



# On hyperplane sections and projections in $l_p^n$

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## Abstract

For  $2 < p < p_0 \simeq 26.265$ , the hyperplane section of the  $l_p^n$ -unit ball  $B_p^n$  perpendicular to  $a^{(n)} = \frac{1}{\sqrt{n}}(1, \dots, 1)$  for large  $n$  has larger volume than the one orthogonal to  $a^{(2)} = \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0)$ , as shown by Oleszkiewicz. This is different from the case of  $l_\infty^n$  considered by Ball. We give a quantitative estimate for which dimensions  $n$  this happens, namely for  $n > c(\frac{1}{p_0-p} + \frac{1}{p-2})$  for some absolute constant  $c > 0$ . Correspondingly for projections of  $B_q^n$  onto hyperplanes, Barthe and Naor showed that projections onto hyperplanes perpendicular to  $a^{(n)}$  have smaller volume for large  $n$  than onto the one orthogonal to  $a^{(2)}$ , if  $\frac{4}{3} < q < 2$ , different from the case  $q = 1$ . We show that this happens for all  $n > 5(\frac{1}{q-\frac{4}{3}} + \frac{1}{2-q})$ .

**Keywords** Volume · Hyperplane sections · Projections ·  $l_p^n$ -ball · Random variables

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## 1 Introduction and main results

In a well-known paper Ball [2] proved that the hyperplane section of the  $n$ -cube  $B_\infty^n := [-1, 1]^n$  perpendicular to  $a^{(2)} = \frac{1}{\sqrt{2}}(1, 1, 0, \dots, 0) \in S^{n-1} \subset \mathbb{R}^n$  has maximal volume among all hyperplane sections. Earlier Hadwiger [6] and Hensley [7] had shown independently of one another that coordinate hyperplanes, e.g. orthogonal to  $a^{(1)} = (1, 0, \dots, 0) \in S^{n-1}$ , yield the minimal  $(n-1)$ -dimensional cubic sections.

For  $0 < p < \infty$ , let  $B_p^n := \{x = (x_j)_{j=1}^n \in \mathbb{R}^n \mid \|x\|_p := (\sum_{j=1}^n |x_j|^p)^{\frac{1}{p}} \leq 1\}$  be the unit ball of  $l_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ . Meyer and Pajor [13] found extremal sections of the  $l_p^n$  balls  $B_p^n$ : They proved that the normalized volume of sections of  $B_p^n$  by a fixed

Dedicated to Bill Johnson on the occasion of his 80th birthday.

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hyperplane is monotone increasing in  $p$ . This implies that coordinate planes provide the minimal sections for  $2 \leq p < \infty$ , as for  $p = \infty$ , and the maximal sections for  $1 \leq p \leq 2$ . The minimal hyperplane sections of  $B_1^n$  are those orthogonal to a main diagonal, e.g.  $a^{(n)} = \frac{1}{\sqrt{n}}(1, \dots, 1) \in S^{n-1}$ , see also [13]. Koldobsky [10] extended this to the full range  $1 \leq p \leq 2$ .

This left open the case of the maximal hyperplane section of  $B_p^n$  for  $2 < p < \infty$ . The situation there is more complicated, since the maximal hyperplane may depend as well on  $p$  as on the dimension  $n$ : Oleszkiewicz [14] proved that Ball's result does not transfer to the balls  $B_p^n$  if  $2 < p < p_0 \simeq 26.265$ : then the intersection of the hyperplane perpendicular to  $a^{(n)}$  has larger volume than the one orthogonal to  $a^{(2)}$ , for sufficiently large dimensions  $n$ . The value  $p_0$  is the unique value in  $(2, \infty)$  such that  $\lim_{n \rightarrow \infty} \frac{\text{vol}_{n-1}(B_p^n \cap (a^{(n)})^\perp)}{\text{vol}_{n-1}(B_p^n \cap (a^{(2)})^\perp)} > 1$  for all  $2 < p < p_0$ . Oleszkiewicz's result is an asymptotic one, not determining dimensions  $n$  for which this happens. We derive a quantitative estimate for dimensions  $n$  such that this holds, namely for  $n > c \left( \frac{1}{p_0-p} + \frac{1}{p-2} \right)$ . On the other hand, recently Eskenazis, Nayar and Tkocz [5] proved that Ball's result is stable for  $l_p^n$  and very large  $p$ :  $(a^{(2)})^\perp \cap B_p^n$  is the maximal hyperplane section of  $B_p^n$  for all dimensions, provided that  $p_1 := 10^{15} \leq p < \infty$ . They call it "resilience of cubic sections".

Dual to hyperplane sections of convex bodies are projections of convex bodies onto hyperplanes. The known results for  $l_p^n$ -balls show a duality between sections and projections, when maximal and minimal directions  $a$  and  $p$  and the conjugate index  $q = \frac{p}{p-1}$  are interchanged. Nevertheless the proofs in both situations are different, since volume does not behave well under duality. Barthe and Naor [3] determined the extremal hyperplane projections of  $l_q^n$ -balls except for the minimal hyperplane projections when  $1 < q < 2$ , corresponding to the dual maximal section case mentioned above when  $2 < p < \infty$ . For  $q = 1$ , the projection of  $B_1^n$  onto the hyperplane perpendicular to  $a^{(1)} = (1, 0, \dots, 0)$  is maximal, the projection onto the hyperplane orthogonal to  $a^{(2)}$  is minimal, which essentially is a consequence of Szarek's result [16] on the best constants in the Khintchine inequality for  $q = 1$ . Barthe and Naor [3] proved that this does not transfer to  $\frac{4}{3} < q < 2$ , namely that the projection onto  $a^{(n)\perp}$  has smaller volume than the one onto  $a^{(2)\perp}$  for large dimensions  $n$ . Here  $q_0 := \frac{4}{3}$  is the unique value in  $(1, 2)$  such that  $\lim_{n \rightarrow \infty} \frac{\text{vol}_{n-1}(P_{(a^{(n)})^\perp}(B_q^n))}{\text{vol}_{n-1}(P_{(a^{(2)})^\perp}(B_q^n))} < 1$  for all  $q_0 < q < 2$ , where  $P_{a^\perp}$  denotes the orthogonal projection onto the hyperplane  $a^\perp$ . In this case, we also give a quantitative estimate for dimensions  $n$  when this happens, namely when  $n > 5 \left( \frac{1}{p-\frac{4}{3}} + \frac{1}{2-p} \right)$ . Note that there is no complete duality here, since  $\frac{4}{3}$  is not the dual index of  $p_0 \simeq 26.265$ .

Let  $a \in S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$  be a direction vector. We introduce the normalized section function

$$A_{n,p}(a) := \frac{\text{vol}_{n-1}(a^\perp \cap B_p^n)}{\text{vol}_{n-1}(B_p^{n-1})}$$

and the normalized projection function

$$P_{n,p}(a) := \frac{\text{vol}_{n-1}(P_{a^\perp}(B_p^n))}{\text{vol}_{n-1}(B_p^{n-1})}.$$

In terms of this notation, Ball's result states  $A_{n,\infty}(a) \leq A_{n,\infty}(a^{(2)}) = \sqrt{2}$  for all  $a \in S^{n-1}$  and Eskenazis, Nayar and Tkocz's result reads  $A_{n,p}(a) \leq A_{n,p}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{p}}$  for all  $a \in S^{n-1}$  and  $10^{15} \leq p < \infty$ . We note that  $A_{n,p}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{p}}$  is independent of  $n \geq 2$ . But as shown by Oleszkiewicz,  $\lim_{n \rightarrow \infty} A_{n,p}(a^{(n)}) > A_{n,p}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{p}}$  for  $2 < p < p_0$ . In the projection case,  $P_{n,1}(a^{(2)}) \leq P_{n,1}(a)$  for all  $a \in S^{n-1}$ , which Eskenazis, Nayar and Tkocz [5] extended to  $P_{n,q}(a^{(2)}) \leq P_{n,q}(a)$  for all  $a \in S^{n-1}$  and  $1 < q \leq 1 + 10^{-12}$ . Again  $P_{n,q}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{q}}$  is independent of  $n \geq 2$ . However, by Barthe and Naor [3],  $P_{n,q}(a^{(2)}) > \lim_{n \rightarrow \infty} P_{n,q}(a^{(n)})$  for  $\frac{4}{3} < q < 2$ .

Our two main results study these limits in more detail.

**Theorem 1.1** *Let  $2 < p < \infty$  and  $n \in \mathbb{N}$ . Then for all  $2 < p < p_0 \simeq 26.265$*

$$\lim_{n \rightarrow \infty} \frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} = \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} > 1.$$

*We have the following quantitative estimate:  $A_{n,p}(a^{(n)}) > A_{n,p}(a^{(2)})$  holds if*

- (a) *either  $5 \leq p < p_0$  and  $n \geq \frac{650}{p_0 - p}$  or*
- (b)  *$2 < p < 5$  and  $n > \frac{65}{p-2}$  is satisfied.*

*Remarks.* (a) The constant 650 in the statement for  $5 \leq p < p_0$  is not optimal, but

by necessity fairly large since the  $p$ -derivative of  $f(p) := \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}}$  at  $p_0$

with  $f(p_0) = 1$  is small,  $f'(p_0) \simeq -\frac{1}{1316}$ . The derivative at 2 is positive and larger in modulus, namely  $f'(2) = \frac{1}{4}(1 - \ln 2) \simeq \frac{1}{13}$ .

(b) The case of complex hyperplane sections of  $l_p^n(\mathbb{C})$  is considered in [8].

**Theorem 1.2** *Let  $1 < q < 2$  and  $n \in \mathbb{N}$ . Then for all  $\frac{4}{3} < q < 2$*

$$\lim_{n \rightarrow \infty} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{1}{\pi}} \sqrt{2^{\frac{2}{q}} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)} < 1.$$

*We have the following quantitative estimate:  $P_{n,q}(a^{(n)}) < P_{n,q}(a^{(2)})$  holds if*

$$n > \frac{\frac{32}{15}}{q - \frac{4}{3}} + \frac{\frac{24}{5}}{2 - q}.$$

*Remark.* For the derivative of  $g(q) := \sqrt{\frac{1}{\pi}} \sqrt{2^{\frac{2}{q}} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)}$  we have  $g'\left(\frac{4}{3}\right) = \frac{9}{32} (4 - \pi - 2 \ln 2) \simeq -\frac{1}{6.73}$  and  $g'(2) = \frac{1}{4} (1 - \ln 2) \simeq \frac{1}{13}$ .

The limits in Theorems 1.1 and 1.2 were already determined by Oleszkiewicz [14] and Barthe, Naor [3]. Meyer and Pajor [13] showed that  $A_{n,p}(a)$  is monotone increasing in  $p$  for any fixed  $n$  and  $a$ . Barthe and Naor proved that  $P_{n,q}(a)$  is monotone increasing in  $q$  for any fixed  $n$  and  $a$ .

## 2 Formulas

Eskenazis, Nayar and Tkocz [5], Proposition 6, proved the following formula for the normalized volume of hyperplane sections.

**Proposition 2.1** *Let  $1 \leq p < \infty$ ,  $n \in \mathbb{N}$  and  $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$ . Then*

$$A_{n,p}(a) = \Gamma\left(1 + \frac{1}{p}\right) \mathbb{E}_{\xi, R} \frac{1}{\|\sum_{j=1}^n a_j R_j \xi_j\|_2} \quad (2.1)$$

where  $(\xi_j)_{j=1}^n$  are i.i.d. random vectors uniformly distributed on the sphere  $S^2 \subset \mathbb{R}^3$  and  $(R_j)_{j=1}^n$  are i.i.d. random variables with density  $c_p^{-1} t^p \exp(-t^p)$  on  $[0, \infty)$ ,  $c_p := \frac{1}{p} \Gamma\left(1 + \frac{1}{p}\right)$ , independent of the  $(\xi_j)_{j=1}^n$ .

For  $p = \infty$  with  $R_j = 1$  one has  $A_{n,\infty}(a) = \mathbb{E}_{\xi} \frac{1}{\|\sum_{j=1}^n a_j \xi_j\|_2}$ , see König, Koldobsky [11]. We will use another formula for  $A_{n,p}(a)$  derived from (2.1).

**Proposition 2.2** *Let  $1 \leq p < \infty$ ,  $n \in \mathbb{N}$  and  $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$ . Then*

$$A_{n,p}(a) = \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \gamma_p(a_j s) ds, \quad (2.2)$$

$$\gamma_p(s) := \frac{1}{\Gamma\left(1 + \frac{1}{p}\right)} \int_0^\infty \cos(sr) \exp(-r^p) dr.$$

**Proof** Define  $\text{sinc}(x) := \frac{\sin x}{x}$  for  $x \neq 0$ ,  $\text{sinc}(0) := 1$ . Let  $t > 0$ ,  $e \in S^2$  be fixed and  $m$  denote the normalized Haar surface measure on  $S^2$ . Then

$$\text{sinc}(t) = \int_{S^2} \exp(it \langle e, u \rangle) dm(u). \quad (2.3)$$

Here  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^n$ . This implies for  $(b_j)_{j=1}^n \in \mathbb{R}^n$

$$\begin{aligned}
\prod_{j=1}^n \operatorname{sinc}(b_j s) &= \int_{(S^2)^n} \exp \left( i s \left\langle e, \sum_{j=1}^n b_j u_j \right\rangle \right) \prod_{j=1}^n dm(u_j) \\
&= \int_{(S^2)^n} \operatorname{sinc} \left( \left\| \sum_{j=1}^n b_j u_j \right\|_2 s \right) \prod_{j=1}^n dm(u_j) = \mathbb{E}_\xi \operatorname{sinc} \left( \left\| \sum_{j=1}^n b_j \xi_j \right\|_2 s \right),
\end{aligned} \tag{2.4}$$

where the second equality follows from (2.3) by integration over  $dm(e)$ . Note that the first equality holds for all  $e \in S^2$ .

For all  $t > 0$  we have  $\frac{2}{\pi} \int_0^\infty \operatorname{sinc}(ts) ds = \frac{1}{t}$  and (2.1) may be rewritten

$$\begin{aligned}
A_{n,p}(a) &= \Gamma \left( 1 + \frac{1}{p} \right) \frac{2}{\pi} \mathbb{E}_{\xi,R} \int_0^\infty \operatorname{sinc} \left( \left\| \sum_{j=1}^n a_j R_j \xi_j \right\|_2 s \right) ds \\
&= \Gamma \left( 1 + \frac{1}{p} \right) \frac{2}{\pi} \int_0^\infty \mathbb{E}_{\xi,R} \operatorname{sinc} \left( \left\| \sum_{j=1}^n a_j R_j \xi_j \right\|_2 s \right) ds.
\end{aligned}$$

The sinc-integral is only a conditionally convergent Riemann integral. The verification that  $\mathbb{E}_{\xi,R}$  and  $\int_0^\infty$  may be interchanged is the same as in the proof of Proposition 3.2 (a) of König, Rudelson [12]. Using (2.4) and the independence of the  $(R_j)_{j=1}^n$ , we get

$$\begin{aligned}
A_{n,p}(a) &= \Gamma \left( 1 + \frac{1}{p} \right) \frac{2}{\pi} \int_0^\infty \mathbb{E}_R \left( \prod_{j=1}^n \operatorname{sinc}(a_j R_j s) \right) ds \\
&= \Gamma \left( 1 + \frac{1}{p} \right) \frac{2}{\pi} \int_0^\infty \prod_{j=1}^n \mathbb{E}_{R_j} \operatorname{sinc}(a_j R_j s) ds.
\end{aligned}$$

Denoting  $\gamma_p(s) := \mathbb{E}_{R_1} \operatorname{sinc}(R_1 s)$ , integration by parts gives

$$\begin{aligned}
\gamma_p(s) &= c_p^{-1} \int_0^\infty \operatorname{sinc}(sr) r^p \exp(-r^p) dr \\
&= c_p^{-1} \frac{1}{p} \int_0^\infty \cos(sr) \exp(-r^p) dr = \frac{1}{\Gamma \left( 1 + \frac{1}{p} \right)} \int_0^\infty \cos(sr) \exp(-r^p) dr.
\end{aligned}$$

□

Equation (2.1) yields  $A_{n,p}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{p}}$ , see [5, Section 3.2].

*Remarks.* (a) Proposition 2.2 is also found in Koldobsky [9, Theorem 3.2], with a different proof.

(b) For  $1 \leq p \leq 2$  the  $\gamma_p$  are just the densities of the (positive)  $p$ -stable random variables. In the case interesting for us, namely  $2 < p < \infty$ , the functions  $\gamma_p$  take

positive and negative values. For  $p \notin 2\mathbb{N}$ ,  $\gamma_p$  has only finitely many real zeros, see Pólya [15], whereas for  $p \in 2\mathbb{N}$ ,  $\gamma_p$  has infinitely many real zeros, see Boyd [4].

Barthe and Naor [3] proved the following formula for the volume of the orthogonal projection of  $B_q^n$  onto hyperplanes.

**Proposition 2.3** *Let  $1 \leq q < \infty$ ,  $p := \frac{q}{q-1}$  be the conjugate index,  $n \in \mathbb{N}$  and  $a = (a_j)_{j=1}^n \in S^{n-1} \subset \mathbb{R}^n$ . Then*

$$P_{n,q}(a) = \Gamma\left(\frac{1}{q}\right) \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| \quad (2.5)$$

where the  $X_j$  are i.i.d. symmetric random variables with density function  $d_q^{-1}|t|^{p-2} \exp(-|t|^p)$ ,  $t \in \mathbb{R}$ ,  $d_q = \frac{2}{p} \Gamma\left(\frac{1}{q}\right)$ . A second formula for  $P_{n,q}(a)$  is

$$\begin{aligned} P_{n,q}(a) &= \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{j=1}^n \delta_q(a_j s)}{s^2} ds, \\ \delta_q(s) &:= \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr) r^{p-2} \exp(-r^p) dr. \end{aligned} \quad (2.6)$$

Note that  $\mathbb{E}|X_1| = \frac{1}{\Gamma\left(\frac{1}{q}\right)}$ . To deduce (2.6) from (2.5), apply the usual formula  $|x| = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 - \operatorname{Re}(\exp(ixs))}{s^2} ds$  to find

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n a_j X_j \right| &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 - \mathbb{E} \exp(i(\sum_{j=1}^n a_j X_j)s)}{s^2} ds = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1 - \prod_{j=1}^n \mathbb{E} \exp(ia_j X_j s)}{s^2} ds \\ &= \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{j=1}^n \mathbb{E} \cos(a_j X_j s)}{s^2} ds = \frac{2}{\pi} \int_0^\infty \frac{1 - \prod_{j=1}^n \delta_q(a_j s)}{s^2} ds, \\ \delta_q(s) &= \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr) r^{p-2} \exp(-r^p) dr. \end{aligned}$$

Differentiation and integration by parts yields a relation between the functions  $\delta_q$  and  $\gamma_p$  in (2.6) and (2.2):

$$\begin{aligned} \delta'_q(s) &= -\frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \sin(sr) r^{p-1} \exp(-r^p) dr \\ &= -\frac{s}{\Gamma\left(\frac{1}{q}\right)} \int_0^\infty \cos(sr) \exp(-r^p) dr = -\frac{\Gamma\left(1 + \frac{1}{p}\right)}{\Gamma\left(1 - \frac{1}{p}\right)} s \gamma_p(s). \end{aligned} \quad (2.7)$$

Since  $\gamma_4''(s) = -\frac{\Gamma(\frac{3}{4})}{4\Gamma(\frac{5}{4})} \delta_{\frac{4}{3}}(s)$ , we have  $\gamma_4'''(s) = \frac{1}{4} s \gamma_4(s)$ . Similarly, for all  $k \in \mathbb{N}$ ,  $\gamma_{2k}^{(2k-1)}(s) = (-1)^k \frac{1}{2k} s \gamma_{2k}(s)$ . Therefore the functions  $\gamma_{2k}$  studied by Boyd [4] satisfy a linear differential equation.

For  $q \searrow 1$ , the variables  $X_j$  tend to the Rademacher variables with  $\delta_q(s) \rightarrow \delta_1(s) = \cos(s)$ , and the best constants in the Khintchine inequality, which were determined by Szarek [16], yield the extrema of  $P_{n,1}: a^{(2)}$  for the minimum and  $a^{(1)}$  for the maximum.

### 3 Prerequisites for the proof of Theorem 1.1

For the proof of Theorem 1.1 we need two lemmas on  $\Gamma$ -functions.

- Lemma 3.1** (a) Let  $f(p) := \frac{\Gamma(1+\frac{3}{p})}{\Gamma(1+\frac{1}{p})}$ . Then  $f(p) \geq 0.9429$  for all  $3 \leq p < \infty$ .
- (b) Let  $g(p) := \left( \frac{3}{\pi} \frac{2^{\frac{2}{p}} \Gamma(1+\frac{1}{p})^3}{\Gamma(1+\frac{3}{p})} \right)^{\frac{1}{2}}$ . Then there is exactly one solution  $p_0 \in (2, \infty)$  of  $g(p) = 1$ ,  $p_0 \simeq 26.265$ . For all  $2 < p < p_0$  we have  $g(p) > 1$ . The function  $g'$  has exactly one zero  $p_1 \in [2, \infty)$ ,  $p_1 \simeq 4.192$ . For  $2 \leq p < p_1$ ,  $g$  is strictly increasing, for  $p_1 < p < \infty$ ,  $g$  is strictly decreasing. The following lower estimates hold:

$$\begin{aligned} g(p) &\geq 1 + \frac{p_0 - p}{1317} \quad p \in [5, p_0], \quad g(p) \\ &> \frac{25}{24} \quad p \in [4, 5], \quad g(p) \geq 1 + \frac{p-2}{44} \quad p \in [2, 4]. \end{aligned}$$

**Proof** (a) In terms of the Digamma function  $\Psi := (\ln \Gamma)'$  we have

$$f'(p) = \frac{f(p)}{p^2} \left( \Psi \left( 1 + \frac{1}{p} \right) - 3\Psi \left( 1 + \frac{3}{p} \right) \right).$$

For  $F(p) := \Psi \left( 1 + \frac{1}{p} \right) - 3\Psi \left( 1 + \frac{3}{p} \right)$  one has  $F'(p) = \frac{1}{p^2} \left( 9\Psi' \left( 1 + \frac{3}{p} \right) - \Psi' \left( 1 + \frac{1}{p} \right) \right)$ . By Abramowitz, Stegun [1, 6.3.16 and 6.4.10] for all  $x > 0$

$$\Psi(1+x) = -\gamma + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}, \quad \Psi'(1+x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} \quad (3.1)$$

where  $\gamma \simeq 0.5772$  denotes the Euler–Mascheroni constant. Therefore  $\Psi'$  is decreasing, and we conclude for all  $0 \leq x \leq 1$  that  $\frac{\pi^2}{6} - 1 = \Psi'(2) \leq \Psi'(1+x) \leq \Psi'(1) = \frac{\pi^2}{6}$ . Hence  $F'(p) \geq \frac{1}{p^2} \left( \frac{4\pi^2}{3} - 9 \right) > 0$  for all  $p \geq 3$ . Thus  $F$  is increasing. Since  $F(9) \simeq -0.012$ ,  $F(10) \simeq 0.084$ ,  $F$  has exactly one zero  $p_1 \in (3, \infty)$ ,

$p_1 \simeq 9.115$ . Hence  $f$  is decreasing in  $(3, p_1)$  and increasing in  $(p_1, \infty)$ . For all  $p \geq 3$ ,  $f(p) \geq f(p_1) > 0.9429$ .

(b) Let  $h(p) := \frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}$ . Then

$$h'(p) = \frac{h(p)}{p^2} \left( 3\Psi\left(1 + \frac{3}{p}\right) - 3\Psi\left(1 + \frac{1}{p}\right) - 2\ln 2 \right).$$

By (3.1) and the geometric series we find for  $p > 3$

$$\begin{aligned} (\ln h)'(p) &= \frac{h'(p)}{h(p)} = \frac{1}{p^2} \left( 3 \sum_{n=1}^{\infty} \left( \frac{\frac{3}{p}}{n(n + \frac{3}{p})} - \frac{\frac{1}{p}}{n(n + \frac{1}{p})} \right) - 2\ln 2 \right) \\ &= \frac{1}{p^2} \left( 3 \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{n^{k+2}} \frac{3^{k+1} - 1}{p^{k+1}} - 2\ln 2 \right) \\ &= \frac{1}{p^2} \left( 3 \sum_{k=0}^{\infty} (-1)^k \zeta(k+2) \frac{3^{k+1} - 1}{p^{k+1}} - 2\ln 2 \right). \end{aligned}$$

The sum is an alternating series with decreasing coefficients. We find that

$$(\ln h)'(p) \leq -\frac{2\ln 2}{p^2} + \frac{\pi^2}{p^3} - \frac{24\zeta(3)}{p^4} + \frac{13\pi^4}{15p^5} - \frac{240\zeta(5)}{p^6} + \frac{242\pi^6}{315p^7} < 0 \quad (3.2)$$

holds for all  $5 \leq p < \infty$ . Therefore  $\ln h$ ,  $h$  and  $g(p) = \sqrt{\frac{3}{\pi} h(p)}$  are strictly decreasing in  $[5, \infty)$ . We have  $\lim_{p \rightarrow \infty} g(p) = \sqrt{\frac{3}{\pi}} < 1$ ,  $g(26) \simeq 1.00020$ ,  $g(27) \simeq 0.99945$ : There is exactly one  $p_0 \in [5, \infty)$  with  $g(p_0) = 1$ ,  $p_0 \simeq 26.265$ , and for  $5 \leq p < p_0$  we have  $g(p) > 1$ . Inequality (3.2) yields for  $5 \leq p \leq p_0$  that  $(\ln h)'(p) \leq -\frac{1.04768}{p^2}$ . Hence for these  $p$

$$g'(p) = g(p)(\ln g)'(p) = \frac{1}{2} g(p)(\ln h)'(p) \leq -\frac{1}{2} \frac{1.04768}{p^2} \leq -\frac{1}{2} \frac{1.04768}{p_0^2} < -\frac{1}{1317}.$$

This implies  $g(p) \geq 1 + \frac{1}{1317}(p_0 - p)$  for all  $5 \leq p \leq p_0$ .

To show  $g(p) > 1$  also for  $2 < p < 5$ , note that  $g(2) = 1$  and

$$g'(p) = \frac{1}{2} g(p)(\ln h)'(p) = \frac{3}{2} \frac{g(p)}{p^2} \left( \Psi\left(1 + \frac{3}{p}\right) - \Psi\left(1 + \frac{1}{p}\right) - \frac{2}{3} \ln 2 \right).$$

Again by (3.1)



$$\Psi\left(1 + \frac{3}{p}\right) - \Psi\left(1 + \frac{1}{p}\right) = \sum_{n=1}^{\infty} \left( \frac{\frac{3}{p}}{n(n + \frac{3}{p})} - \frac{\frac{1}{p}}{n(n + \frac{1}{p})} \right) = \sum_{n=1}^{\infty} \frac{2p}{(np + 1)(np + 3)}.$$

All summands are decreasing in  $p$ . Thus  $k(p) := \Psi(1 + \frac{3}{p}) - \Psi(1 + \frac{1}{p}) - \frac{2}{3} \ln 2$  is strictly decreasing in  $p$ , with  $k(4) = \pi - \frac{8}{3} - \frac{2}{3} \ln 2 \simeq 0.0128 > 0$  and  $k(5) \simeq -0.0470 < 0$ . Thus  $g'$  has exactly one zero  $p_1 \in (2, \infty)$ ,  $p_1 \simeq 4.193$ , and  $g$  is strictly increasing in  $(2, p_1)$  and strictly decreasing in  $(p_1, \infty)$ . We know already that  $g(5) > 1$  and hence  $g(p) > 1$  for all  $2 < p \leq 5$ . For  $p \in [4, 5]$ ,  $g(p) \geq \min(g(4), g(5)) = g(5) > \frac{25}{24}$ . Further

$$\left( \frac{g(p)}{p^2} \right)' = \frac{3}{2} \frac{g(p)}{p^4} \left( k(p) - \frac{4}{3} p \right) < 0$$

since  $k(p) - \frac{4}{3} p \leq k(2) - \frac{8}{3} = -(2 + \frac{2}{3} \ln(2)) < 0$ . Therefore  $\frac{g(p)}{p^2}$  and  $k(p)$  are both strictly decreasing and positive for  $p \in [2, p_1]$ , and with  $g'(p) = \frac{g(p)}{p^2} k(p)$ ,  $g'$  is decreasing and hence  $g$  is concave in  $[2, p_1]$ . Therefore for  $2 \leq p \leq 4$

$$g(p) \geq 1 + \frac{g(4) - 1}{2} (p - 2) > 1 + \frac{p - 2}{44}$$

which proves all lower estimates stated in Lemma 3.1.  $\square$

For  $p \rightarrow \infty$ , the functions  $\gamma_p$  in (2.2) tend to  $\gamma_\infty$ ,  $\gamma_\infty(s) = \text{sinc}(s)$ . We estimate their difference for  $p \geq 2$ .

**Lemma 3.2** *Let  $2 \leq p < \infty$ . Then for all  $s > 0$*

$$\left| \text{sinc}(s) - \int_0^\infty \cos(sr) \exp(-r^p) dr \right| \leq 0.3926.$$

*This implies  $\gamma_p(s) > 0$  for all  $0 \leq s \leq \frac{2}{3}\pi$ .*

**Proof** We have  $\int_0^\infty \exp(-s^p) ds = \Gamma\left(1 + \frac{1}{p}\right) < 1$ . Since  $\text{sinc}(s) = \int_0^1 \cos(sr) dr$ , we find

$$\begin{aligned} & \left| \text{sinc}(s) - \int_0^\infty \cos(sr) \exp(-r^p) dr \right| \\ &= \left| \int_0^1 \cos(sr) (1 - \exp(-r^p)) dr - \int_1^\infty \cos(sr) \exp(-r^p) dr \right| \\ &\leq \left( 1 - \Gamma\left(1 + \frac{1}{p}\right) \right) + 2 \int_1^\infty \exp(-r^p) dr \\ &= \left( 1 - \Gamma\left(1 + \frac{1}{p}\right) \right) + \frac{2}{p} \int_1^\infty u^{\frac{1}{p}-1} \exp(-u) du =: \phi(p). \end{aligned}$$

Then  $\phi' < 0$ , since for  $p \geq 2$

$$\phi'(p) = -\frac{1}{p^2} \left( 2 \int_1^\infty u^{\frac{1}{p}-1} \left( 1 + \frac{\ln(u)}{p} \right) \exp(-u) du - \Gamma' \left( 1 + \frac{1}{p} \right) \right).$$

Since  $\Gamma(x)$  has a (positive) minimum at  $\tilde{x} \simeq 1.4616$ ,  $\Gamma'(1 + \frac{1}{p}) = 0$  for  $\tilde{p} \simeq 2.1662$ . For  $p > \tilde{p}$ ,  $\Gamma'(1 + \frac{1}{p}) < 0$  and for  $2 \leq p < \tilde{p}$ ,  $0 \leq \Gamma'(1 + \frac{1}{p}) < \frac{3}{100}$ . The last inequality holds since  $\Gamma$  is log-convex and positive in  $(1, 2)$ , thus convex, and for  $2 \leq p < \tilde{p}$ ,  $\Gamma'(1 + \frac{1}{p})$  is decreasing, with  $\Gamma'(\frac{3}{2}) = \Gamma(\frac{3}{2}) \Psi(\frac{3}{2}) = \sqrt{\pi}(1 - \ln 2 - \frac{\gamma}{2}) < \frac{3}{100}$ , see Abramowitz, Stegun [1, 6.3.4]. Therefore for all  $p \geq 2$

$$\phi'(p) \leq -\frac{1}{p^2} \left( 2 \int_1^\infty u^{-1} \exp(-u) du - \frac{3}{100} \right) < -\frac{2}{5} \frac{1}{p^2} < 0$$

using that  $\int_1^\infty u^{-1} \exp(-u) du \simeq 0.219$ . We conclude  $\phi(p) \leq \phi(2) < 0.3926$ .

This yields for all  $0 \leq s \leq \frac{2}{3}\pi$  and  $p \geq 2$

$$\begin{aligned} \Gamma \left( 1 + \frac{1}{p} \right) \gamma_p(s) &= \int_0^\infty \cos(sr) \exp(-r^p) dr \geq \text{sinc}(s) \\ -0.3926 &\geq \frac{3\sqrt{3}}{4\pi} - 0.3926 > \frac{1}{50} > 0. \end{aligned}$$

□

*Remark.* The derivative of  $p\phi(p)$  is increasing with

$$\lim_{p \rightarrow \infty} (p\phi(p))' = \gamma + 2 \int_1^\infty \frac{1}{u} \exp(-u) du \leq 1.016.$$

Thus for all  $p \geq 1$ ,  $|\text{sinc}(s) - \int_0^\infty \cos(sr) \exp(-r^p) dr| \leq \frac{1.016}{p}$ . Actually,  $\gamma_p(s) > 0$  for all  $p \geq 1$  and  $s \in [0, \pi]$ . However, we do not need this.

## 4 Proof of Theorem 1.1

### Proof of Theorem 1.1.

(i) To estimate  $A_{n,p}(a^{(n)})$  from below, we first find a lower bound for  $\gamma_p\left(\frac{s}{\sqrt{n}}\right)$  for all  $s \leq \frac{3}{2}\sqrt{n}$ . By the series representation for  $\cos x$  we have for  $0 \leq x \leq \frac{3}{2}$ , by pairing up terms in the alternating series,

$$\cos x - \left( 1 - \frac{x^2}{2} + \frac{x^4}{26} \right) = \frac{x^4}{312} - \frac{x^6}{720} + \sum_{k=4}^\infty (-1)^m \frac{x^{2m}}{(2m)!} > 0$$

$1 - \frac{x^2}{2} + \frac{x^4}{26} > 0$ . Therefore for  $s \leq \frac{3}{2}\sqrt{n}$

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right) \geq \frac{1}{\Gamma\left(1 + \frac{1}{p}\right)} \left[ \int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}} \left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26n^2}\right) \exp(-r^p) dr + \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty} \cos\left(\frac{sr}{\sqrt{n}}\right) \exp(-r^p) dr \right].$$

To estimate this from below, write  $\int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}} = \int_0^{\infty} - \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty}$  and use that  $|\cos(x)| \leq 1$ ,

$$\begin{aligned} \gamma_p\left(\frac{s}{\sqrt{n}}\right) &\geq \frac{1}{\Gamma\left(1 + \frac{1}{p}\right)} \left[ \int_0^{\infty} \left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26n^2}\right) \exp(-r^p) dr - R \right] \\ &= \frac{1}{\Gamma\left(1 + \frac{1}{p}\right)} \left[ \Gamma\left(1 + \frac{1}{p}\right) - \frac{\Gamma\left(1 + \frac{3}{p}\right) s^2}{6n} + \frac{\Gamma\left(1 + \frac{5}{p}\right) s^4}{130n^2} - R \right], \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} R &:= \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty} \left(2 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26n^2}\right) \exp(-r^p) dr \\ &= \frac{1}{p} \int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}^{\infty} \left(2u^{\frac{1}{p}-1} - \frac{s^2}{2n} u^{\frac{3}{p}-1} + \frac{s^4}{26n^2} u^{\frac{5}{p}-1}\right) \exp(-u) du; \quad u \geq 1. \end{aligned} \quad (4.2)$$

(ii) We first consider the case  $p \geq 5$ . Since  $u \geq 1$  in the above integral,  $u^{\frac{5}{p}-1} \leq 1$ ,  $2u^{\frac{1}{p}-1} \leq 2\left(\frac{2}{3}\frac{s}{\sqrt{n}}\right)^{p-1}$  and  $\int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}^{\infty} \exp(-u) du = \exp\left(-\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p\right)$ . The remainder

term  $R$  will be smaller than the fourth order term  $\frac{\Gamma\left(1 + \frac{5}{p}\right) s^4}{130n^2}$  provided that

$$\frac{1}{p\Gamma\left(1 + \frac{5}{p}\right)} \left(2\left(\frac{2}{3}\frac{s}{\sqrt{n}}\right)^{p-1} + \frac{s^4}{26n^2}\right) \exp\left(-\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p\right) < \frac{s^4}{130n^2}.$$

For  $s \leq \frac{3}{2}\sqrt{n}$  the left side is decreasing in  $p$ , and therefore this condition is the strongest for  $p = 5$ . Writing  $y := \frac{s}{\sqrt{n}}$ , it means

$$\frac{1}{5} \left( \frac{32}{81} y^4 + \frac{1}{26} y^4 \right) \exp\left(-\left(\frac{3}{2} y\right)^5\right) < \frac{1}{130} y^4$$

or  $\frac{913}{81} < \exp\left(\left(\frac{3}{2}\frac{1}{y}\right)^5\right)$ ,  $y < \frac{3}{2} \frac{1}{\ln\left(\frac{913}{81}\right)^{\frac{1}{5}}} \simeq 1.2567$ . Choosing  $s \leq \frac{7}{6}\sqrt{n}$ ,  $R \leq \frac{\Gamma\left(1+\frac{5}{p}\right)s^4}{130n^2}$  is satisfied and therefore

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right) \geq 1 - c \frac{s^2}{n}, \quad c := \frac{1}{6} \frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}.$$

By the proof of Lemma 3.1 (a),  $\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)}$  is decreasing for  $5 \leq p \leq p_1 \simeq 9.115$  and increasing for  $p > p_1$ . Its value at  $p_0$  is less than the one at 5, so that  $c \leq 0.1622 < \frac{8}{49}$ , for its value at  $p = 5$ . Then for  $s \leq \frac{7}{6}\sqrt{n}$ ,  $x := c \frac{s^2}{n} \leq \frac{8}{49} \frac{49}{36} = \frac{2}{9}$ . We have

$$\ln(1-x) = -\sum_{j=1}^{\infty} \frac{x^j}{j} \geq -x - \frac{1}{2}x^2 \sum_{k=0}^{\infty} x^k = -x - \frac{1}{2} \frac{x^2}{1-x} \geq -x - \frac{9}{14}x^2$$

and hence

$$\gamma_p\left(\frac{s}{\sqrt{n}}\right)^n \geq \left(1 - c \frac{s^2}{n}\right)^n \geq \exp\left(-cs^2 - \frac{9}{14}c^2 \frac{s^4}{n}\right) \geq \exp(-cs^2) \left(1 - \frac{9}{14}c^2 \frac{s^4}{n}\right).$$

By Lemma 3.1 (a),  $\frac{\Gamma\left(1+\frac{3}{p}\right)}{\Gamma\left(1+\frac{1}{p}\right)} \geq 0.9429$  and hence  $c \geq 0.1571$  and for  $s \leq \frac{7}{6}\sqrt{n}$

$$\begin{aligned} \int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\geq \int_0^{\frac{7}{6}\sqrt{n}} \exp(-cs^2) \left(1 - \frac{9}{14}c^2 \frac{s^4}{n}\right) ds \\ &= \int_0^{\infty} \exp(-cs^2) \left(1 - \frac{9}{14}c^2 \frac{s^4}{n}\right) ds - \int_{\frac{7}{6}\sqrt{n}}^{\infty} \exp(-cs^2) \left(1 - \frac{9}{14}c^2 \frac{s^4}{n}\right) ds \end{aligned}$$

For  $s \geq \frac{7}{6}\sqrt{n}$  and  $c \geq 0.1571$  we have  $1 - \frac{9}{14}c^2 \frac{s^4}{n} < 0$  for all  $n \geq 35$ . Actually, evaluating the last integral in terms of the error function shows that the integral is already negative for  $n \geq 24$ . Thus for  $n \geq 24$

$$\begin{aligned} \int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\geq \int_0^{\infty} \exp(-cs^2) \left(1 - \frac{9}{14}c^2 \frac{s^4}{n}\right) ds \\ &= \frac{1}{2} \sqrt{\frac{\pi}{c}} \left(1 - \frac{27}{56} \frac{1}{n}\right) = \sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(1+\frac{3}{p}\right)}} \left(1 - \frac{27}{56} \frac{1}{n}\right), \end{aligned}$$

where we used that  $\int_0^\infty \exp(-cs^2)ds = \frac{1}{2}\sqrt{\frac{\pi}{c}}$  and  $\int_0^\infty \exp(-cs^2)c^2 s^4 ds = \frac{3}{8}\sqrt{\frac{\pi}{c}}$ . By Lemma 3.2 we have  $\gamma_p(s) > 0$  for all  $0 \leq s \leq 2$ . Hence

$$0 < \sqrt{n} \int_{\frac{7}{6}\sqrt{n}}^2 \gamma_p(s)^n du = \int_{\frac{7}{6}\sqrt{n}}^{2\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds.$$

By the proof of Proposition 2.2

$$\begin{aligned} |\gamma_p(s)| &= \left| \frac{p}{\Gamma\left(1 + \frac{1}{p}\right)} \int_0^\infty \text{sinc}(sr) r^p \exp(-r^p) dr \right| \\ &= \left| \frac{1}{s\Gamma\left(1 + \frac{1}{p}\right)} \int_0^\infty \sin(sr) pr^{p-1} \exp(-r^p) dr \right|, \quad |\sin(sr)| \leq 1 \\ &\leq \frac{1}{s\Gamma\left(1 + \frac{1}{p}\right)} \int_0^\infty \exp(-u) du = \frac{1}{s\Gamma\left(1 + \frac{1}{p}\right)}. \end{aligned}$$

This yields the tail estimate for  $p \geq 5$

$$\begin{aligned} \int_{2\sqrt{n}}^\infty \left| \gamma_p\left(\frac{s}{\sqrt{n}}\right) \right|^n ds &= \sqrt{n} \int_2^\infty |\gamma_p(s)|^n ds \leq \frac{\sqrt{n}}{\Gamma\left(1 + \frac{1}{p}\right)^n} \int_2^\infty s^{-n} ds \\ &= \frac{2\sqrt{n}}{n-1} \left( \frac{1}{2\Gamma\left(1 + \frac{1}{p}\right)} \right)^n < \frac{2\sqrt{n}}{n-1} 0.5446^n. \end{aligned}$$

We conclude for  $p \geq 5$  and  $n \geq 24$  that

$$\begin{aligned} \int_0^\infty \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds &\geq \int_0^{\frac{7}{6}\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds - \int_{2\sqrt{n}}^\infty \left| \gamma_p\left(\frac{s}{\sqrt{n}}\right) \right|^n ds \\ &\geq \sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma\left(1 + \frac{1}{p}\right)}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{27}{56} \frac{1}{n}\right) - \frac{2\sqrt{n}}{n-1} 0.5446^n \\ &\geq \sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma\left(1 + \frac{1}{p}\right)}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{27}{56} \frac{1}{n} - \frac{\sqrt{n}}{n-1} 0.5446^n\right), \end{aligned}$$

using  $\sqrt{\frac{3\pi}{2}} \sqrt{\frac{\Gamma(1+\frac{1}{p})}{\Gamma(1+\frac{3}{p})}} > 2$ . For  $n \geq 24$ ,  $\frac{\sqrt{n}}{n-1} 0.5446^n < \frac{10^{-5}}{n}$  and  $\frac{27}{56} + 10^{-5} < \frac{193}{400}$ , so that

$$\begin{aligned} A_{n,p}(a^{(n)}) &= \Gamma\left(1 + \frac{1}{p}\right) \frac{2}{\pi} \int_0^\infty \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds \\ &\geq \sqrt{\frac{6}{\pi}} \sqrt{\frac{\Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{193}{400} \frac{1}{n}\right). \end{aligned}$$

This is  $> A_{n,p}(a^{(2)}) = 2^{\frac{1}{2}-\frac{1}{p}}$ , provided that

$$\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} \geq \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{193}{400} \frac{1}{n}\right) > 1.$$

By Lemma 3.1 (b), the quotient  $g(p) := \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}}$  is  $> 1$  for all  $2 < p < p_0 \simeq 26.265$ , with  $g(p) \geq 1 + \frac{1}{1317}(p_0 - p)$  for all  $5 \leq p \leq p_0$ . We find for  $p \geq 5$  and  $n \geq 24$ ,

$$\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} \geq \left(1 + \frac{1}{1317}(p_0 - p)\right) \left(1 - \frac{193}{400} \frac{1}{n}\right).$$

This is  $> 1$  provided that  $5 \leq p \leq p_0$  and  $n \geq \frac{650}{p_0 - p}$ ;  $n \geq 24$  being automatically satisfied.

(iii) Secondly we consider the case  $2 < p \leq 5$ . To estimate the remainder  $R$  in (4.2), we use that in this case  $u^{\frac{5}{p}-1} \leq u^{\frac{3}{2}}$ . For  $x > 0$

$$\begin{aligned} \int_x^\infty u^{\frac{3}{2}} \exp(-u) du &\leq \left( \left( \int_x^\infty u \exp(-u) du \right) \left( \int_x^\infty u^2 \exp(-u) du \right) \right)^{\frac{1}{2}} \\ &= \left( (1+x)(2+2x+x^2) \right)^{\frac{1}{2}} \exp(-x), \end{aligned}$$

Since the function  $\phi(x) := \frac{(1+x)(2+2x+x^2)}{x^4} = \left(\frac{1}{x} + \frac{1}{x^2}\right) \left(2 + \frac{2}{x} + \frac{1}{x^2}\right)$  is decreasing in  $x$  with  $\phi\left(\frac{9}{2}\right) = \left(\frac{5\sqrt{110}}{81}\right)^2 < \left(\frac{13}{20}\right)^2$ , we conclude that  $\int_x^\infty u^{\frac{3}{2}} \exp(-u) du \leq \frac{13}{20} x^2 \exp(-x)$  for all  $x \geq \frac{9}{2}$ . Now choose  $s \leq \sqrt{\frac{n}{2}}$  for  $2 \leq p \leq 5$ . Then  $x :=$

$$\left(\frac{3}{2} \frac{\sqrt{n}}{s}\right)^p \geq \left(\frac{3}{\sqrt{2}}\right)^p \geq \frac{9}{2} \text{ and}$$

$$R \leq \frac{1}{p} \left( 2 \left( \frac{2}{3} \frac{s}{\sqrt{n}} \right)^{p-1} + \frac{s^4}{26n^2} \frac{13}{20} \left( \frac{3}{2} \frac{\sqrt{n}}{s} \right)^{2p} \right) \exp \left( - \left( \frac{3}{2} \frac{\sqrt{n}}{s} \right)^p \right).$$

Again we want this to be smaller than the fourth order term  $\Gamma(1 + \frac{5}{p}) \frac{s^4}{130n^2}$ , a condition which is strongest for  $p = 2$ . We then require for  $y := \frac{s}{\sqrt{n}}$

$$\frac{1}{2} \left( \frac{4}{3} y + \frac{81}{640} \right) \exp \left( - \left( \frac{3}{2} \frac{1}{y} \right)^2 \right) < \Gamma \left( \frac{7}{2} \right) \frac{y^4}{130}.$$

This is equivalent to  $g(y) := \frac{1}{\sqrt{\pi}} \left( 26 \frac{16}{9} \frac{1}{y^3} + 13 \frac{27}{80} \frac{1}{y^4} \right) \exp \left( - \left( \frac{3}{2} \frac{1}{y} \right)^2 \right) < 1$ . The function  $g$  is increasing and positive in  $y > 0$ , with  $g(0.7161) < 1$ . Hence the condition is satisfied for all  $0 < y \leq 0.7161$ , and in particular for our choice  $y = \frac{s}{\sqrt{n}} \leq \frac{1}{\sqrt{2}}$ .

Therefore for  $s \leq \sqrt{\frac{n}{2}}$ , as in part (ii),

$$\gamma_p \left( \frac{s}{\sqrt{n}} \right) \geq 1 - c \frac{s^2}{n}, \quad c := \frac{1}{6} \frac{\Gamma \left( 1 + \frac{3}{p} \right)}{\Gamma \left( 1 + \frac{1}{p} \right)}.$$

We have  $c \leq \frac{1}{4}$  for  $2 \leq p \leq 5$  and  $x := c \frac{s^2}{n} \leq \frac{1}{8}$ . Similarly as in (ii),

$\ln(1 - x) \geq -x - \frac{x^2}{1-x} \geq -x - \frac{4}{7}x^2$  and

$$\begin{aligned} \gamma_p \left( \frac{s}{\sqrt{n}} \right) &\geq \left( 1 - c \frac{s^2}{n} \right)^n = \exp \left( n \ln \left( 1 - c \frac{s^2}{n} \right) \right) \\ &\geq \exp \left( -cs^2 - \frac{4}{7} c^2 \frac{s^4}{n} \right) \geq \exp(-cs^2) \left( 1 - \frac{4}{7} c^2 \frac{s^4}{n} \right) \end{aligned}$$

and

$$\begin{aligned} \int_0^{\sqrt{\frac{n}{2}}} \gamma_p \left( \frac{s}{\sqrt{n}} \right)^n ds &\geq \int_0^{\sqrt{\frac{n}{2}}} \exp(-cs^2) \left( 1 - \frac{4}{7} c^2 \frac{s^4}{n} \right) ds \\ &\geq \int_0^{\infty} \exp(-cs^2) \left( 1 - \frac{4}{7} c^2 \frac{s^4}{n} \right) ds - \int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2) ds \\ &= \frac{1}{2} \sqrt{\frac{\pi}{c}} \left( 1 - \frac{3}{7} \frac{1}{n} \right) - \int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2) ds. \end{aligned}$$

To estimate the error term, note that  $c \geq .16219$ , its approximate value for  $p = 5$ , and

$$\begin{aligned} \int_{\sqrt{\frac{n}{2}}}^{\infty} \exp(-cs^2) ds &= \frac{1}{2\sqrt{c}} \int_{\frac{\sqrt{n}}{2}}^{\infty} \frac{1}{\sqrt{u}} \exp(-u) du \\ &\leq \frac{1}{c\sqrt{2n}} \int_{\frac{\sqrt{n}}{2}}^{\infty} \exp(-u) du = \frac{1}{c\sqrt{2n}} \exp\left(-\frac{c}{2}n\right) \leq \frac{4.36}{\sqrt{n}} 0.9222^n. \end{aligned}$$

Again  $\int_{\sqrt{\frac{n}{2}}}^{2\sqrt{n}} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds \geq 0$ , since by Lemma 3.2  $\gamma_p(x) > 0$  for all  $0 \leq x \leq 2$  and, as in (ii),  $\int_{2\sqrt{n}}^{\infty} \left|\gamma_p\left(\frac{s}{\sqrt{n}}\right)\right|^n ds \leq \frac{2\sqrt{n}}{n-1} 0.5446^n$ , so that for  $2 \leq p \leq 5$

$$\int_0^{\infty} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds \geq \frac{1}{2} \sqrt{\frac{\pi}{c}} \left(1 - \frac{3}{7} \frac{1}{n}\right) - \left(\frac{4.36}{\sqrt{n}} 0.9222^n + \frac{2\sqrt{n}}{n-1} 0.5446^n\right).$$

Since  $c \leq \frac{1}{4}$ ,  $\frac{1}{2} \sqrt{\frac{\pi}{c}} \geq \sqrt{\pi}$ . Further for  $n \geq 33$

$$\frac{3}{7} + \frac{1}{\sqrt{\pi}} \left(4.36\sqrt{n} 0.9222^n + \frac{2n\sqrt{n}}{n-1} 0.5446^n\right) \leq 1.405$$

so that  $\int_0^{\infty} \gamma_p\left(\frac{s}{\sqrt{n}}\right)^n ds \geq \frac{1}{2} \sqrt{\frac{\pi}{c}} \left(1 - \frac{1.405}{n}\right)$ . For  $p \in [2, 4]$ , using Lemma 3.1 (b),

$$\begin{aligned} \frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} &\geq \sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} \left(1 - \frac{1.405}{n}\right) \\ &\geq \left(1 + \frac{p-2}{44}\right) \left(1 - \frac{1.405}{n}\right). \end{aligned}$$

For  $n \geq \frac{65}{p-2}$  this is  $> 1$ , with  $n \geq 33$  being automatically satisfied. For  $p \in [4, 5]$ ,

again by Lemma 3.1 (b),  $\sqrt{\frac{3}{\pi}} \sqrt{\frac{2^{\frac{2}{p}} \Gamma\left(1 + \frac{1}{p}\right)^3}{\Gamma\left(1 + \frac{3}{p}\right)}} > \frac{25}{24}$  and  $\frac{A_{n,p}(a^{(n)})}{A_{n,p}(a^{(2)})} > 1$  is satisfied for  $n \geq \frac{65}{p-2}$ , too.  $\square$

## 5 Prerequisites for the proof of Theorem 1.2

We need two lemmas for the proof of Theorem 1.2.

**Lemma 5.1** (a) For  $q \in [1, 2]$ , let  $f(q) := \frac{\Gamma(2-\frac{1}{q})}{\Gamma(\frac{1}{q})}$ . Then  $f$  is decreasing, with  $f(1) = 1$ ,  $f(2) = \frac{1}{2}$  and  $f(\frac{4}{3}) \leq 0.7397$ .



- (b) For  $q \in [1, 2]$ , let  $g(q) := \sqrt{\frac{2}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)}$ . Then  $g'$  has exactly one zero in  $q_1 \in (1, 2)$ ,  $q_1 \simeq 1.612$ , and  $g$  is strictly decreasing in  $[1, q_1]$  and strictly increasing in  $(q_1, 2]$ , with  $g\left(\frac{4}{3}\right) = g(2) = 1$ . For  $q \in \left(\frac{4}{3}, 2\right)$ , we have

$$g(q) \leq 1 - M \left( \frac{1}{q} - \frac{1}{2} \right) \left( \frac{3}{4} - \frac{1}{q} \right), \quad M = 0.86326.$$

**Proof** (a) Differentiation gives  $f'(q) = \frac{f(q)}{q^2} \left( \Psi\left(\frac{1}{q}\right) + \Psi\left(2 - \frac{1}{q}\right) \right)$ . Since  $\Gamma$  is logarithmic convex,  $\Psi$  is increasing. Hence  $\Psi\left(\frac{1}{q}\right) \leq \Psi(1) = -\gamma$  and  $\Psi\left(2 - \frac{1}{q}\right) \leq \Psi\left(\frac{3}{2}\right) = 2(1 - \ln(2)) - \gamma$ ,  $\Psi\left(\frac{1}{q}\right) + \Psi\left(2 - \frac{1}{q}\right) \leq -2(\gamma + \ln(2) - 1) < 0$ . Therefore  $f$  is decreasing in  $[1, 2]$ . Moreover,  $f\left(\frac{4}{3}\right) \leq 0.7397$ .

(b) For  $g$  we find  $(\ln(g))'(q) = \frac{1}{2q^2} \left( \Psi\left(2 - \frac{1}{q}\right) - \Psi\left(\frac{1}{q}\right) - 2\ln(2) \right)$ . The function  $h(q) := \Psi\left(2 - \frac{1}{q}\right) - \Psi\left(\frac{1}{q}\right) - 2\ln(2)$  is strictly increasing, since with  $\frac{1}{q} + \frac{1}{p} = 1$  we have, using (3.1),

$$h'(q) = \frac{1}{q^2} \left( \Psi'\left(2 - \frac{1}{q}\right) + \Psi'\left(\frac{1}{q}\right) \right) = \frac{1}{q^2} \sum_{n=1}^{\infty} \left( \frac{1}{(n + \frac{1}{p})^2} + \frac{1}{(n - \frac{1}{p})^2} \right) > 0.$$

Moreover  $h(1) = -2\ln(2) < 0$ ,  $h(2) = 2(1 - \ln(2)) > 0$ . Thus  $h$  has exactly one zero  $q_1 \in (1, 2)$ ,  $q_1 \simeq 1.612$ . We get that  $g$  is decreasing in  $[1, q_1]$  and increasing in  $(q_1, 2]$ . We have  $g(1) = \frac{2}{\sqrt{\pi}} > 1$ ,  $g\left(\frac{4}{3}\right) = g(2) = 1$  and  $g(q) < 1$  for  $q \in \left(\frac{4}{3}, 2\right)$ .

For  $\frac{4}{3} < q < 2$ , choose  $\theta \in (0, 1)$  with  $\frac{1}{q} = (1 - \theta)\frac{1}{2} + \theta\frac{3}{4}$ ,  $\theta = \frac{4}{q} - 2$ ,  $1 - \theta = 3 - \frac{4}{q}$ . Since  $\Gamma$  is logarithmic convex,  $F := \ln \Gamma$  satisfies  $F'' = \Psi' > 0$  and by Taylor's formula with second degree remainder we have for some  $\eta \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$$F\left(\frac{1}{q}\right) \leq (1 - \theta)F\left(\frac{1}{2}\right) + \theta F\left(\frac{3}{4}\right) - \frac{\Psi'(\eta)}{2} \left( \frac{1}{q} - \frac{1}{2} \right) \left( \frac{3}{4} - \frac{1}{q} \right).$$

Since by (3.1)  $\Psi'$  is decreasing,  $\min_{\eta \in [\frac{1}{2}, \frac{3}{4}]} \Psi'(\eta) = \Psi'\left(\frac{3}{4}\right)$  and

$$F\left(\frac{1}{q}\right) \leq \left(3 - \frac{4}{q}\right) F\left(\frac{1}{2}\right) + \left(\frac{4}{q} - 2\right) F\left(\frac{3}{4}\right) - \frac{\Psi'\left(\frac{3}{4}\right)}{2} \left( \frac{1}{q} - \frac{1}{2} \right) \left( \frac{3}{4} - \frac{1}{q} \right).$$

Similarly, for  $\frac{5}{4} < 2 - \frac{1}{q} < \frac{3}{2}$ , choose  $\theta \in (0, 1)$  with  $2 - \frac{1}{q} = (1 - \theta)\frac{5}{4} + \theta\frac{3}{2}$ ,  $\theta = 3 - \frac{4}{q}$ ,  $1 - \theta = \frac{4}{q} - 2$ , such that with  $\min_{\eta \in [\frac{5}{4}, \frac{3}{2}]} \Psi'(\eta) = \Psi'\left(\frac{3}{2}\right)$

$$F\left(2 - \frac{1}{q}\right) \leq \left(\frac{4}{q} - 2\right) F\left(\frac{5}{4}\right) + \left(3 - \frac{4}{q}\right) F\left(\frac{3}{2}\right) - \frac{\Psi'\left(\frac{3}{2}\right)}{2} \left( \frac{1}{q} - \frac{1}{2} \right) \left( \frac{3}{4} - \frac{1}{q} \right).$$

This yields after exponentiation with  $c := \frac{\Psi'(\frac{3}{4}) + \Psi'(\frac{3}{2})}{4}$

$$\Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right) \leq \left(\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right)\right)^{3 - \frac{4}{q}} \\ \times \left(\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right)\right)^{\frac{4}{q} - 2} \exp\left(-2c\left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{3}{4} - \frac{1}{q}\right)\right).$$

Clearly  $\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}\right) = \frac{\pi}{2}$ , and by the complement formula for the  $\Gamma$ -function, see Abramowitz, Stegun [1],  $\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{5}{4}\right) = \frac{\pi}{4 \sin(\frac{\pi}{4})} = \frac{\pi}{2\sqrt{2}}$ , so that

$$g(q) = \sqrt{\frac{2^{\frac{2}{q}}}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)} \leq \exp\left(-c\left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{3}{4} - \frac{1}{q}\right)\right) =: k(q).$$

Let  $\varepsilon := \left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{3}{4} - \frac{1}{q}\right)$ . Then  $\varepsilon \leq \frac{1}{64}$ . Numerical evaluation yields  $c \geq 0.86917$ . By Taylor expansion  $k(q) \leq 1 - c\varepsilon + \frac{c^2\varepsilon^2}{2} = 1 - c\left(1 - \frac{c\varepsilon}{2}\right)\varepsilon \leq 1 - d\varepsilon$ ,  $d := c - \frac{c^2}{128} \geq 0.86326 =: M$ . Therefore  $g(q) \leq 1 - M\left(\frac{1}{q} - \frac{1}{2}\right)\left(\frac{3}{4} - \frac{1}{q}\right)$ .  $\square$

**Lemma 5.2** For all  $\frac{4}{3} \leq q \leq 2$  and  $0 \leq s \leq \frac{16}{5}$ , the functions  $\delta_q$  of (2.6) satisfy  $\delta_{\frac{4}{3}}(s) \leq \delta_q(s) \leq \delta_2(s) = \exp\left(-\frac{s^2}{4}\right)$ . Further  $\delta_{\frac{4}{3}}\left(\frac{48}{25}\right) > 0$  and  $\delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$ .

**Proof** Let  $2 \leq p = \frac{q}{q-1} \leq 4$  be the conjugate index of  $q$  and  $s \in [\frac{48}{25}, \frac{16}{5}]$ . We will show that  $\frac{d}{dp} \delta_q(s) > 0$ , or equivalently  $\frac{d}{dp} \delta_q(s) < 0$ . For  $m > -1$  we have

$$\int_0^\infty r^m \exp(-r^p) dr = \frac{1}{p} \int_0^\infty u^{\frac{m+1}{p}-1} \exp(-u) du = \frac{1}{p} \Gamma\left(\frac{m+1}{p}\right). \quad (5.1)$$

Expanding  $\cos(sr)$  into its Taylor series at zero, we find using (5.1)

$$\delta_q(s) = \sum_{n=0}^\infty \left( f_{2n}(p) \frac{s^{4n}}{(4n)!} - f_{2n+1}(p) \frac{s^{4n+2}}{(4n+2)!} \right) =: \sum_{n=0}^\infty F_n(p, s) \quad (5.2)$$

where  $f_{2n}(p) := \frac{\Gamma(1 + \frac{4n-1}{p})}{\Gamma(1 - \frac{1}{p})}$ ,  $f_{2n+1}(p) := \frac{\Gamma(1 + \frac{4n+1}{p})}{\Gamma(1 - \frac{1}{p})}$ . Since  $\Gamma$  is logarithmic convex, we have for  $x > 0$  and  $0 \leq \theta \leq 1$  that  $\Gamma(x + \theta) \leq \Gamma(x)^{1-\theta} \Gamma(x+1)^\theta = x^\theta \Gamma(x)$ . Let  $n \geq 2$ ,  $x := 1 + \frac{4n-1}{p} \geq 1$  and  $\theta := \frac{2}{p}$ . We claim that  $x^\theta = \left(1 + \frac{4n-1}{p}\right)^\theta \leq \frac{4n+1}{p}$ . This is equivalent to  $(4n + p - 1)^{\frac{2}{p}} p^{1 - \frac{2}{p}} \leq 4n + 1$ . Applying the inequality  $ab \leq \frac{a^r}{r} + \frac{b^{r'}}{r'}$  with  $r := \frac{p}{2}$  and  $r' = \frac{p}{p-2}$ , we get  $(4n + p - 1)^{\frac{2}{p}} p^{1 - \frac{2}{p}} \leq \frac{2}{p}(4n + p - 1) + p - 2 =$

$\frac{2}{p}(4n-1) + p$  which is  $\leq 4n+1$  if and only if  $n \geq \frac{p+1}{4}$ , which is satisfied, since  $p \leq 4$  and  $n \geq 2$ . Therefore

$$\frac{f_{2n+1}(p)}{f_{2n}(p)} = \frac{\Gamma\left(1 + \frac{4n+1}{p}\right)}{\Gamma\left(1 + \frac{4n-1}{p}\right)} \leq \frac{4n+1}{p} \quad (5.3)$$

and

$$\begin{aligned} F_n(p, s) &= f_{2n}(p) \frac{s^{4n}}{(4n)!} \left(1 - \frac{f_{2n+1}(p)}{f_{2n}(p)} \frac{s^2}{(4n+1)(4n+2)}\right) \\ &\geq f_{2n}(p) \frac{s^{4n}}{(4n)!} \left(1 - \frac{s^2}{2p(2n+1)}\right) > 0 \end{aligned}$$

for  $n \geq 2$  and  $s \leq \frac{16}{5} < \sqrt{20}$ . Hence for all  $m \geq 1$ ,  $\delta_q(s) \geq \sum_{n=0}^m F_n(p, s)$ . In particular, for  $q = \frac{4}{3}$ , we find by numerical evaluation  $\delta_{\frac{4}{3}}\left(\frac{48}{25}\right) > 0.0026 > 0$ , choosing  $m = 2$ , and  $\delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$ , choosing  $m = 4$ . Formula (5.2) implies

$$\begin{aligned} \frac{d}{dp} \delta_q(s) &= \sum_{n=0}^{\infty} \frac{d}{dp} F_n(p, s) =: -\frac{1}{p^2} \sum_{n=0}^{\infty} G_n(p, s) \\ &=: -\frac{1}{p^2} \sum_{n=0}^{\infty} \left( f_{2n}(p) g_{2n}(p) \frac{s^{4n}}{(4n)!} - f_{2n+1}(p) g_{2n+1}(p) \frac{s^{4n+2}}{(4n+2)!} \right), \end{aligned} \quad (5.4)$$

where in terms of the Digamma function  $\Psi$ ,  $g_{2n}(p) = (4n-1)\Psi\left(1 + \frac{4n-1}{p}\right) + \Psi\left(1 - \frac{1}{p}\right)$ ,  $g_{2n+1}(p) = (4n+1)\Psi\left(1 + \frac{4n+1}{p}\right) + \Psi\left(1 - \frac{1}{p}\right)$ . By Abramowitz, Stegun [1], 6.3,  $\Psi'$  is positive, decreasing and  $\Psi'(1+x) \leq \frac{1}{x+\frac{1}{2}}$ . Therefore

$$\begin{aligned} \Psi\left(1 + \frac{4n+1}{p}\right) &\leq \Psi\left(1 + \frac{4n-1}{p}\right) + \frac{2}{p} \Psi'\left(1 + \frac{4n-1}{p}\right) \\ &\leq \Psi\left(1 + \frac{4n-1}{p}\right) + \frac{1}{2n} (4n+1) \Psi\left(1 + \frac{4n+1}{p}\right) \\ &\leq (4n-1) \Psi\left(1 + \frac{4n-1}{p}\right) + 2 \Psi\left(1 + \frac{4n-1}{p}\right) + \frac{4n+1}{2n} \end{aligned}$$

so that

$$\frac{g_{2n+1}(p)}{g_{2n}(p)} \leq 1 + \frac{2 \Psi\left(1 + \frac{4n-1}{p}\right) + 2 + \frac{1}{2n}}{g_{2n}(p)} = 1 + \frac{2}{4n-1} + C_n(p) \quad (5.5)$$

where  $C_n(p) := \frac{2 - \frac{2\Psi(1-\frac{1}{p})}{4n-1} + \frac{1}{2n}}{g_{2n}(p)}$ . The function  $k(x) := \Psi(1+x) - \ln(1+x)$  is increasing, since using (3.1) we find

$$k'(x) = \sum_{n=1}^{\infty} \frac{1}{(n+x)^2} - \frac{1}{1+x} > \int_1^{\infty} \frac{dy}{(y+x)^2} - \frac{1}{1+x} = 0.$$

Further, by Abramowitz, Stegun [1, 6.3.18],  $\lim_{x \rightarrow \infty} \Psi(1+x) - \ln(1+x) = 0$ . Therefore for any  $x > 0$ ,  $k(x) < 0$ . For  $x \geq \frac{7}{4}$ ,  $k(x) \geq k(\frac{7}{4}) \simeq -0.193 > -\frac{1}{5}$ . Hence for  $n \geq 2$ ,  $\Psi\left(1 + \frac{4n-1}{p}\right) \geq \ln\left(1 + \frac{4n-1}{p}\right) - \frac{1}{5}$  and

$$C_n(p) \leq \frac{2 - \frac{2\Psi(1-\frac{1}{p})}{4n-1} + \frac{1}{2n}}{(4n-1) \ln\left(1 + \frac{4n-1}{p}\right) - \frac{4n-1}{5} + \Psi\left(1 - \frac{1}{p}\right)} =: R.$$

We have that  $\Psi\left(1 - \frac{1}{p}\right) \in [-1.964, -1.085]$ . The right side is maximal for  $p = 4$ , with  $\Psi\left(\frac{3}{4}\right) \simeq -1.086$ , the logarithmic term in the denominator being minimal then,

$$C_n(p) \leq \frac{2 - \frac{2\Psi(\frac{3}{4})}{4n-1} + \frac{1}{2n}}{(4n-1) \ln\left(n + \frac{3}{4}\right) - \frac{4n-1}{5} + \Psi\left(\frac{3}{4}\right)}.$$

A tedious calculation yields that  $C_n(p) \leq c_n(p) := \frac{2}{3n \ln(n)}$  and  $\frac{2}{4n-1} + c_n(2) \leq \frac{1}{n+\frac{1}{4}}$  for all  $n \geq 5$ . For  $n = 2, 3, 4$ , keeping a dependence on  $p$ , we may estimate  $C_n(p) \leq R \leq c_n(p)$ , where

$$c_2(p) = \frac{1}{6} + \frac{p}{10}, c_3(p) = \frac{1}{11} + \frac{p}{36}, c_4(p) = \frac{1}{18} + \frac{p}{72}.$$

Using this and Eqs. (5.5) and (5.3) we find for  $n \geq 2$

$$\begin{aligned} \frac{f_{2n+1}(p)g_{2n+1}(p)}{f_{2n}(p)g_{2n}(p)} &\leq \frac{4n+1}{p} \left(1 + \frac{2}{4n-1} + c_n(p)\right) \leq \frac{4n+1}{2} \left(1 + \frac{2}{4n-1} + c_n(2)\right) \\ &=: q_n \leq \frac{4n+1}{2} \left(1 + \frac{1}{n+\frac{1}{4}}\right) = \frac{4n+5}{2}, \end{aligned}$$

the last estimate for  $q_n$  holding for  $n \geq 5$ , whereas  $q_2 \leq \frac{15}{2}$ ,  $q_3 \leq \frac{25}{3}$  and  $q_4 \leq \frac{21}{2}$ . This implies

$$G_n(p, s) \geq f_{2n}(p)g_{2n}(p) \frac{s^{4n}}{(4n)!} \left(1 - \frac{q_n s^2}{(4n+1)(4n+2)}\right) =: \tilde{G}_n(p, s) > 0$$

for all  $n \geq 2$  and  $s \leq \frac{16}{5} < \sqrt{12}$ . Hence by (5.4) for all  $m \geq 1$

$$\frac{d}{dp} \delta_q(s) \leq -\frac{1}{p^2} \sum_{n=0}^m G_n(p, s).$$

For  $m = 1$ , with  $g_0(s) = 0$ , we have

$$\frac{d}{dp} \delta_q(s) \leq +\frac{s^2}{2p^2} \left[ a(p) - b(p) \frac{s^2}{12} + c(p) \frac{s^4}{360} \right] =: \phi(p, s)$$

with  $-0.972 \leq a(p) := f_1(p)g_1(p) \leq -0.954$  varying very little,  $-0.255 \leq b(p) := f_2(p)g_2(p) \leq 0.114$ ,  $0 < b(p)$  for  $p \leq 2.83$  and  $c(p) := f_3(p)g_3(p)$  decreasing in  $p \in [2, 4]$ , with value 6.66 at  $p = 2$  and 1.64 at  $p = 4$ . Therefore  $360|a(p)| \geq 343.5$  and  $\phi(p, s) < 0$  will be satisfied if

$$s^2 < 15 \frac{b(p)}{c(p)} + \sqrt{\left(15 \frac{b(p)}{c(p)}\right)^2 + \frac{343.5}{c(p)}}.$$

This holds for all  $0 \leq s \leq \frac{16}{5}$ , if  $c(p) \leq 3.275$ , i.e.  $p \geq 2.81$ . For  $0 \leq p \leq 2$ , the right side is minimal for  $p = 2$  and we require  $s \leq 2.72$ . If  $p < 2.81$  and  $s > 2.72$  one needs two more terms,  $m = 3$ , to show  $\frac{d}{dp} \delta_q(s) < 0$ ,

$$\frac{d}{dp} \delta_q(s) \leq -\frac{1}{p^2} \left[ G_0(p, s) + G_1(p, s) + \tilde{G}_2(p, s) + \tilde{G}_3(p, s) \right] < 0.$$

□

**Corollary 5.3** For all  $\frac{4}{3} \leq q \leq 2$  and  $\frac{48}{25} \leq s \leq \frac{16}{5}$ ,  $|\delta_q(s)| \leq 0.588$ .

**Proof** By Lemma 5.2,  $\delta_{\frac{4}{3}}(s) \leq \delta_q(s) \leq \delta_2(s) = \exp\left(-\frac{s^2}{4}\right) \leq \exp\left(-\left(\frac{24}{25}\right)^2\right) < \frac{2}{5}$  for all  $s \in [\frac{48}{25}, \frac{16}{5}]$ . By (2.7),  $\delta'_{\frac{4}{3}}(s) = -\frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} s \gamma_4(s)$ . According to Boyd [4], page 83,  $\gamma_4(s) > 0$  for all  $0 \leq s \leq 3.45$ , the first positive zero of  $\gamma_4$  being at  $s_1 \simeq 3.4535$ . Therefore  $\delta_{\frac{4}{3}}$  is strictly decreasing in  $[\frac{48}{25}, \frac{16}{5}]$ , with  $\frac{2}{5} > \delta_{\frac{4}{3}}(s) \geq \delta_{\frac{4}{3}}\left(\frac{16}{5}\right) > -0.588$  by Lemma 5.2. We conclude that

$$\max \left\{ |\delta_q(s)| \mid q \in \left[\frac{4}{3}, 2\right], s \in \left[\frac{48}{25}, \frac{16}{5}\right] \right\} \leq 0.588.$$

□

*Remark.* For  $q \searrow 1$ ,  $\delta_q(s) \rightarrow \cos(s)$ , so that  $|\delta_q(\pi)| \rightarrow 1$ . Corollary 5.3 does not extend to the range  $1 \leq q < \frac{4}{3}$ .

## 6 Proof of Theorem 1.2

*Proof of Theorem 1.1.*

Barthe and Naor [3] showed for  $1 \leq q < \infty$  that

$$\lim_{n \rightarrow \infty} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{2^{\frac{2}{q}}}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)}$$

and this is  $< 1$  if and only if  $\frac{4}{3} < q < 2$ , see Lemma 5.1 (b). Now consider  $\frac{4}{3} < q < 2$  and let  $p = \frac{q}{q-1}$  be the conjugate index,  $2 < p < 4$ . As in the proof of Theorem 1.1, we use  $\cos(x) \geq 1 - \frac{x^2}{2} + \frac{x^4}{26} > 0$  for all  $0 \leq x \leq \frac{3}{2}$ , so that by (2.6) for all  $s \leq \frac{3}{2}\sqrt{n}$

$$\begin{aligned} \delta_q\left(\frac{s}{\sqrt{n}}\right) &\geq \frac{p}{\Gamma\left(\frac{1}{q}\right)} \left( \int_0^{\frac{3}{2}\frac{\sqrt{n}}{s}} \left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26n^2}\right) r^{p-2} \exp(-r^p) dr \right. \\ &\quad \left. + \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty} \cos\left(\frac{sr}{\sqrt{n}}\right) r^{p-2} \exp(-r^p) dr \right) \\ &\geq \frac{p}{\Gamma\left(\frac{1}{q}\right)} \left( \int_0^{\infty} \left(1 - \frac{s^2 r^2}{2n} + \frac{s^4 r^4}{26n^2}\right) r^{p-2} \exp(-r^p) dr - R \right) \\ &= \frac{1}{\Gamma\left(1 - \frac{1}{p}\right)} \left( \Gamma\left(1 - \frac{1}{p}\right) - \Gamma\left(1 + \frac{1}{p}\right) \frac{s^2}{2n} + \Gamma\left(1 + \frac{3}{p}\right) \frac{s^4}{26n^2} - R \right), \\ R &:= \int_{\frac{3}{2}\frac{\sqrt{n}}{s}}^{\infty} \left(2 - \frac{s^2}{2n} r^2 + \frac{s^4}{26n^2} r^4\right) r^{p-2} \exp(-r^p) dr \\ &= \frac{1}{p} \int_{\left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p}^{\infty} \left(2u^{-\frac{1}{p}} - \frac{s^2}{2n} u^{\frac{1}{p}} + \frac{s^4}{26n^2} u^{\frac{3}{p}}\right) \exp(-u) du. \end{aligned}$$

Then  $u^{-\frac{1}{p}} \leq \frac{2}{3} \frac{s}{\sqrt{n}}$  and  $u^{\frac{3}{p}} \leq u^{\frac{3}{2}}$ . As in the proof of Theorem 1 (iii), for  $x \geq \frac{9}{2}$

$$\int_x^{\infty} u^{\frac{3}{2}} \exp(-u) du \leq \left( (1+x)(2+2x+x^2) \right)^{\frac{1}{2}} \exp(-x) \leq \frac{13}{20} x^2 \exp(-x).$$

Choose again  $s \leq \sqrt{\frac{n}{2}}$ . Then with  $x := \left(\frac{3}{2}\frac{\sqrt{n}}{s}\right)^p \geq \left(\frac{3}{\sqrt{2}}\right)^2 = \frac{9}{2}$  and  $y := \frac{s}{\sqrt{n}}$

$$R \leq \frac{1}{p} \left( 2\frac{2}{3}y + \frac{y^4}{26} \frac{13}{20} \left(\frac{3}{2}\frac{1}{y}\right)^{2p} \right) \exp\left(-\left(\frac{3}{2}\frac{1}{y}\right)^p\right).$$

We want  $R \leq \Gamma\left(1 + \frac{3}{p}\right) \frac{y^4}{26}$ , a condition which is strongest for  $p = 2$  when it means

$$\frac{1}{2} \left( \frac{4}{3} y + \frac{81}{640} \right) < \frac{\Gamma\left(\frac{5}{2}\right)}{26} y^4 \exp\left(\frac{9}{4} \frac{1}{y^2}\right)$$

which is the same requirement as in (iii) of the proof of Theorem 1.1, being valid for  $0 \leq y \leq 0.7161$ . Thus the choice of  $s \leq \sqrt{\frac{n}{2}}$  is allowed and then

$$\delta_q\left(\frac{s}{\sqrt{n}}\right) \geq 1 - c \frac{s^2}{n}, \quad c := \frac{1}{2} \frac{\Gamma\left(1 + \frac{1}{p}\right)}{\Gamma\left(1 - \frac{1}{p}\right)} = \frac{1}{2} \frac{\Gamma\left(2 - \frac{1}{q}\right)}{\Gamma\left(\frac{1}{q}\right)}.$$

We have by Lemma 5.1 (a)  $\frac{1}{4} \leq c \leq 0.3699 < \frac{37}{100}$ , the lower estimate attained for  $p = q = 2$ , the maximum for  $c$  attained for  $p = 4$ ,  $q = \frac{4}{3}$ . Therefore for  $x := c \frac{s^2}{n} \leq \frac{37}{200}$ ,  $\ln(1-x) \geq -x - \frac{1}{2} \frac{x^2}{1-x} \geq -x - \frac{100}{163} x^2$  and  $\left(1 - c \frac{s^2}{n}\right)^n \geq \exp(-cs^2) \left(1 - \frac{100}{163} c^2 \frac{s^4}{n}\right)$ . This yields the estimate

$$\begin{aligned} \int_0^{\sqrt{\frac{n}{2}}} \frac{1 - \delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2} ds &\leq \int_0^{\sqrt{\frac{n}{2}}} \frac{1 - \exp(-cs^2) \left(1 - \frac{100}{163} c^2 \frac{s^4}{n}\right)}{s^2} ds \\ &\leq \int_0^\infty \frac{1 - \exp(-cs^2)}{s^2} ds - \int_{\sqrt{\frac{n}{2}}}^\infty \frac{1 - \exp(-cs^2)}{s^2} ds + \frac{100}{163} \frac{c^2}{n} \int_0^\infty s^2 \exp(-cs^2) ds \\ &= \sqrt{\pi c} \left(1 + \frac{25}{163} \frac{1}{n}\right) - \int_{\sqrt{\frac{n}{2}}}^\infty \frac{1 - \exp(-cs^2)}{s^2} ds. \end{aligned}$$

Therefore, using (2.6),

$$\begin{aligned} P_{n,q}(a^{(n)}) &= \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} \int_0^\infty \frac{1 - \delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2} ds \\ &= \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} \left( \int_0^{\sqrt{\frac{n}{2}}} \frac{1 - \delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2} ds + \int_{\sqrt{\frac{n}{2}}}^\infty \frac{1 - \delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2} ds \right) \\ &\leq \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} \left( \sqrt{\pi c} \left(1 + \frac{25}{163} \frac{1}{n}\right) + \int_{\sqrt{\frac{n}{2}}}^\infty \frac{\exp(-cs^2) - \delta_q\left(\frac{s}{\sqrt{n}}\right)^n}{s^2} ds \right) \\ &= \sqrt{\frac{2}{\pi}} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right) \left(1 + \frac{25}{163} \frac{1}{n}\right) + \Gamma\left(\frac{1}{q}\right) \frac{2}{\pi} S, \end{aligned}$$

$S := \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp(-cu^2)^n - \delta_q(u)^n}{u^2} du$ . Since  $c \geq \frac{1}{4}$  and  $\delta_q(s) > 0$  for all  $0 \leq u \leq \frac{48}{25}$  by Lemma 5.2, we find

$$S \leq \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp\left(-\frac{u^2}{4}n\right)}{u^2} du + \frac{1}{\sqrt{n}} \int_{\frac{48}{25}}^{\infty} \frac{|\delta_q(u)|^n}{u^2} du.$$

For  $n \geq 8$ ,  $v = \frac{u^2}{4}n \geq 1$  and

$$\begin{aligned} \frac{1}{\sqrt{n}} \int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{\exp\left(-\frac{u^2}{4}n\right)}{u^2} du &= \frac{1}{4} \int_{\frac{n}{8}}^{\infty} \frac{\exp(-v)}{v^{\frac{3}{2}}} dv \\ &\leq \frac{1}{4} \left(\frac{8}{n}\right)^{\frac{3}{2}} \int_{\frac{n}{8}}^{\infty} \exp(-v) dv = \frac{4\sqrt{2}}{n^{\frac{3}{2}}} \exp\left(-\frac{n}{8}\right) \leq \frac{4\sqrt{2}}{n^{\frac{3}{2}}} 0.8825^n. \end{aligned}$$

By Corollary 5.3 we have  $|\delta_q(s)| \leq 0.588$  for all  $\frac{48}{25} \leq u \leq \frac{16}{5}$ . Therefore

$$\frac{1}{\sqrt{n}} \int_{\frac{48}{25}}^{\frac{16}{5}} \frac{|\delta_q(u)|^n}{u^2} du \leq \frac{1}{\sqrt{n}} 0.588^n \int_{\frac{48}{25}}^{\frac{16}{5}} \frac{du}{u^2} = \frac{5}{24} \frac{1}{\sqrt{n}} 0.588^n.$$

Integration by parts shows for  $\frac{4}{3} \leq q < 2$ ,  $2 < p \leq 4$  that

$$\begin{aligned} |\delta_q(u)| &= \left| \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^{\infty} \frac{\sin(ur)}{u} (p-2-pr^p) r^{p-3} \exp(-r^p) dr \right| \\ &\leq \frac{1}{u} \frac{p}{\Gamma\left(\frac{1}{q}\right)} \int_0^{\infty} |p-2-pr^p| r^{p-3} \exp(-r^p) dr \\ &= \frac{1}{u} \frac{2p}{\Gamma\left(1-\frac{1}{p}\right)} \left(\frac{1-\frac{2}{p}}{e}\right)^{1-\frac{2}{p}} =: \phi(p) \frac{1}{u}, \end{aligned}$$

using that  $\int (p-2-pr^p) r^{p-3} \exp(-r^p) dr = r^{p-2} \exp(-r^p)$ . For  $\phi$ , we have that  $(\ln \phi)'(p) = \frac{1}{p^2} \left(2 \ln\left(1-\frac{2}{p}\right) + p - \Psi\left(1-\frac{1}{p}\right)\right)$  and  $\lambda(p) := 2 \ln\left(1-\frac{2}{p}\right) + p - \Psi\left(1-\frac{1}{p}\right)$  is increasing, since  $\lambda'(p) = \frac{p(3+(p-1)^2)-(p-2)\Psi'\left(1-\frac{1}{p}\right)}{p^2(p-2)}$  and  $\Psi'\left(1-\frac{1}{p}\right) \leq \Psi'\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$ , so that  $\lambda'(p) \geq \frac{p(3+(p-1)^2)-(p-2)\frac{\pi^2}{2}}{p^2(p-2)} > 0$ . We have  $\lambda(2) < 0$ ,  $\lambda(4) > 0$ , so that  $\phi$  is first decreasing and then increasing in  $[2, 4]$ . Since  $\phi(2) < \phi(4) < \frac{14}{5}$ ,



we conclude that  $|\delta_q(u)| < \frac{14}{5} \frac{1}{u}$ . Then

$$\begin{aligned} \frac{1}{\sqrt{n}} \int_{\frac{16}{5}}^{\infty} \frac{|\delta_q(u)|^n}{u^2} du &\leq \frac{1}{\sqrt{n}} \left(\frac{14}{5}\right)^n \int_{\frac{16}{5}}^{\infty} \frac{du}{u^{n+2}} \\ &= \frac{5}{16} \frac{1}{\sqrt{n}(n+1)} \left(\frac{7}{8}\right)^n \leq \frac{5}{16} \frac{1}{n^{\frac{3}{2}}} 0.875^n. \end{aligned}$$

We finally get that

$$S \leq \frac{4\sqrt{2}}{n^{\frac{3}{2}}} 0.8825^n + \frac{5}{24} \frac{1}{\sqrt{n}} 0.588^n + \frac{5}{16} \frac{1}{n^{\frac{3}{2}}} 0.875^n.$$

By Lemma 5.1 (a),  $\sqrt{\frac{2}{\pi} \frac{\Gamma(\frac{1}{q})}{\Gamma(2-\frac{1}{q})}} \leq \frac{2}{\sqrt{\pi}}$ , with equality for  $p = q = 2$ , and  $\frac{2}{\sqrt{\pi}} S \leq \frac{0.10559}{n}$  for all  $n > 20$ . We conclude with  $\frac{25}{163} + 0.10559 < 0.25896$  that

$$P_{n,q}(a^{(n)}) \leq \sqrt{\frac{2}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right) \left(1 + \frac{0.25896}{n}\right)}.$$

By Barthe, Naor [3]  $P_{n,q}(a^{(2)}) = 2^{\frac{1}{2} - \frac{1}{q}}$ , so that together with Lemma 5.2 (b)

$$\begin{aligned} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} &\leq \sqrt{\frac{2^{\frac{2}{q}}}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right) \left(1 + \frac{0.25896}{n}\right)} \\ &\leq \left(1 - M \left(\frac{1}{q} - \frac{1}{2}\right) \left(\frac{3}{4} - \frac{1}{q}\right)\right) \left(1 + \frac{0.25896}{n}\right), \quad M = 0.86326. \end{aligned}$$

Suppose that  $n$  satisfies  $n \geq \frac{4}{5} \frac{q^2}{(q-\frac{4}{3})(2-q)}$ . Then the last product is  $< 1$ , since  $\frac{0.25896}{n} \leq$

$M \left(\frac{1}{q} - \frac{1}{2}\right) \left(\frac{3}{4} - \frac{1}{q}\right)$  suffices and this requires  $n > \frac{\frac{8}{3} \frac{0.25896}{M} q^2}{(q-\frac{4}{3})(2-q)}$ , with  $\frac{8}{3} \frac{0.25896}{M} < \frac{4}{5}$ .

The condition  $n \geq \frac{4}{5} \frac{q^2}{(q-\frac{4}{3})(2-q)}$  will be satisfied if  $n > \frac{\frac{32}{15}}{q-\frac{4}{3}} + \frac{\frac{24}{5}}{2-q} = \frac{\frac{8}{3} q - \frac{32}{15}}{(q-\frac{4}{3})(2-q)} \geq$

$\frac{\frac{4}{5} q^2}{(q-\frac{4}{3})(2-q)}$ , where the last inequality is an equality for  $q = \frac{4}{3}$  and  $q = 2$ . Hence for

$n > \frac{\frac{32}{15}}{q-\frac{4}{3}} + \frac{\frac{24}{5}}{2-q} =: \phi(q)$  we have  $P_{n,q}(a^{(n)}) < P_{n,q}(a^{(2)})$ . The restriction  $n > 20$  is automatically satisfied since the minimum of  $\phi$  is  $\phi(\frac{8}{3}) = 20$ . We have the equality

$$\lim_{n \rightarrow \infty} \frac{P_{n,q}(a^{(n)})}{P_{n,q}(a^{(2)})} = \sqrt{\frac{2^{\frac{2}{q}}}{\pi} \Gamma\left(\frac{1}{q}\right) \Gamma\left(2 - \frac{1}{q}\right)}. \quad \square$$

*Remark.* Barthe and Naor [3] show that  $P_{n,q}(a) \leq P_{n,q}(a^{(n)})$  for all  $a \in S^{n-1}$  and  $q \geq 2$ . Since the quantities are differentiable and coincide for  $q = 2$ , they conclude that  $\frac{d}{dq} P_{n,q}(a)|_{q=2} \leq \frac{d}{dq} P_{n,q}(a^{(n)})|_{q=2}$  holds. If this inequality were strict,

$P_{n,q}(a) \geq P_{n,q}(a^{(n)})$  might hold for some  $2 - \varepsilon_n < q < 2$ . In this case, for  $q$  close to 2,  $a^{(n)}$  might always yield the minimum, and no singularity  $O(\frac{1}{2-q})$  would occur in Theorem 1.2.

Similarly, in the case of sections, Koldobsky [10] showed that  $A_{n,p}(a^{(n)}) \leq A_{n,p}(a)$  holds for all  $1 < p < 2$  and  $a \in S^{n-1}$ , which together with  $A_{n,2}(a^{(n)}) = A_{n,2}(a)$  implies  $\frac{d}{dp} A_{n,p}(a)|_{p=2} \leq \frac{d}{dp} A_{n,p}(a^{(n)})|_{p=2}$ . If this inequality were strict, it might be that  $A_{n,p}(a) \leq A_{n,p}(a^{(n)})$  for  $p$  close to 2,  $2 < p < 2 + \varepsilon_n$ .

However, numerical evaluation of (2.3) yields for  $p = 6$ ,  $p = 8$  and  $n = 3, 4$  that  $a^{(n)}$  is not maximal, since

$$\begin{aligned} A_{3,6}(a^{(2)}) &= 2^{\frac{1}{3}} \simeq 1.260 > A_{3,6}(a^{(3)}) \simeq 1.250 \\ A_{4,8}(a^{(2)}) &= 2^{\frac{3}{8}} \simeq 1.297 > A_{4,8}(a^{(4)}) \simeq 1.295 > A_{4,8}(a^{(3)}) \simeq 1.270. \end{aligned}$$

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