E^{(i)}(t)$	112
$P_{\text{ess}}^{(i)}(t)$	essential part of $P^{(i)}(t)$	118
P_J	standard parabolic subgroup of G of type J	51
$\mathcal{P}_{\text{vis}}(I)$	set of visible subsets of I	9
Q, Q_+, Q_-	(positive/negative) root lattice	22
$Q^\vee, Q_+^\vee, Q_-^\vee$	(positive/negative) coroot lattice	22
R	set of reflections of (W, S)	3
R_γ	$k(\cos(\gamma), \sin(\gamma))$ for $\gamma \in [0, 2\pi)$	66
$\text{relint}(P)$	relative interior of a convex set P	82
r^h	group homomorphism $\Lambda \rightarrow \mathbb{K}^*, \lambda \mapsto r^{\langle \lambda, h \rangle}$	48
$\Sigma(W, S)$	Coxeter complex of (W, S)	5
$\Sigma^*(W, S)$	reversed Coxeter complex of (W, S)	5
$\tilde{\Sigma}^*(W, S)$	$\Sigma^*(W, S)/\sim$	104
$S = \{s_1, \dots, s_n\}$	set of simple reflections	3
S_n	symmetric group on n letters	4
$S(w)$	simple factors of $w \in W$	4
S_+, S_-	subsemigroup of G generated by K and A_+° (resp. A_-°)	169
$\text{St}(\mathbb{K})$	Steinberg group over \mathbb{K}	48
Θ	compact involution of G	58
θ	compact involution of \mathfrak{g}	45
τ	type function $\mathcal{F}(\Xi_h) \rightarrow \mathcal{P}_{\text{vis}}(I), w(F_J) \mapsto J_{\text{vis}}$	95
$T = T_\Lambda(\mathbb{K})$	\mathbb{K} -split torus	48
$T_h^{(i)}$	$\{t \in \mathbb{R} : P^{(i)}(t) \neq \emptyset\}$	112
\mathcal{T}	maximal \mathbb{K} -split torus of \mathcal{G}	52
U_α	root group of $\alpha \in \Phi$	48
U_+, U_-	positive/negative unipotent subgroup of G	49
$U_{J,+}^\perp$	unipotent radical of P_J	51
$u(z)$	$x_+(z)$ for $z \in \mathbb{K}$	66
$\mathcal{U}(\mathfrak{g})$	universal enveloping algebra of \mathfrak{g}	31
\mathfrak{u}_α	root space of $\alpha \in \Phi^\infty$	24
$\mathfrak{u}_+, \mathfrak{u}_-$	direct sum of positive/negative root spaces of \mathfrak{g}	25
V, V^*	dual \mathbb{K} -vector spaces	xviii

(V, ρ)	\mathfrak{g} -module	30
V_μ	weight space of $\mu \in \text{wt}(V, \rho)$	30
(W, S)	Coxeter system	3
W_J, wW_J, W_Jw	standard subgroup/left coset/right coset of type J	4
$W = W(\mathbb{A})$	Weyl group of \mathbb{A}	32
W_h	stabilizer of $h \in \mathfrak{a}$ under the W -action	37
W^*	subgroup of $\text{Aut}(\mathfrak{g})$ generated by $\{w^* \mid w \in W\}$	34
\tilde{W}	extended Weyl group	49
w^*	extension of $w \in W$ to an automorphism of \mathfrak{g}	34
\tilde{w}	canonical representative of $w \in W$ in \tilde{W}	50
w_+	longest word of a finite Coxeter group	5
$\text{wt}(V, \rho)$	set of weights of (V, ρ)	30
Ξ_h	$\text{conv}(W \cdot h)$ for $h \in \mathfrak{a}$	80
X_+, X_-	positive/negative Tits cone	37
X_+°, X_-°	topological interior of X_+ (resp. X_-)	38
$X(\mathcal{T})$	character group of \mathcal{T}	52
$x_\alpha(z)$	element of U_α for $z \in \mathbb{K}$	48
$x_+(z), x_-(z)$	$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}$ for $z \in \mathbb{K}$	56
$[x, y], (x, y), [x, y), (x, y]$	closed/open/half-open line segment between x and y	82
(\mathcal{X}, μ)	symmetric space	162
Z	center of G	50
\mathfrak{z}	center of \mathfrak{g}	25

Bibliography

- [AB08] Peter Abramenko and Kenneth S. Brown. *Buildings: Theory and applications*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008.
- [Ati82] Michael F. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.
- [BB05] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.
- [Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 1991.
- [BP02] Stephen Berman and Karen Hunger Parshall. Victor Kac and Robert Moody: their paths to Kac-Moody Lie algebras. *Math. Intelligencer*, 24(1):50–60, 2002.
- [Bri96] Brigitte Brink. On centralizers of reflections in Coxeter groups. *Bull. London Math. Soc.*, 28(5):465–470, 1996.
- [BvdB16] Dana Bălibanu and Erik P. van den Ban. Convexity theorems for semisimple symmetric spaces. *Forum Math.*, 28(6):1167–1204, 2016.
- [Cap05] Pierre-Emmanuel Caprace. Primitive symmetric spaces. *Bull. Belg. Math. Soc. Simon Stevin*, 12(3):321–328, 2005.
- [Cap09] Pierre-Emmanuel Caprace. “Abstract” homomorphisms of split Kac-Moody groups. *Mem. Amer. Math. Soc.*, 198(924):xvi+84, 2009.
- [Deo89] Vinay V. Deodhar. A note on subgroups generated by reflections in Coxeter groups. *Arch. Math.*, 53(6):543–546, 1989.
- [DMGH09] Tom De Medts, Ralf Gramlich, and Max Horn. Iwasawa decompositions of split Kac-Moody groups. *J. Lie Theory*, 19(2):311–337, 2009.
- [Dui84] Johannes J. Duistermaat. On the similarity between the Iwasawa projection and the diagonal part. *Mémoires de la Société Mathématique de France*, 15:129–138, 1984.
- [Dye90] Matthew Dyer. Reflection subgroups of Coxeter systems. *J. Algebra*, 135(1):57–73, 1990.
- [Ebe96] Patrick B. Eberlein. *Geometry of nonpositively curved manifolds*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1996.
- [FHHK20] Walter Freyn, Tobias Hartnick, Max Horn, and Ralf Köhl. Kac-Moody symmetric spaces. *Münster J. Math.*, 13(1):1–114, 2020.
- [Gru07] Peter M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren der mathematischen Wissenschaften*. Springer, Berlin, 2007.

- [GS82] Victor Guillemin and Shlomo Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.
- [GS84] Victor Guillemin and Shlomo Sternberg. Convexity properties of the moment mapping. II. *Invent. Math.*, 77(3):533–546, 1984.
- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.
- [Hec80] Gert J. Heckman. Projections of orbits and asymptotic behaviour of multiplicities for compact Lie groups, 1980. Ph.D. University of Leiden.
- [Hel78] Sigurður Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press, Inc., New York-London, 1978.
- [Hit10] Petra Hitzelberger. Kostant convexity for affine buildings. *Forum Math.*, 22(5):959–971, 2010.
- [Hit11] Petra Hitzelberger. Non-discrete affine buildings and convexity. *Adv. Math.*, 227(1):210–244, 2011.
- [HN12] Joachim Hilgert and Karl-Hermann Neeb. *Structure and geometry of Lie groups*. Springer Monographs in Mathematics. Springer, New York, 2012.
- [Hor54a] Alfred Horn. Doubly stochastic matrices and the diagonal of a rotation matrix. *Amer. J. Math.*, 76:620–630, 1954.
- [Hor54b] Alfred Horn. On the eigenvalues of a matrix with prescribed singular values. *Proc. Amer. Math. Soc.*, 5:4–7, 1954.
- [Hor17] Max Horn. Decompositions of Kac-Moody groups, 2017. arXiv:1708.05566v1, <https://arxiv.org/abs/1708.05566v1>.
- [Hos02] Tetsuya Hosaka. Parabolic subgroups of finite index in Coxeter groups. *J. Pure and Appl. Algebra*, 169(2-3):215–227, 2002.
- [Hum75] James E. Humphreys. *Linear algebraic groups*, volume 21 of *Graduate Texts in Mathematics*. Springer, New York-Heidelberg, 1975.
- [Kac90] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.
- [Kir84] Frances Kirwan. Convexity properties of the moment mapping. III. *Invent. Math.*, 77(3):547–552, 1984.
- [Kna02] Anthony W. Knap. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.
- [Kos73] Bertram Kostant. On convexity, the Weyl group and the Iwasawa decomposition. *Ann. Sci. École Norm. Sup. (4)*, 6:413–455, 1973.
- [KP84] Victor G. Kac and Dale H. Peterson. Unitary structure in representations of infinite-dimensional groups and a convexity theorem. *Invent. Math.*, 76(1):1–14, 1984.
- [KP85] Victor G. Kac and Dale H. Peterson. Defining relations of certain infinite-dimensional groups. *Astérisque*, (S131):165–208, 1985.
- [Kum02] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*, volume 204 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2002.

- [KV24] Ralf Köhl and Christian Vock. Kac-Moody symmetric spaces: arbitrary symmetrizable complex or almost split real type, 2024. arXiv:2309.02176v3, <https://arxiv.org/abs/2309.02176v3>.
- [Lee18] John M. Lee. *Introduction to Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2018.
- [Loo69] Ottmar Loos. *Symmetric spaces. I: General theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1969.
- [LR91] Jiang-Hua Lu and Tudor Ratiu. On the nonlinear convexity theorem of Kostant. *J. Amer. Math. Soc.*, 4(2):349–363, 1991.
- [Mar18] Timothée Marquis. *An introduction to Kac-Moody groups over fields*. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2018.
- [Max89] George Maxwell. Wythoff’s construction for Coxeter groups. *J. Algebra*, 123(2):351–377, 1989.
- [Mor88] Jun Morita. Root strings with three or four real roots in Kac-Moody root systems. *Tohoku Math. J. (2)*, 40(4):645–650, 1988.
- [MP95] Robert V. Moody and Arturo Pianzola. *Lie algebras with triangular decompositions*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1995.
- [Nee94] Karl-Hermann Neeb. A convexity theorem for semisimple symmetric spaces. *Pacific Journal of Mathematics*, 162:305–349, 1994.
- [PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994.
- [Rém02] Bertrand Rémy. Groupes de Kac-Moody déployés et presque déployés. *Astérisque*, 277:viii+348, 2002.
- [Roc70] R. Tyrrell Rockafellar. *Convex analysis*, volume 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.
- [Sch90] Rudolf Scharlau. Geometrical realizations of shadow geometries. *Proc. London Math. Soc. (3)*, 61(3):615–656, 1990.
- [Spr09] Tonny A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.
- [Ste68] Robert Steinberg. *Lectures on Chevalley groups*. Yale University, New Haven, CT, 1968. Notes prepared by John Faulkner and Robert Wilson.
- [Tam97] Tin-Yau Tam. Kostant’s convexity theorem and the compact classical groups. *Linear and Multilinear Algebra*, 43(1-3):87–113, 1997.
- [Tho72] Colin J. Thompson. Inequalities and partial orders on matrix spaces. *Indiana Univ. Math. J.*, 21:469–480, 1972.
- [Tit74] Jacques Tits. *Buildings of spherical type and finite BN-pairs*, volume 386 of *Lecture Notes in Mathematics*. Springer, Berlin-New York, 1974.
- [Tit87] Jacques Tits. Uniqueness and presentation of Kac-Moody groups over fields. *J. Algebra*, 105(2):542–573, 1987.
- [vdB86] Erik P. van den Ban. A convexity theorem for semisimple symmetric spaces. *Pacific J. Math.*, 124(1):21–55, 1986.

Eidesstattliche Erklärung

Hiermit erkläre ich, dass die vorliegende Dissertation mit dem Titel *Kostant Convexity for split real and complex Kac-Moody groups* die Ergebnisse meiner eigenen Forschung darstellt. Jegliche Hilfe und Beratung durch meinen Betreuer, Prof. Dr. Ralf Köhl, beschränkte sich auf unterstützende Anmerkungen und fachliche Ratschläge.

Des Weiteren versichere ich, dass diese Dissertation weder in ihrer Gesamtheit noch in Teilen bereits Gegenstand eines anderen Prüfungsverfahrens war. Einige Ergebnisse der vorliegenden Dissertation sind in folgendem Artikel enthalten:

Paul Zellhofer und Ralf Köhl, *On convexity and the Iwasawa decomposition of split real and complex Kac-Moody groups* (2024).
(Preprint, verfügbar auf [arXiv:2312.02584](https://arxiv.org/abs/2312.02584), zur Publikation eingereicht)

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Die vorliegende Arbeit wurde unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft verfasst. Alle Quellen und Hilfsmittel, die ich verwendet habe, sind korrekt und vollständig angegeben. Alle Abbildungen wurden von mir persönlich angefertigt.

Ich versichere darüberhinaus, dass mir kein akademischer Grad entzogen wurde.

Kiel, 15. April 2025

Paul Zellhofer