

Essays on Nonlinear Contracting
under Uncertainty and Asymmetric Information:
Trade-off between Risk Sharing and Moral Hazard Problem

Inaugural-Dissertation
zur Erlangung des akademischen Grades eines Doktors
der Wirtschafts- und Sozialwissenschaften
der Wirtschafts- und Sozialwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

vorgelegt von
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Kiel, 2025

Gedruckt mit Genehmigung der
Wirtschafts- und Sozialwissenschaftlichen Fakultät
der Christian-Albrechts-Universität zu Kiel

Erstbegutachtung: Prof. Dr. Peter Nippel

Zweitbegutachtung: Prof. Dr. Alexander Klos

Tag der mündlichen Prüfung: 21.11.2025

Vorwort

Die vorliegende Dissertation wurde während meiner Tätigkeit als wissenschaftlicher Mitarbeiter an der Professur für Finanzwirtschaft von Herrn Prof. Dr. Peter Nippel verfasst. Herrn Prof. Dr. Peter Nippel spreche ich meinen aufrichtigen Dank für die fachliche Betreuung und die anregenden wissenschaftlichen Diskussionen aus, die maßgeblich zur Weiterentwicklung dieser Arbeit beigetragen haben. Mein besonderer Dank gilt zudem Herrn Prof. Dr. Alexander Klos für die Übernahme des Zweitgutachtens.

Ebenso danke ich meinen Kollegen Dr. Martin Krebs und Victor Farahwaran für den stets konstruktiven fachlichen Austausch sowie die angenehme Zusammenarbeit während unserer gemeinsamen Zeit am Lehrstuhl. Weiterhin möchte ich den Inhabern und den Promovierenden der anderen Professuren für den Austausch wertvoller Erkenntnisse und Ideen während unserer gemeinsamen Doktorand:innenseminare danken.

Für die jederzeit zuverlässige und engagierte Unterstützung bei organisatorischen und administrativen Belangen danke ich Frau Barbara Geisler und Frau Hille Rowehl ausdrücklich. Ihre Hilfsbereitschaft hat meine Arbeit in vielerlei Hinsicht erleichtert.

Die freundliche und produktive Arbeitsatmosphäre am Lehrstuhl wird mir in sehr guter Erinnerung bleiben.

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List of Abbreviations

<i>CAR</i>	Capital Asset Ratio
<i>CE</i>	Certainty Equivalent
<i>EAD</i>	Exposure at Default
<i>EL</i>	Expected Loss
<i>FOA</i>	First-Order Approach
<i>LEN</i>	Linear-Exponential-Normal
<i>LGD</i>	Loss Given Default
<i>LHS</i>	Left-Hand Side
<i>NB</i>	Net Benefit
<i>NPE</i>	Net Profit under Equity
<i>NPM</i>	Net Profit under <i>Musharaka</i>
<i>NPV</i>	Net Present Value
<i>PD</i>	Probability of Default
<i>PLS</i>	Profit and Loss Sharing
<i>RAR</i>	Risk-Adjusted Return
<i>RHS</i>	Right-Hand Side
<i>ROA</i>	Return on Assets
<i>RP</i>	Risk Premium
<i>RPE</i>	Risk Premium under Equity
<i>RPM</i>	Risk Premium under <i>Musharaka</i>
<i>RR</i>	Recovery Rate
<i>SME</i>	Small and Medium-Sized Enterprises

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Overview

The papers in this thesis explore optimal contracting under uncertainty and asymmetric information and the trade-off between the benefits of risk sharing and the drawbacks resulting from moral hazard problem in various agency relationships. A contract involving profit and loss sharing results in an agency relationship, characterized by risk sharing when the parties to the contract share in an uncertain outcome. A risk-averse contracting party will benefit from risk sharing; however, if one (or more) parties to the contract can influence the outcome with their (costly) effort which is unobservable by their counterparts, profit and loss sharing will diminish the incentives of the agent(s) to extend the first-best level of effort since they will have to share the benefit of their effort with their counterparts while bearing the cost of effort on their own. Therefore, the moral hazard problem (also known as the hidden action problem) arising from informational asymmetries between parties to a profit-and-loss-sharing arrangement results in a trade-off between the benefits of risk sharing and the drawbacks from diluted incentives to extend effort.

This trade-off has been extensively explored in agency-theoretic literature. It has been suggested that a linear incentive scheme in a principal-agent relationship, where the principal's objective is to maximize their utility by the choice of an incentive scheme meanwhile the agent maximizes their utility with the choice of their hidden effort, cannot be Pareto-optimal. The trade-off between risk sharing and moral hazard will diminish the agent's incentives to extend the first-best level of effort and result in a suboptimal level of risk sharing. It has also been argued that an incentive scheme involving a fixed and an outcome-dependent payment could alleviate the adverse effects of the moral hazard problem in a principal-agent relationship. The outcome-dependent payment would then constitute a linear incentive scheme. With a few exceptions, incentive schemes involving nonlinear sharing of "good" and "bad" outcomes (in addition to the fixed payment) have not been sufficiently analyzed even though it has been generally shown that the optimal incentive schemes in the presence of moral hazard problem are nonlinear in output.

This dissertation, with the aim of finding an incentive scheme that can eliminate the adverse effects of the moral hazard problem in agency relationships involving profit and loss sharing, explores nonlinear profit-and-loss-sharing arrangements inspired by Islamic profit-and-loss-sharing contracts between parties that entail sharing of profits and losses from a project based on possibly different ratios. In such arrangements, the losses are shared between

the parties based on their initial capital contributions to the project while the profits are shared based on a pre-agreed ratio. Furthermore, Islamic PLS arrangements can be used to partially finance a business, the use of which is the topic of the first paper. The nonlinear (or piecewise-linear) sharing of profits and losses is applied in the second and third papers.

Paper	Title	Author(s)
1	Islamic Profit-and-Loss-Sharing Contracting versus Regular Equity in Entrepreneurial Finance: Risk Sharing and Managerial Incentives <ul style="list-style-type: none"> published in <i>The Journal of Entrepreneurial Finance</i>: Hadizada, A. and Nippel, P. (2022). Islamic profit and loss sharing contracting versus regular equity in entrepreneurial finance: Risk sharing and managerial incentives. <i>The Journal of Entrepreneurial Finance</i>, 24(2), 209-247. doi:10.57229/2373-1761.1449 	Abdulali Hadizada and Peter Nippel
2 – Part I	Nonlinear Profit and Loss Sharing versus Equity Financing in Entrepreneurial Finance: Risk Sharing and Managerial Incentives in a Principal-Agent Relationship When the Agent Is Risk-Neutral <ul style="list-style-type: none"> published in <i>New Challenges in Accounting and Finance</i>: Hadizada, A. (2025). Nonlinear profit-and-loss-sharing contracting versus equity in entrepreneurial finance: Risk sharing and managerial incentives in a principal-agent relationship when the agent is risk-neutral. <i>New Challenges in Accounting and Finance</i>, 13, 28-51. doi:10.32038/NCAF.2025.13.02 	Abdulali Hadizada
2 – Part II	Nonlinear Profit and Loss Sharing versus Equity Financing in Entrepreneurial Finance: Risk Sharing and Managerial Incentives in a Principal-Agent Relationship When the Agent Is Risk-Averse	Abdulali Hadizada
3	Bank Loan Credit Risk Pooling: Risk Diversification versus Moral Hazard	Abdulali Hadizada

The first paper analyzes the consequences of risk sharing and moral hazard problem when an entrepreneur, starting a business, shares not the total business risk, but only the risk associated with a certain area of the firm with an outside financier. Such a financing alternative is compared to equity financing, and it is shown that when the entrepreneur finances only a part of the business with capital from an outside partner, hence sharing only a portion of the business risk, the entrepreneur's incentive to extend effort is less diluted in comparison to financing with equity capital that involves sharing of the outcome of the whole business and of the total risk associated with it. Moreover, partial risk sharing turns out to be more advantageous for the

entrepreneur than total risk sharing with the equity partner if the portion of the business risk not shared with an outside financier is, *ceteris paribus*, relatively small. The suggested financing alternative is referred to as a *musharaka* contract, a profit-and-loss-sharing contract in Islamic finance, even though in practice, a *musharaka* contract slightly differs from the contract analyzed in this paper.

In the first paper, the financing alternative being compared to equity financing involves linear sharing of profits and losses resulting from the part of the business being financed with capital from the outside financier. The second and third papers apply the nonlinear profit-and-loss-sharing aspect of Islamic PLS contracts in two different settings. The second paper explores the principal-agent relationship between an entrepreneur and a financier who provides capital for the entrepreneur's business whose outcome is shared nonlinearly depending on whether a profit or a loss occurs. The third paper analyzes a similar nonlinear sharing rule in a bank loan credit risk pooling arrangement. While the profits and losses in the second paper refer to the increase or decrease in the value of the business, the occurrence of a loan default is the determinant of the sharing rule in the context of credit risk pooling.

Paper 2 is presented in two parts. The agent (i.e., the entrepreneur) is assumed to be risk-neutral and risk-averse in the first and second parts, respectively. The textual similarities and overlaps are due to the fact that they were originally conceived as one and later separated. Both are to be perceived as standalone papers.

The second paper introduces a nonlinear profit-and-loss-sharing arrangement with a fixed payment component that can eliminate (or alleviate) the adverse effects of the moral hazard problem in a principal-agent relationship. In such an arrangement, the profits and losses defined as the increase and decrease in the value of the business or project that an entrepreneur finances with capital from an outside financier are shared between the entrepreneur and the financier based on different ratios. The losses (i.e., the value of the business in case of a decrease in value) are shared proportionally to initial capital contributions while the profits (i.e., the value of the business in case of an increase in value) are shared based on a pre-agreed ratio. When the entrepreneur is assumed to be risk-neutral, nonlinear profit and loss sharing can be arranged so that the incentives of the entrepreneur to extend effort are not diluted. In case of risk neutrality, the entrepreneur does not benefit from risk sharing *per se*, however, the nonlinear PLS arrangement is shown to be a feasible alternative to financing with usually unavailable risk-free debt capital. When the entrepreneur is assumed to be risk-averse, the suggested nonlinear PLS arrangement diminishes the incentives of the entrepreneur to a lesser

extent than financing with equity capital that involves linear sharing of profits and losses. It is also shown that the optimally arranged nonlinear profit and loss sharing is superior to debt or equity financing for the risk-averse entrepreneur.

The third paper presents such a nonlinear sharing of credit risks in a bank loan credit risk pooling arrangement that the negative effects of the moral hazard problem associated with credit risk transfer instruments are eliminated. In such an arrangement, banks pool their loans together and monitor the loans they contribute to the pool by extending (costly) effort. They share the payments from the loan pool based on a nonlinear sharing rule. In case a loan in the pool defaults, the originating bank of the loan receives a share of the recovered amount from the loan in proportion to its contribution to the pool (in terms of face value). In case of no default, the share of the payment from the loan the originating bank receives is selected so that the incentives of the bank to monitor its loan are not diminished. Thus, the adverse effects of the moral hazard problem are eliminated. Furthermore, the contributions of the participant banks to the loan pool can be arranged in such a way that they all benefit from the diversification of credit risk.

The risk sharing arrangements analyzed in these papers alleviate or completely eliminate the negative effects of the moral hazard problem in various agency relationships. Furthermore, such arrangements, when optimally structured, are shown to be the superior alternative (albeit only under certain conditions in the first paper) when it comes to the trade-off between risk sharing (or risk diversification) and moral hazard. The major limitations of the results obtained in these papers can be tied to the underlying assumptions. Most importantly, it is assumed throughout the thesis that there are no informational asymmetries other than hidden action (i.e., the unobservable effort of the agent(s)). Furthermore, the parties to the risk sharing arrangements considered in this thesis are assumed to have unlimited liability. While they provide analytical tractability, these assumptions can be relaxed, by introducing hidden information and limited liability, in further agency-theoretic analyses of nonlinear profit-and-loss-sharing arrangements to better assess their practical applicability.

Islamic Profit-and-Loss-Sharing Contracting versus Regular Equity in Entrepreneurial Finance: Risk Sharing and Managerial Incentives

Islamic PLS versus Equity: Risk Sharing and Managerial Incentives

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Published in *The Journal of Entrepreneurial Finance*

Link: <https://doi.org/10.57229/2373-1761.1449>

Abstract

An entrepreneur shares business risk with the investors providing capital for her firm. Risk sharing is *per se* beneficial, but also results in an agency problem from diminished incentives for the entrepreneur. This classical trade-off depends on the financial contracting between the entrepreneur and the financier. As an alternative to debt or equity, we consider *musharaka* financing, an Islamic profit-and-loss-sharing contract. First, we show that debt is inferior to equity or *musharaka* even though debt financing ensures first-best efforts in our model. Whether financing with equity or by use of *musharaka* results in higher utility for the entrepreneur depends on how the firm's risks are related and on the structure of the costs the entrepreneur has to bear when spending effort.

Keywords: entrepreneurial finance, equity, *musharaka*, risk sharing, managerial incentives, profit and loss sharing

1. Introduction

An entrepreneur needs capital and has to provide effort to advance the success of her business. Capital can be raised primarily as debt or equity; however, hybrid instruments might also be available. In this paper, we focus on a near-equity financial instrument known as *musharaka* in Islamic finance.¹ A *musharaka* contract results in profit and loss sharing not unlike a partnership with an equity investor. *Musharaka* is, however, more similar to project financing in which the investor provides capital to be invested in some particular tangible assets or a joint venture (Jobst, 2017), and consequently, the profits and losses from this particular project are shared. In contrast, providing equity capital results in co-ownership of the entire business, and the equity holders share all profits or losses and any increase or decrease in total value of the firm.

Therefore, *musharaka* and equity financing differ with respect to risk sharing and result in different incentives for the entrepreneur. We compare the alternative financial contracts in an agency-theoretic model with risk sharing and managerial incentives for the entrepreneur as the main determinants of the entrepreneur's utility. We start with debt financing and then consider *musharaka* and equity financing from the perspective of an entrepreneur in a principal-agent relationship.

In our model, although (risk-free) debt financing results in first-best effort of the entrepreneur, this alternative of raising capital turns out to be always inferior to *musharaka* and equity financing; With debt financing, the entrepreneur foregoes the benefits of risk sharing, which cannot be compensated for by the effect of higher incentives. The direct comparison of *musharaka* and equity financing reveals no definitive ranking. Which of the alternative financial instruments results in higher utility for the entrepreneur depends on the structure of the risks from the entrepreneurial business and on the structure of the costs the entrepreneur has to bear when spending effort. The partial profit and loss sharing via *musharaka* financing results in suboptimal risk sharing. On the other hand, *musharaka* comes with better incentives for the entrepreneur and results in a higher increase in expected firm value net of (private) costs under certain conditions. We consider this trade-off analytically and present a simple numerical example for illustration.

¹ Prohibition of activities deemed sinful by Islam governs Islamic banking and finance. From a financial perspective, such prohibitions apply to interest, speculation, excessive uncertainty, and short selling (Iqbal & Mirakhor, 2011).

The rest of this paper is structured as follows. In the next section, we provide a short introduction into Islamic (corporate) finance and compare *musharaka* and equity financing with respect to contractual matters. Section 3 provides the review of the literature on the agency relationships and risk sharing pertaining to Islamic mode of financing. In section 4, we present the theoretical model used for our analyses. Section 5 comparatively explores the alternative methods of financing with debt, *musharaka*, and equity financing in this theoretical framework. Equity financing is compared to *musharaka* in section 6. The insights of this theoretical comparison are illustrated by means of a numerical analysis in section 7. Section 8 provides concluding remarks.

2. Profit and Loss Sharing in Islamic Finance

Islamic methods of financing include “synthetic loans”, lease contracts, and profit-sharing contracts (Jobst, 2017). The profit-sharing contracts adhere the most to the three principles underlying Islamic finance: principle of equity (wealth distribution), principle of participation (risk sharing), and principle of ownership (asset-based financing) (Hussain et al., 2015). Two types of profit-sharing contracts are “profit-sharing-and-loss-bearing” *mudaraba* contracts and “profit-and-loss-sharing” (PLS) *musharaka* contracts (Hussain et al., 2015).

In a *mudaraba* transaction, an investor provides capital for a project or an investment whereas an entrepreneur plays the role of a manager. The investor is not entitled to take part in the management process, and the profits generated from the investment are distributed between the investor and the entrepreneur based on a pre-agreed ratio while only the investor bears any losses unless they result from the negligence of the entrepreneur (Jobst, 2017). *Mudaraba* financing is used mostly by banks, where the bank can act both as an investor (on the assets side) and as an entrepreneur (on the liabilities side).

Musharaka can be defined as “an equity partnership agreement with one or more partners to jointly finance an investment project” (Jobst, 2017). In a *musharaka* transaction, both the investor and the entrepreneur provide capital to finance a project or an investment. *Musharaka* financing can be used for corporate financing by non-financial companies, and it is a preferred way of Islamic financing due to its features of profit and loss and risk sharing.

Financing with regular equity, as well as by means of a *musharaka* contract, results in a partnership relationship between an entrepreneur and investor(s) (for *musharaka* as partnership, see, e.g., Usmani (1999), Mirakhor and Zaidi (2007), or Jobst (2017)). One of the parties to the partnership, the entrepreneur, is running a business or planning to start one. The

other party invests in the business and shares in profits and losses. With regular equity, the investment is open-ended, and all the partners hold a share in the whole business and participate in the firm's profits and the increase (or decrease) in its value. Unlike equity, *musharaka* contracts are not necessarily open-ended or can be terminated by notice of one partner. However, if one of the partners wishes to terminate his/her participation, the remaining partners may buy him/her out of the partnership (Usmani, 1999). Both types of partnership entail the right, but not the obligation of the investor(s) to take part in the management process.

The main difference between equity and *musharaka* financing is, or at least can be, in the extent of profit and loss sharing. Equity capital is used for financing the entire business of a firm, and each of the equity partners holds a share in the success of that business in total. On the other hand, *musharaka* financing can be used for financing a specific project or investment, fixed assets, real estate, commodities, working capital, or other tangible assets to be used in production or trade (see Usmani (1999), Habib (2018), Rahman et al. (2020)). Yildirim (2021) identifies *musharaka* as a project-specific instrument that does not entitle the investors as the owners of the entire firm. Jobst (2017) considers a *musharaka* contract being similar to a joint venture. While a share in the firm's equity entitles the financier to a claim on all of the firm's assets, a *musharaka* contract would allow a claim only on the profit generated by the investment being financed (Bacha et al., 2015). Then a *musharaka* investor does not profit from an increase in firm value in total, but only from the revenues and the increase in the value of the financed assets or projects.

This kind of partnership with sharing of profits and losses from specific assets requires identifiability and measurability, which are least problematic for tangible assets. Therefore, in our analysis, we make a distinction between tangible and intangible assets, wherein the *musharaka* contract involves an investment in tangible assets only, and consequently, sharing the profits and losses from only those assets. The profits from the increase in the value of intangible assets or growth opportunities are not shared between the *musharaka* investor and the entrepreneur.

3. Literature on Agency Aspects and Risk Sharing in (Islamic) Finance

We consider a principal-agent relationship between a risk-averse entrepreneur and a risk-averse external financier (investor/partner). The entrepreneur as the manager of the firm can spend effort to increase its future value. The effort comes at private costs for the entrepreneur. In that case, the contract between the principal and the agent determines not only

the allocation of risk, but also the managerial incentives and hence, the agent's effort. This principal-agent setting is similar to those in the seminal papers by Holmström (1979), Shavell (1979), and Grossman and Hart (1983).

In corporate finance, principal-agent relationships between managers (agents) and shareholders or bondholders as principals have been extensively explored, with Jensen and Meckling (1976) as one of the most influential papers. In that literature, external shareholders and bondholders are assumed to be largely diversified, hence interested in the market value of their claims. Then the optimal capital structure mainly has to trade off the agency costs of debt and equity, tax benefits, and bankruptcy costs. The allocation of risk by financial contracting and the firm's capital structure are *per se* not important if all investors manage their portfolio risk via buy-or-sell transactions on a broad capital market.²

This is different for small and medium-sized enterprises (SME), i.e., in entrepreneurial context. The entrepreneur often has a considerable share of her wealth invested in her firm and therefore, cares about the direct impact of the risk on her utility and not only the (risk-dependent) market value of her firm. Similarly, the investor's utility depends directly on the risk he has to bear as financier if his financial claims are not traded on a frictionless capital market. This kind of financial relation between an individual entrepreneur and a financier, both risk-averse and both with a future uncertain wealth that depends primarily on the firm's performance, is what we are going to analyze.

Such a principal-agent relationship within the framework of Islamic financial transactions is seldom represented in literature even though there has been an extensive amount of research in the field of Islamic finance and banking. Most of this research, however, consists of various ways of comparing Islamic and conventional banks with respect to their financial characteristics and performance using econometric methods (e.g., Čihák and Hesse (2010), Beck et al. (2013), Johnes et al. (2014), Pappas et al. (2016)). The theoretical framework of Islamic financing instruments, especially, PLS contracts, with consideration for risk sharing and managerial incentives is hardly explored, with a few notable exceptions outlined below.

Mirakhor and Zaidi (2007) classify and describe Islamic financial instruments, including the profit-sharing contracts of *mudaraba* and *musharaka*. They define *mudaraba*

² With investor's unlimited access to frictionless secondary markets, a firm's capital structure is irrelevant for the ultimate allocation of its business risk. Therefore, the assumption of unlimited access to frictionless secondary markets is one of the building blocks of the Modigliani and Miller (1958) theorem.

financing as “financing by way of trust” and *musharaka* financing as a “partnership”. They also refer to moral hazard problems in the context of *musharaka* and hint to the issue that risk aversion of individual entrepreneurs may reduce the efficiency of outcome-based incentive systems, which force them to absorb the risk that their income may vary owing to factors beyond their control. The authors qualitatively infer, high-powered incentives and efficient allocation of risk are, therefore, in conflict. In support of such conflict, Dar (2007) claims that the profit-sharing contracts in Islamic financing such as *mudaraba* and *musharaka* contracts lead to an incentive incompatibility between the entrepreneur and the investor. Our model analyzes this conflict and compares *musharaka* financing with debt or equity with respect to the entrepreneur’s utility.

Formal models of *musharaka* financing with agency conflicts can be found in, e.g., Yousfi (2013), Elfakir and Tkiouat (2015), and Arbi (2021). Yousfi (2013) considers PLS financing in a double-sided moral hazard problem with risk neutrality. Hence, risk sharing is not an issue. Nevertheless, the first-best solution is not reachable under *musharaka* financing since PLS implies partially externalizing the marginal returns of each agent’s effort. Elfakir and Tkiouat (2015) examine *musharaka* financing in an agency model, where either effort is assumed to be observable or the assumed distribution of output allows for a forcing contract. With all parties assumed to be risk-neutral, Arbi (2021) analyzes diminishing *musharaka* contracts, which resemble leasing contracts. He considers a moral hazard problem when asset maintenance is not observable and concludes that if the lessor plans to buy out the lessee at the end of the contract, he/she purposefully neglects the maintenance of the asset.

Different from our problem of unobservable effort in combination with risk sharing is the moral hazard problem that may result if output or the state of nature under which production takes place is unobservable for the principal (costly state verification). That kind of moral hazard problem with contracts as revelation mechanisms is considered by, e.g., Haque and Mirakhor (1986), Presley and Sessions (1994), Yustiardi et al. (2020), and Ajmi et al. (2020).

In overview, the existing theoretical literature mostly focuses on *mudaraba* financing when it comes to Islamic profit-sharing contracts and the informational asymmetries inherent in such contracts. *Musharaka* financing has not been compared to other forms of financing with a consideration for risk sharing and effort-based moral hazard problem in a theoretical framework. Our analysis contributes to the existing literature by theoretically deriving the implications of risk sharing and incentives in a partnership relationship between an entrepreneur and a *musharaka* investor.

4. The Model

We employ the LEN (Linear-Exponential-Normal) model, which is well-known from agency theoretic frameworks dating back to seminal papers Holmstrom and Milgrom (1987), Spremann (1988), and Holmstrom and Milgrom (1991). For the entrepreneur E (she) seeking financing of her business, as well as for potential financiers, we assume an exponential utility function:

Assumption 1.

$$U(W_{1,i}) = -e^{(-\theta_i W_{1,i})}, \quad (1)$$

where $W_{1,i}$ is the future uncertain wealth of individual i (i.e., the entrepreneur or a financier) in $t = 1$, and $\theta_i > 0$ is i 's (constant) absolute risk aversion coefficient (Pratt, 1964).

The future wealth $W_{1,E}$ of the entrepreneur depends on the outcome of her business and on how that outcome is divided between herself and an outside investor/partner (he) as a financier. This division depends on the financial contract between the entrepreneur and the financier. The financier's future wealth also depends on the firm's success and how it is shared.

The entrepreneur's business needs an initial investment of capital C . We assume that this investment is in tangible assets and intangible assets or provides for growth opportunities. The tangible assets are of value \tilde{a} in $t = 1$, and the value of the intangible assets or growth opportunities in $t = 1$ is \tilde{b} . Both values depend on the entrepreneur's unobservable effort. With efforts e_a and e_b , respectively, she can shift the distributions of \tilde{a} and \tilde{b} to the right:

Assumption 2.

Asset values depend on the entrepreneur's effort and are jointly normally distributed:

$$\tilde{a} = x + e_a + \tilde{\varepsilon}_a, \quad (2)$$

and

$$\tilde{b} = y + e_b + \tilde{\varepsilon}_b, \quad (3)$$

where $x > 0$ and $y > 0$ are constants, e_a and e_b indicate the entrepreneur's efforts, $\tilde{\varepsilon}_a$ and $\tilde{\varepsilon}_b$ are jointly normally distributed random variables with $E(\tilde{\varepsilon}_a) = 0$, $E(\tilde{\varepsilon}_b) = 0$, $Var(\tilde{\varepsilon}_a) > 0$, and $Var(\tilde{\varepsilon}_b) > 0$.

The entrepreneur not only chooses the total effort $e_a + e_b$, but also has to decide how to allocate her effort between different tasks, i.e., providing for the value \tilde{a} of tangible assets or for the value of intangible assets or growth opportunities, \tilde{b} . With her effort, the entrepreneur increases the expected value of the firm, but that comes at private costs.

Assumption 3.

e_a and e_b are unobservable for anyone but the entrepreneur. The entrepreneur's private cost of exerting effort is

$$c(e_a, e_b) = \frac{e_a^2 + e_b^2}{2} + \gamma e_a e_b, \quad (4)$$

where $-1 < \gamma < 1$ represents the interdependence between the efforts e_a and e_b .³

The non-observability of the efforts results in a moral hazard problem with the entrepreneur as the agent and an outside financier as the principal. We assume increasing marginal costs and a positive or negative cross derivative, $\frac{d^2c}{de_a de_b} = \gamma$, with the cost function in (4). For $\gamma > 0$, a higher effort e_a to increase the value \tilde{a} of tangible assets increases the marginal cost of effort e_b to increase the value \tilde{b} of intangible assets, and *vice versa*. Efforts are independent when $\gamma = 0$. For $\gamma < 0$, a higher effort e_a decreases the marginal cost of effort e_b , and *vice versa*.

Limited management capacity of the entrepreneur probably results in a negative relation between e_a and e_b , i.e., a higher effort in one area goes hand in hand with a lower effort in the other. We find that for $\gamma > 0$ in our model. But a positive relation between e_a and e_b may also prevail, i.e., a higher effort in one area results in lower effort costs and hence, in a higher effort in the other, e.g., due to complementarities or learning effects. We find that for $\gamma < 0$ in our model.

³ We restrict γ to the interval $]-1;1[$ to rule out nonsensical results: See Appendix A.

Given the assumptions 1 and 2 above, the certainty equivalent of the entrepreneur's uncertain future wealth is

$$CE_E = E(W_{1,E}) - \frac{1}{2} \theta_E \text{Var}(W_{1,E}) \quad (5)$$

(Freund, 1956), where the expected value $E(W_{1,E})$ and the variance $\text{Var}(W_{1,E})$ depend on the financial contract entered by the entrepreneur and her financier. This financial contract directly affects the entrepreneur's utility by allocating the expected value of the firm and the business risk. Furthermore, the financial contract also has incentive effects on the entrepreneur's effort decision and hence indirectly determines her utility.

Assumption 4.

The entrepreneur chooses the financial structure that maximizes the certainty equivalent (5) of her future uncertain wealth, i.e., her expected utility, by deciding between alternative financial contracts, and subsequently, she chooses the effort that maximizes her utility. Financiers are also risk-averse and rational and anticipate the entrepreneur's effort decision correctly.

In the following, we consider three variants of financial structure for the entrepreneurial firm: (i) debt financing; (ii) financing through a *musharaka* contract with an investor I who will hold a share α_I in \tilde{a} (but not a share in \tilde{b} due to the nature of the *musharaka* contract as discussed in section 2); (iii) financing with equity, i.e., a partnership with an equity partner P who will hold a share α_P in the firm, that is, in both asset classes.

The entrepreneur's initial wealth $W_{0,E}$ in $t = 0$ may or may not be sufficient to finance the investment in her business, i.e., possibly, $W_{0,E} \geq C$ or $W_{0,E} < C$. In case of sufficient capital, $W_{0,E} \geq C$, she does not need an outside financier, but, as we will see, will nevertheless benefit from contracting with an outside financier because of the merits of risk sharing. In case of insufficient capital, $W_{0,E} < C$, the entrepreneur needs external financing to start her business. The necessary funds, $(C - W_{0,E})$, can be raised by debt financing, through a *musharaka* contract, or by equity financing. If the entrepreneur refrains from starting her business, she invests all her capital in risk-free assets on the market, earning a return rate of r_f . Even with investing in the firm after raising external capital with *musharaka* contract or as equity, she might have some money left to invest outside her firm at the rate of r_f . We assume that the

same return rate is charged on any debt of the entrepreneur's firm. This implies that the debt is risk-free, and no transaction costs or other market imperfections result in differences between lending and borrowing rates. We state this assumption to concentrate on risk-sharing benefits and adverse incentive effects instead of other implications of market imperfections.

Assumption 5.

In case the entrepreneur raises any debt to finance her firm, this debt is free of default risk, and the interest rate r_f on debt is the same as the rate of return on alternative (risk-free) investments outside the firm.

5. Alternative Methods of Financing

5.1. Debt financing

We first consider debt financing for the entrepreneur with initial wealth $W_{0,E} < C$. In this case, the entrepreneur's wealth in $t = 1$ is

$$\tilde{W}_{1,E} = \tilde{a} + \tilde{b} - c(e_a, e_b) - (1 + r_f)(C - W_{0,E}), \quad (6)$$

with $(C - W_{0,E})$ being the debt raised.

The certainty equivalent of this future wealth in (6) is

$$CE_E = E(\tilde{a} + \tilde{b}) - c(e_a, e_b) - (1 + r_f)(C - W_{0,E}) - \frac{1}{2} \theta_E \text{Var}(\tilde{a} + \tilde{b}). \quad (7)$$

With (2) and (3) for the asset values from Assumption 2 and (4) for the cost function from Assumption 3, we see how this certainty equivalent depends on the entrepreneur's effort:

$$CE_E = x + y + e_a + e_b - \frac{1}{2}(e_a^2 + e_b^2) - \gamma e_a e_b - (1 + r_f)(C - W_{0,E}) - \frac{1}{2} \theta_E \text{Var}(\tilde{a} + \tilde{b}). \quad (8)$$

After the business is initiated with debt financing, the entrepreneur chooses the efforts e_a and e_b that maximize her certainty equivalent from (8). This optimization leads to

$$e_a = \frac{1}{1 + \gamma}, \quad e_b = \frac{1}{1 + \gamma}. \quad (9)$$

These are the first-best efforts since with risk-free debt, the entrepreneur not only bears the total marginal costs, but also the total marginal benefit of her efforts. Both efforts from (9)

decrease in the parameter γ that measures the interdependence of the marginal costs. The more expensive one effort makes (additional) effort in the other part of the firm, the lower are the optimal efforts in both parts. Note that both efforts have the same value under debt financing. With the optimal efforts from (9), the certainty equivalent from (8) is

$$CE_E = x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}). \quad (10)$$

The entrepreneur will start her business with debt financing only if this certainty equivalent (10) exceeds her future wealth in case she refrains from it:

$$x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) > (1+r_f)W_{0,E} \quad (11)$$

$$\Leftrightarrow \frac{1}{1+r_f} \left[x + y + \frac{1}{1+\gamma} - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) \right] > C. \quad (12)$$

Ultimately, this condition (12) is an *NPV* criterion: The present value of expected future value of the firm minus deduction for risk (on the LHS) must exceed the invested capital (on the RHS). With debt financing, no risk sharing is provided, so the deduction for risk (i.e., the risk premium) depends on the total risk $\text{Var}(\tilde{a} + \tilde{b})$ and the entrepreneur's individual risk preference θ_E .

In the following, we will consider how alternative financing contracts affect the entrepreneur's incentives and risk sharing and what certainty equivalent results for the entrepreneur.

5.2. *Musharaka financing*

The entrepreneur is now facing the opportunity to finance tangible assets with capital contribution from an investor through a *musharaka* profit-and-loss-sharing contract. Based on this contract, the *musharaka* investor I provides capital in the amount of S_I and is entitled to a share α_I in the value \tilde{a} of tangible assets. Since the assets in which the investor has a stake are tangible, their value is relatively easy to observe and hence to be divided based on the contractual share α_I separately from the firm's total value.

The investor's certainty equivalent CE_I of his future wealth depending on the share α_I and the capital S_I invested in the entrepreneurial firm is

$$CE_I = \alpha_I E(\tilde{a}) - (1 + r_f)(S_I - W_{0,I}) - \frac{1}{2} \theta_I \alpha_I^2 \text{Var}(\tilde{a}), \quad (13)$$

with $W_{0,I}$ for the investor's initial wealth in $t = 0$ and $\theta_I > 0$ for his risk aversion factor.

We assume that the *musharaka* contract results in a certainty equivalent for the investor exactly as high as with his outside option.

Assumption 6.

$$\begin{aligned} \alpha_I E(\tilde{a}) - (1 + r_f)(S_I - W_{0,I}) - \frac{1}{2} \theta_I \alpha_I^2 \text{Var}(\tilde{a}) &= (1 + r_f) W_{0,I} \\ \Leftrightarrow \alpha_I E(\tilde{a}) - \frac{1}{2} \theta_I \alpha_I^2 \text{Var}(\tilde{a}) &= (1 + r_f) S_I \end{aligned} \quad (14)$$

This assumption is consistent with Assumption 5 that has the same implication for the firm's lender. Neither the lender nor the now considered investor I earns more (on a risk-adjusted basis) than with their outside option. Without this similarity, differences in risk sharing and incentive effects would be contaminated with differences in distribution of wealth.

Sharing the value \tilde{a} in *musharaka* financing and with the investor's capital provision S_I , the entrepreneur's certainty equivalent is

$$\begin{aligned} CE_E &= (1 - \alpha_I) E(\tilde{a}) + E(\tilde{b}) - c(e_a, e_b) - (1 + r_f)(C - W_{0,E} - S_I) \\ &\quad - \frac{1}{2} \theta_E (1 - \alpha_I)^2 \text{Var}(\tilde{a}) - \frac{1}{2} \theta_E \text{Var}(\tilde{b}) - \theta_E (1 - \alpha_I) \text{Cov}(\tilde{a}, \tilde{b}). \end{aligned} \quad (15)$$

If $(C - W_{0,E} - S_I) > 0$, the entrepreneur additionally requires some debt financing. On the other hand, if $(C - W_{0,E} - S_I) < 0$, she invests the surplus on the market. As stated in Assumption 5, the interest rate r_f on debt is the same as the rate of return on alternative (risk-free) investments outside the firm. Therefore, we do not have to differentiate cases with respect to the sign of $(C - W_{0,E} - S_I)$.

With (2) and (3) for the asset values from Assumption 2 and (4) for the cost function from Assumption 3, the entrepreneur's certainty equivalent becomes

$$CE_E = (1 - \alpha_I)(x + e_a) + (y + e_b) - \frac{1}{2}(e_a^2 + e_b^2) - \gamma e_a e_b - (1 + r_f)(C - W_{0,E} - S_I) - \frac{1}{2}\theta_E(1 - \alpha_I)^2 \text{Var}(\tilde{a}) - \frac{1}{2}\theta_E \text{Var}(\tilde{b}) - \theta_E(1 - \alpha_I) \text{Cov}(\tilde{a}, \tilde{b}) \quad (16)$$

This is the entrepreneur's certainty equivalent to be maximized by the choice of efforts e_a and e_b after the *musharaka* financing contract has been signed, that is, with given α_I and S_I . For the entrepreneur, the following efforts are optimal:

$$e_a = \frac{1 - \gamma - \alpha_I}{1 - \gamma^2}, \quad e_b = \frac{1 - \gamma + \gamma\alpha_I}{1 - \gamma^2}. \quad (17)$$

Note that financing the firm with the considered *musharaka* contract results in lower effort e_a compared to the case of debt financing (see (9)) for all $\alpha_I > 0$. Since the entrepreneur has to share the fruits of her effort in raising the value \tilde{a} of the tangible assets, her incentives to do so are diluted. And a larger share α_I of the *musharaka* investor results in higher dilution of incentives and in lower effort e_a . This is the classical moral hazard problem in a principal-agent relationship. On the other hand, e_b is higher than with debt financing if $\gamma > 0$ (and *vice versa* for $\gamma < 0$). But the total effort $e_a + e_b$ is less than under debt financing for all $\alpha_I > 0$:

$$e_a + e_b = \frac{2 - \alpha_I}{1 + \gamma} < \frac{2}{1 + \gamma}. \quad (18)$$

In this respect, a dilution of overall incentives by *musharaka* financing is to be observed.

If the investor correctly anticipates the entrepreneur's efforts based on his share α_I , we can substitute for S_I from the investor's participation constraint (14) and the efforts from (17) in (16):

$$CE_E = x + y + \frac{1}{1 + \gamma} - \frac{\alpha_I^2}{2(1 - \gamma^2)} - (1 + r_f)(C - W_{0,E}) - \frac{1}{2}[\theta_E(1 - \alpha_I)^2 + \theta_I\alpha_I^2] \text{Var}(\tilde{a}) - \frac{1}{2}\theta_E \text{Var}(\tilde{b}) - \theta_E(1 - \alpha_I) \text{Cov}(\tilde{a}, \tilde{b}) \quad (19)$$

Now our last step (in fact, the entrepreneur's first decision) is to find the optimal share α_I that maximizes her certainty equivalent in (19). This maximization leads to

$$\alpha_I^* = \frac{\theta_E [Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b})]}{\frac{1}{1-\gamma^2} + (\theta_E + \theta_I) Var(\tilde{a})}. \quad (20)$$

Since we want to make sure that this inner solution lies in the economically sensible interval $]0;1[$, i.e., $0 < \alpha_I^* < 1$, we state

*Assumption 7.*⁴

$$Cov(\tilde{a}, \tilde{b}) > -Var(\tilde{a}), \quad (21)$$

and

$$Cov(\tilde{a}, \tilde{b}) < \frac{1}{1-\gamma^2} + \theta_I Var(\tilde{a}). \quad (22)$$

From condition (21) follows $\alpha_I^* > 0$. This condition holds for any positive covariance and even with a negative covariance if the variance $Var(\tilde{a})$ of the value of the tangible assets is not “small” compared to the variance $Var(\tilde{b})$ of the value of the other (intangible) assets.⁵

From condition (22) follows $\alpha_I^* < 1$.

Note that α_I^* takes the maximum value for $\gamma = 0$. In that case, the effort e_b from (17) is independent of α_I , and therefore, there is no effort distortion in that area. That allows for more risk sharing, that is, a higher value of α_I . But since the total effort $e_a + e_b$ decreases in α_I (see (18)), the choice of α_I always has to trade off this incentive effect against the benefits of risk sharing.

⁴ Otherwise, we would have to continue with the boundary solution of either $\alpha_I = 0$ or $\alpha_I = 1$.

⁵ If $Var(\tilde{a}) > \rho_{(\tilde{a}, \tilde{b})}^2 Var(\tilde{b})$, then $\sigma_{\tilde{a}} > \left| \rho_{(\tilde{a}, \tilde{b})} \right| \sigma_{\tilde{b}} \Leftrightarrow \sigma_{\tilde{a}}^2 > \left| \rho_{(\tilde{a}, \tilde{b})} \right| \sigma_{\tilde{a}} \sigma_{\tilde{b}}$. Adding the covariance to both sides results in $\sigma_{\tilde{a}}^2 + \rho_{(\tilde{a}, \tilde{b})} \sigma_{\tilde{a}} \sigma_{\tilde{b}} = Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b}) > \left[\left| \rho_{(\tilde{a}, \tilde{b})} \right| + \rho_{(\tilde{a}, \tilde{b})} \right] \sigma_{\tilde{a}} \sigma_{\tilde{b}} \geq 0$. Therefore, $Var(\tilde{a}) > \rho_{(\tilde{a}, \tilde{b})}^2 Var(\tilde{b})$ suffices for $Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b}) > 0 \Leftrightarrow Cov(\tilde{a}, \tilde{b}) > -Var(\tilde{a})$.

With the optimal share α_i^* from (20), the entrepreneur's maximized certainty equivalent from (19) is

$$CE_E = x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2}\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right] \underbrace{\frac{\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right]}{\frac{1}{1-\gamma^2} + (\theta_E + \theta_I)\text{Var}(\tilde{a})}}_{\alpha_i^*}. \quad (23)$$

As in (11), we now compare this certainty equivalent from (23) with her future wealth in case she refrains from the business. Under the *musharaka* financing, the entrepreneur gains if

$$x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2}\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right] \frac{\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right]}{\frac{1}{1-\gamma^2} + (\theta_E + \theta_I)\text{Var}(\tilde{a})} > (1+r_f)W_{0,E} \quad (24)$$

$$\Leftrightarrow \frac{1}{1+r_f} \left[x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2}\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right] \frac{\theta_E \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right]}{\frac{1}{1-\gamma^2} + (\theta_E + \theta_I)\text{Var}(\tilde{a})} \right] > C. \quad (25)$$

The LHS of condition (25) is again a risk-adjusted present value, given that the entrepreneur optimally chooses the terms of the *musharaka* contract and individually optimizes her effort. If this present value exceeds the total investment needed, the business has a positive *NPV* adjusted for risk.

More importantly we address the question, which of the so far considered financing contracts results in a higher utility for the entrepreneur. To answer this question, we have to compare certainty equivalent in (23) under the optimal *musharaka* contract with the certainty equivalent in (10) in case of debt financing.

Proposition 1.

Musharaka financing is always more advantageous for the entrepreneur compared to debt since the certainty equivalent in (23) exceeds the certainty equivalent in (10) if

$$\frac{\theta_E^2 \left[\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b}) \right]^2}{\frac{1}{1-\gamma^2} + (\theta_E + \theta_I) \text{Var}(\tilde{a})} > 0, \quad (26)$$

which holds for all parameters compatible with our assumptions.

Therefore, despite diluted incentives to extend effort, *musharaka* financing always results in a higher certainty equivalent for the entrepreneur than debt financing due to the benefits from risk sharing.

5.3. Equity financing

Instead of financing with capital from an investor by means of a *musharaka* profit-and-loss-sharing contract as considered before, we now analyze financing with regular equity capital. Equity capital comes from a partner P who provides capital in amount of S_p and then holds a share α_p in the firm, i.e., he becomes a regular shareholder. Therefore, as opposed to the *musharaka* investor, the partner P does not profit from the returns of some (tangible) assets only, but from the overall increase or decrease in firm value. This difference affects the incentives for the entrepreneur to spend effort and results in an optimal value for the partner's share α_p that differs from the *musharaka* investor's share α_I (in (20)) and consequently, in a different allocation of risk.

We will now analyze how financing with equity capital affects the entrepreneur's utility and then compare the results with those from debt financing and *musharaka* profit and loss sharing.

We assume that the partner P is also risk-averse with an exponential utility function and a risk aversion factor of $\theta_p > 0$ (see Assumption 1). Therefore, the certainty equivalent CE_p of his future wealth depending on the share α_p and the capital S_p invested in the firm is

$$CE_p = \alpha_p E(\tilde{a} + \tilde{b}) - (1 + r_f)(S_I - W_{0,p}) - \frac{1}{2} \theta_p \alpha_p^2 \text{Var}(\tilde{a} + \tilde{b}), \quad (27)$$

with $W_{0,p}$ for his initial wealth in $t = 0$.

We assume (similarly to Assumption 6) that the equity partnership results in a certainty equivalent for the partner exactly as high as with his outside option.

Assumption 8.

$$\begin{aligned} \alpha_p E(\tilde{a} + \tilde{b}) - (1 + r_f)(S_I - W_{0,P}) - \frac{1}{2} \theta_p \alpha_p^2 \text{Var}(\tilde{a} + \tilde{b}) &= (1 + r_f) W_{0,P} \\ \Leftrightarrow \alpha_p E(\tilde{a} + \tilde{b}) - \frac{1}{2} \theta_p \alpha_p^2 \text{Var}(\tilde{a} + \tilde{b}) &= (1 + r_f) S_p \end{aligned} \quad (28)$$

The reason for this assumption is again that none of the considered financiers should earn more (on a risk-adjusted basis) than with their outside option. Without this similarity, differences in risk sharing and incentive effects would be contaminated with differences in distribution of wealth.

With the partner P providing capital S_p and holding a share α_p in the firm, the entrepreneur's certainty equivalent is

$$\begin{aligned} CE_E &= (1 - \alpha_p) E(\tilde{a} + \tilde{b}) - c(e_a, e_b) - (1 + r_f)(C - W_{0,E} - S_p) \\ &\quad - \frac{1}{2} \theta_E (1 - \alpha_p)^2 \text{Var}(\tilde{a} + \tilde{b}) \end{aligned} \quad (29)$$

For $(C - W_{0,E} - S_p) > 0$, the entrepreneur additionally requires some debt financing with an interest rate r_f . On the other hand, if $(C - W_{0,E} - S_p) < 0$, she invests the remaining capital at the risk-free rate r_f (see the discussion of (15) above).

With (2) and (3) for the asset values from Assumption 2 and (4) for the cost function from Assumption 3 follows:

$$\begin{aligned} CE_E &= (1 - \alpha_p)(x + e_a + y + e_b) - \frac{1}{2}(e_a^2 + e_b^2) - \gamma e_a e_b - (1 + r_f)(C - W_{0,E} - S_p) \\ &\quad - \frac{1}{2} \theta_E (1 - \alpha_p)^2 \text{Var}(\tilde{a} + \tilde{b}) \end{aligned} \quad (30)$$

This is the entrepreneur's certainty equivalent to be maximized by the choice of efforts e_a and e_b after the partnership contract has been signed, that is, with given α_p and S_p . For the entrepreneur, the following efforts are optimal:

$$e_a = \frac{1 - \alpha_p}{1 + \gamma}, \quad e_b = \frac{1 - \alpha_p}{1 + \gamma}. \quad (31)$$

Note that the equity financing results in the same efforts ($e_a = e_b$) in both of the entrepreneur's tasks. This is the result of the fact that the entrepreneur evenly shares the fruits of all her efforts with the partner, so that the incentives for providing effort e_a and effort e_b are evenly diluted. And a larger share α_p of the equity partner results in higher dilution of incentives and hence in lower effort in both tasks. These efforts in (31) fall short of the first-best efforts (see (9)) for any $\alpha_p > 0$.

If the partner correctly anticipates the entrepreneur's efforts based on his share α_p , we can substitute S_p from the partner's participation constraint (28) and the efforts from (31) in (30):

$$CE_E = x + y + \frac{1}{1+\gamma} - \frac{\alpha_p^2}{1+\gamma} - \frac{1}{2} \left[\theta_E (1-\alpha_p)^2 + \theta_P \alpha_p^2 \right] \text{Var}(\tilde{a} + \tilde{b}). \quad (32)$$

The first-order condition for the maximization of (32) determines the optimal α_p :

$$\alpha_p^* = \frac{\theta_E \text{Var}(\tilde{a} + \tilde{b})}{\frac{2}{1+\gamma} + (\theta_E + \theta_P) \text{Var}(\tilde{a} + \tilde{b})}. \quad (33)$$

Note that α_p^* increases in γ . The reason is that α_p^* has a negative impact on effort: $\frac{de_a}{d\alpha_p} = \frac{de_b}{d\alpha_p} = -\frac{1}{1+\gamma}$. But the larger γ , the smaller this negative marginal effect in absolute terms. Hence, with larger values of γ , it becomes marginally less costly to choose higher values for α_p^* to improve on risk sharing.

With this optimal share α_p^* from (33), the entrepreneur's certainty equivalent from (32) is

$$CE_E = x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2} \theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2} \theta_E \text{Var}(\tilde{a} + \tilde{b}) \underbrace{\frac{\theta_E \text{Var}(\tilde{a} + \tilde{b})}{\frac{2}{1+\gamma} + (\theta_E + \theta_P) \text{Var}(\tilde{a} + \tilde{b})}}_{\alpha_p^*}. \quad (34)$$

In order to render the business with equity partnership advantageous for the entrepreneur, this certainty equivalent (34) must exceed her future wealth in case she refrains from the business:

$$x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) \frac{\theta_E \text{Var}(\tilde{a} + \tilde{b})}{\frac{2}{1+\gamma} + (\theta_E + \theta_p) \text{Var}(\tilde{a} + \tilde{b})} > (1+r_f)W_{0,E} \quad (35)$$

$$\Leftrightarrow \frac{1}{1+r_f} \left[x + y + \frac{1}{1+\gamma} - \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) + \frac{1}{2}\theta_E \text{Var}(\tilde{a} + \tilde{b}) \frac{\theta_E \text{Var}(\tilde{a} + \tilde{b})}{\frac{2}{1+\gamma} + (\theta_E + \theta_p) \text{Var}(\tilde{a} + \tilde{b})} \right] > C \quad (36)$$

The LHS of (36) is again a risk-adjusted present value, given that the entrepreneur optimally chooses the terms of the partnership and individually optimizes her effort. If this present value exceeds the total investment needed, the business has a positive *NPV* adjusted for risk.

By comparing the certainty equivalent of the entrepreneur under equity partnership with a shareholder from (34) with the result for debt (10), we find

Proposition 2.

Equity financing results in a higher utility for the entrepreneur compared to debt since the certainty equivalent in (34) exceeds the certainty equivalent in (10) if

$$\frac{\theta_E^2 [\text{Var}(\tilde{a} + \tilde{b})]^2}{\frac{2}{1+\gamma} + (\theta_E + \theta_p) \text{Var}(\tilde{a} + \tilde{b})} > 0, \quad (37)$$

which holds for any $\gamma > -1$.

Therefore, as in the case with *musharaka* financing, despite diminished incentives to exert effort, equity financing dominates debt financing. *Musharaka* and equity financing differ in how risk is shared. But with both financial alternatives, the entrepreneur benefits from the risk sharing to an extent that compensates for the losses from diminished effort incentives.

6. *Musharaka* versus Equity

Both *musharaka* and equity financing are variants of profit-and-loss-sharing contracts and consequently, result in risk sharing. But the terms of risk sharing and incentive effects are different. For the optimal value of α_l in *musharaka* financing (see (20)), we calculated the entrepreneur's certainty equivalent in (23). For the optimal value of α_p in partnership with equity financing (see (33)), we calculated the entrepreneur's certainty equivalent in (34). For the sake of comparability, we further assume

Assumption 9.

The equity partner and the *musharaka* investor are equally risk-averse: $\theta_p = \theta_l \equiv \theta$.

This assumption ensures that none of the alternative financial contracts leads to more (or less) benefits from risk sharing only because of lower (or higher) risk aversion of the financier with whom the entrepreneur shares risk.

Comparing the certainty equivalent of the entrepreneur under partnership with an equity shareholder from (34) with the certainty equivalent under the *musharaka* contract (23) by applying Assumption 9, we find

Proposition 3.

Musharaka financing may or may not result in higher utility for the entrepreneur compared to financing with regular equity depending on the structure of risks and the effort cost function. *Musharaka* results in a higher certainty equivalent for the entrepreneur than financing with equity, i.e., *musharaka* is more advantageous if

$$\left(1 - \rho_{(\tilde{a}, \tilde{b})}^2\right) \left\{ (1 + \gamma)(\theta_E + \theta) \text{Var}(\tilde{a} + \tilde{b}) + 2 \right\} < \frac{\text{Var}(\tilde{a} + \tilde{b})}{\text{Var}(\tilde{b})} \left[2 - \frac{\text{Var}(\tilde{a} + \tilde{b})}{(1 - \gamma)\text{Var}(\tilde{a})} \right], \quad (38)$$

with $\rho_{(\tilde{a}, \tilde{b})}$ for the correlation between the two components of the firm's value.⁶

If condition (38) holds, that is, if *musharaka* financing results in a higher utility for the entrepreneur than equity financing, it must be because of better incentive effects overcompensating for the disadvantage in how risk is shared.

⁶ The formal comparison of the certainty equivalents from (34) and (23) to deduce (38) is a straightforward, but lengthy task. A step-by-step illustration by the authors is available upon request.

If incentives are not relevant and moral hazard problem does not exist, equity financing with linear sharing of total business risk dominates *musharaka* financing as considered here, with sharing of risk in only \tilde{a} :⁷

Lemma 1.

If the financial structure does not have an impact on the entrepreneur's effort (i.e., in the absence of agency problems), and the first-best efforts $e_a = e_b = \frac{1}{1+\gamma}$ are implemented, risk sharing with equity is superior to risk sharing under *musharaka* financing: $CE_E(\text{equity}) > CE_E(\text{musharaka})$ for all $|\rho_{(\tilde{a}, \tilde{b})}| < 1$. Proof: see Appendix B.

In that case, only for $|\rho_{(\tilde{a}, \tilde{b})}| = 1$, both alternatives of raising capital result in the same certainty equivalent for the entrepreneur since with perfect positive or negative correlation, there is, in fact, only one risk. The entrepreneur can, therefore, optimally share total business risk with the *musharaka* investor via his share in \tilde{a} . A certain share of the investor in \tilde{a} only results in the same risk for the entrepreneur as the optimal share in both \tilde{a} and \tilde{b} of an equity partner if the risks are perfectly correlated. Then no advantage of equity financing with respect to risk sharing remains.

For all non-perfectly correlated risks in \tilde{a} and \tilde{b} , i.e., $|\rho_{(\tilde{a}, \tilde{b})}| < 1$, sharing total business risk $Var(\tilde{a} + \tilde{b})$ with an equity partner, *ceteris paribus*, dominates partial risk sharing with *musharaka* financing, and the relative advantage of equity with respect to risk sharing increases in the ratio $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$. The higher $Var(\tilde{a} + \tilde{b})$ is compared to $Var(\tilde{a})$, the more important it is for the entrepreneur to share total risk instead of only $Var(\tilde{a})$ (see Appendix C). The ratio $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$ of total business risk to the variance $Var(\tilde{a})$ is lowest if the risk in \tilde{b} is negligible.

If the risk in \tilde{b} , i.e., $Var(\tilde{b})$, is negligible (and so is the covariance), the entrepreneur does not

⁷ The optimal division of the sum of two risky assets, independently of the specific distributions, was already determined in Borch (1960). For the optimality of linear sharing rules, see also Borch (1968), Wilson (1968), Drèze (1990), and Lemaire (1990).

benefit much from sharing this risk with a partner instead of bearing it alone with *musharaka* financing under which only the risk in \tilde{a} is shared.

As a counterbalance to the suboptimal risk sharing, *musharaka* financing comes with superior incentive effects under certain conditions, which can be determined through the comparison of the profits from effort net of the entrepreneur's costs under *musharaka* and equity financing.

$$\begin{aligned}
NPM &= \frac{1-\gamma-\alpha_I}{1-\gamma^2} + \frac{1-\gamma+\gamma\alpha_I}{1-\gamma^2} \\
&\quad - \frac{1}{2} \left[\left(\frac{1-\gamma-\alpha_I}{1-\gamma^2} \right)^2 + \left(\frac{1-\gamma+\gamma\alpha_I}{1-\gamma^2} \right)^2 \right] - \gamma \left(\frac{1-\gamma-\alpha_I}{1-\gamma^2} \right) \left(\frac{1-\gamma+\gamma\alpha_I}{1-\gamma^2} \right) \\
&= \frac{1}{1+\gamma} - \frac{\alpha_I^2}{2(1-\gamma^2)}
\end{aligned} \tag{39}$$

is the net profit from extending effort under *musharaka*, and

$$\begin{aligned}
NPE &= \frac{1-\alpha_P}{1+\gamma} + \frac{1-\alpha_P}{1+\gamma} \\
&\quad - \frac{1}{2} \left[\left(\frac{1-\alpha_P}{1+\gamma} \right)^2 + \left(\frac{1-\alpha_P}{1+\gamma} \right)^2 \right] - \gamma \left(\frac{1-\alpha_P}{1+\gamma} \right) \left(\frac{1-\alpha_P}{1+\gamma} \right) \\
&= \frac{1}{1+\gamma} - \frac{\alpha_P^2}{1+\gamma}
\end{aligned} \tag{40}$$

is the net profit from extending effort under equity financing. Comparing those net profits from effort in (39) and (40) we find

Lemma 2.

$$NPM > NPE \quad \text{for} \quad \frac{\alpha_I^2}{\alpha_P^2} < 2(1-\gamma). \tag{41}$$

$$\text{Proof: } NPM > NPE \Leftrightarrow \frac{\alpha_I^2}{2(1-\gamma^2)} < \frac{\alpha_P^2}{1+\gamma} \Leftrightarrow \frac{\alpha_I^2}{\alpha_P^2} < 2(1-\gamma).$$

A higher net profit from effort under *musharaka* financing compared to equity, i.e., $NPM > NPE$, is a necessary (but not sufficient) condition for *musharaka* to dominate equity. A higher net profit under *musharaka* is necessary to compensate for the disadvantage in risk sharing. However, even if the net profit under *musharaka* is higher, the entrepreneur's utility might nevertheless be lower due to the advantage in risk sharing of equity financing. Generally,

there is a trade-off between risk-sharing benefits and net profit of effort. For given (first-best) efforts, equity financing results in higher utility because of better risk sharing (see Lemma 1). On the other hand, with the agency problem at work, *musharaka* financing results in higher net profit of effort under the condition specified in Lemma 2.

The advantage of *musharaka* financing with respect to net profit of effort (see Lemma 2) depends (*inter alia*) on γ . For *musharaka* to result in a certainty equivalent at least as high as equity financing, *musharaka* must come with an advantage with respect to net profit: $NPM > NPE$. The higher γ gets, the smaller this advantage becomes; the difference in net profits (from (39) and (40)), *ceteris paribus*, decreases in γ :

$$\frac{d(NPM - NPE)}{d\gamma} = -\frac{\alpha_p^2}{(1+\gamma)^2} - \frac{\gamma\alpha_i^2}{(1-\gamma^2)^2} < 0 \quad (42)$$

for all γ .⁸ The relative advantage of *musharaka* with respect to net profit decreases in γ , *ceteris paribus*, since a higher γ implies such a change of marginal cost of effort (see (4)) that, for given efforts, costs would increase. As a reaction, the entrepreneur adjusts her effort, and the net profit from effort changes. This adjustment is more detrimental for net profits in case of *musharaka* financing.

To take a closer look at the trade-off between risk sharing and incentive-dependent net profits from effort by means of a comparative statics analysis, we consider the case of indifference, that is, a constellation in which *musharaka* and equity financing result in the same utility for the entrepreneur, and therefore, (38) holds as an equation:

$$\left(1 - \rho_{(\tilde{a}, \tilde{b})}^2\right) \left\{ (1+\gamma)(\theta_E + \theta) \text{Var}(\tilde{a} + \tilde{b}) + 2 \right\} = \frac{\text{Var}(\tilde{a} + \tilde{b})}{\text{Var}(\tilde{b})} \left[2 - \frac{\text{Var}(\tilde{a} + \tilde{b})}{(1-\gamma)\text{Var}(\tilde{a})} \right]. \quad (43)$$

As already discussed, an increase in γ decreases the relative advantage of *musharaka* with respect to net profit. This is reflected in the fact that the LHS of (43) increases in γ while

⁸ Proof: The partial derivative in (42) is obviously negative for all $\gamma \geq 0$. For $\gamma < 0$, we consider the equivalent condition $-\frac{\gamma\alpha_i^2}{\alpha_p^2} - (1-\gamma)^2 < 0$. From Lemma 2, we know that $NPM > NPE$ requires $\frac{\alpha_i^2}{\alpha_p^2} < 2(1-\gamma)$.

This latter condition implies $-\frac{\gamma\alpha_i^2}{\alpha_p^2} < -2(1-\gamma)\gamma$ if $\gamma < 0$. Therefore, $-\frac{\gamma\alpha_i^2}{\alpha_p^2} - (1-\gamma)^2 < -2(1-\gamma)\gamma - (1-\gamma)^2 < 0$

for all $-1 < \gamma < 0$. Consequently, $\frac{d(NPM - NPE)}{d\gamma} < 0$ when $NPM > NPE$.

the RHS decreases in γ . As a counterbalance, the relative advantage of equity with respect to risk sharing must decrease to restore indifference. This relative risk sharing advantage of equity

depends on the ratio $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$ (see the discussion to Lemma 1). So, if we consider $Var(\tilde{a})$

and $\rho_{(\tilde{a}, \tilde{b})}^2$ as given, a larger value of γ calls for a smaller $Var(\tilde{b})$ to restore indifference. If we instead consider both variances as given, a larger value of γ calls for a smaller $\rho_{(\tilde{a}, \tilde{b})}^2$.

These comparative statics with respect to the parameter γ from the cost function and the relation between the risk $Var(\tilde{b})$ in the intangible assets or growth opportunities and the risk $Var(\tilde{a})$ in the tangible assets are now to be illustrated in the next section by means of a numerical analysis.

7. Numerical Analysis

Proposition 3 states the condition for *musharaka* financing to result in a higher utility for the entrepreneur:

$$\left(1 - \rho_{(\tilde{a}, \tilde{b})}^2\right) \left\{ (1 + \gamma)(\theta_E + \theta) Var(\tilde{a} + \tilde{b}) + 2 \right\} < \frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{b})} \left[2 - \frac{Var(\tilde{a} + \tilde{b})}{(1 - \gamma) Var(\tilde{a})} \right]. \text{ see (38)}$$

In the following numerical analysis, we want to illustrate constellations of the variance ratio

$\frac{Var(\tilde{b})}{Var(\tilde{a})}$, the correlation coefficient $\rho_{(\tilde{a}, \tilde{b})}$, and the parameter γ from the entrepreneur's cost

function for which this condition (38) holds. We consider the risk $Var(\tilde{b})$ in intangible assets

or growth opportunities relative to the risk $Var(\tilde{a})$ in tangible assets instead of the total risk

ratio $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$ or $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{b})}$ (which enter condition (38)) to disentangle the effects of the

individual risks in intangible assets or growth opportunities, $Var(\tilde{b})$, and in tangible assets,

$Var(\tilde{a})$, and the correlation between those risks, measured by $\rho_{(\tilde{a}, \tilde{b})}$.

Since the risk aversion factors θ_E and θ do not enter condition (38) separately, we only need to specify the sum of both; we assume, $\theta_E + \theta = 1$. For further simplification, we can assume a normalized $Var(\tilde{a}) = 1$ since our main focus is the variance ratio $\frac{Var(\tilde{b})}{Var(\tilde{a})}$. Figures 1-3 illustrate the entrepreneur's indifference curves with γ on the horizontal axis and $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ on vertical axis, for three different values of $\rho_{(\tilde{a},\tilde{b})}$, i.e., the constellations for which (43) holds.

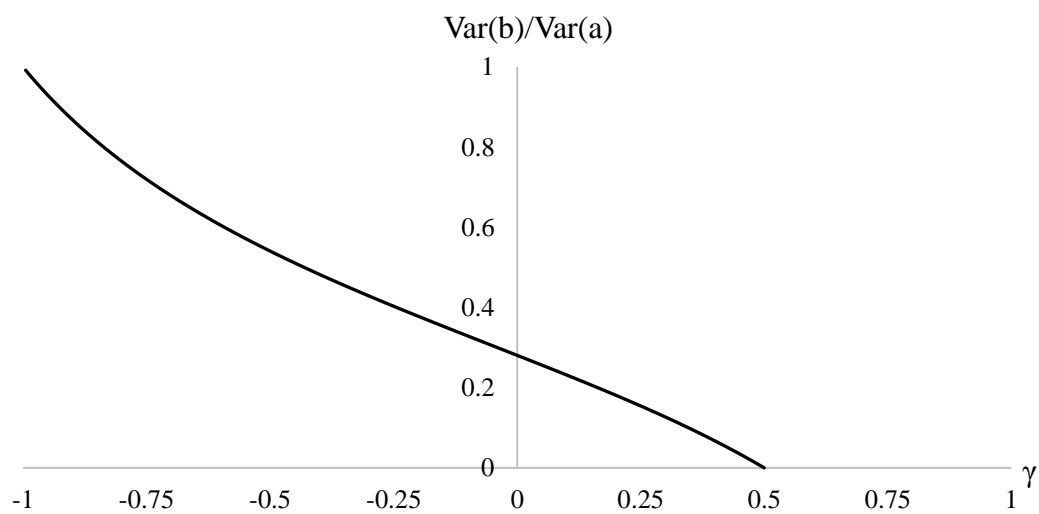


Figure 1. Indifference Curve: $\rho_{(\tilde{a},\tilde{b})} = 0$

We start with the case of uncorrelated risks: $\rho_{(\tilde{a},\tilde{b})} = 0$. For this case, the curve in Figure

1 represents all combinations of γ and $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ that result in the entrepreneur being indifferent between *musharaka* and equity financing. Below (above) this indifference curve, *musharaka* leads to a higher (lower) utility for the entrepreneur, given the respectively optimal contractual design.

For *musharaka* to be advantageous, the risk $Var(\tilde{b})$ in the value \tilde{b} of intangible assets or growth opportunities must be small relative to the risk $Var(\tilde{a})$ in the tangible assets since only the latter is shared with the *musharaka* investor. That observation is consistent with our discussion in section 6; for *musharaka* to be advantageous, total business risk $Var(\tilde{a} + \tilde{b})$ must

not be much higher relative to the risk in \tilde{a} , i.e., $Var(\tilde{a})$. The reasoning is in the relative importance of risk sharing. Only if the risk $Var(\tilde{b})$ in intangible assets or growth opportunities is relatively small (and so is $Var(\tilde{a} + \tilde{b})$), the renunciation of sharing this particular risk in case of *musharaka* financing results in a disadvantage sufficiently small to be compensated for by better incentives. If the risk $Var(\tilde{b})$ in intangible assets or growth opportunities is relatively large (and so is $Var(\tilde{a} + \tilde{b})$), the entrepreneur prefers equity financing because of the higher importance of comprehensive risk sharing ensured by equity.

In Figure 1, we observe that the indifference curve is downward sloping. This is due to the trade-off between risk sharing and incentives as already discussed in section 5. A higher γ decreases the advantage of *musharaka* with respect to net profits from effort and calls for a lower risk $Var(\tilde{b})$ in intangible assets or growth opportunities relative to the risk $Var(\tilde{a})$ in

tangible assets, that is, a lower $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ ratio, for any given correlation. A lower $Var(\tilde{b})$ makes sharing of total risk $Var(\tilde{a} + \tilde{b})$ with equity less important, *ceteris paribus*, relative to the sharing of only $Var(\tilde{a})$ with *musharaka*.

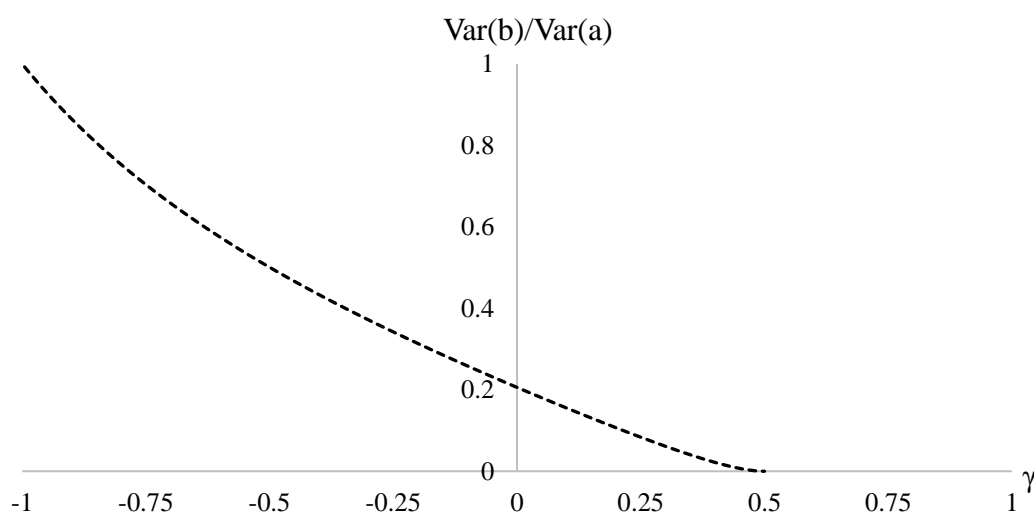


Figure 2. Indifference Curve: $\rho_{(\tilde{a},\tilde{b})} = 0.5$

Similar observations can be made for the case of a positive correlation: $\rho_{(\tilde{a},\tilde{b})} = 0.5$. In Figure 2, the indifference curve is also downward sloping. Again, below (above) this

indifference curve, *musharaka* leads to higher (lower) utility for the entrepreneur, given the respectively optimal contractual design.

Note that the indifference curve in Figure 2 lies below the one in Figure 1, that is, the indifference curve shifts downwards when $\rho_{(\tilde{a}, \tilde{b})}$ increases. The reason is that for any given combination of individual risks $Var(\tilde{a})$ and $Var(\tilde{b})$, a higher correlation results in a higher total risk $Var(\tilde{a} + \tilde{b})$, and a higher total risk implies that the comprehensive risk sharing ensured by equity is more important for the entrepreneur. Therefore, the relative risk sharing disadvantage of *musharaka* can only be compensated for in a smaller range of $\frac{Var(\tilde{b})}{Var(\tilde{a})}$.

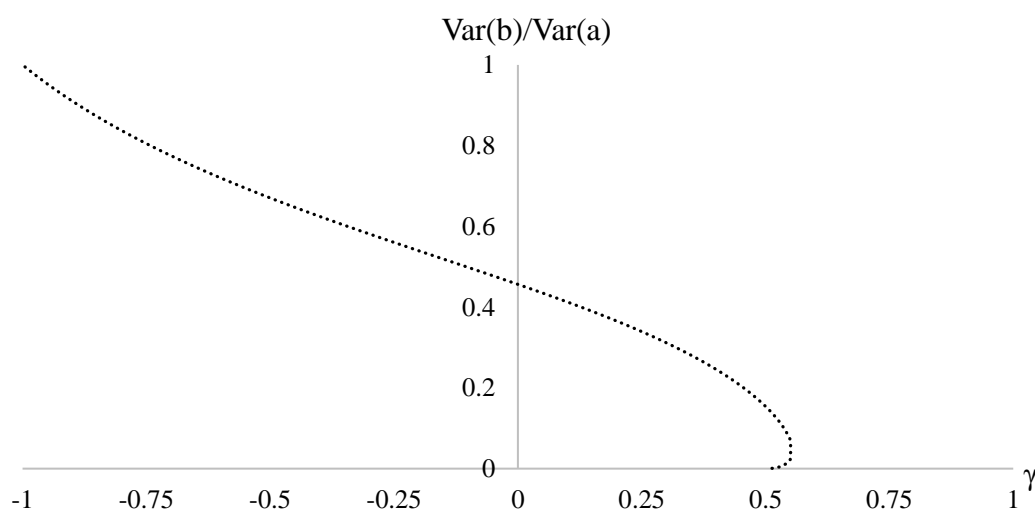


Figure 3. Indifference Curve: $\rho_{(\tilde{a}, \tilde{b})} = -0.5$

Lastly, we consider the case of a negative correlation: $\rho_{(\tilde{a}, \tilde{b})} = -0.5$. Under this assumption, we identify the combinations of γ and $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ that lead to the indifference curve depicted in Figure 3.

The fundamental observations we made regarding the indifference curves in figures 1 and 2 apply again. Note that, in this case with $\rho_{(\tilde{a}, \tilde{b})} = -0.5$, the indifference curve lies above the one in Figure 1 for the case of non-correlated risks. The reasoning is the converse of the one given in the comparison of Figure 2 with Figure 1. Here with $\rho_{(\tilde{a}, \tilde{b})} = -0.5$, for any given

combination of individual risks $Var(\tilde{a})$ and $Var(\tilde{b})$, this lower (negative) correlation results in a smaller total risk $Var(\tilde{a} + \tilde{b})$, and a smaller total risk implies that comprehensive risk sharing ensured by equity is less important for the entrepreneur. Therefore, the relative risk sharing disadvantage of *musharaka* is smaller and can be compensated for in a larger range of $\frac{Var(\tilde{b})}{Var(\tilde{a})}$.

Furthermore, it is noticeable that for relatively large values of γ , the indifference curve in Figure 3 bends backwards. That has a mere technical reason. The larger γ is, the smaller the ratio $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$ should be to result in indifference for reasons explained above. But in the range of very small values of $Var(\tilde{b})$, total risk $Var(\tilde{a} + \tilde{b})$ is decreasing in $Var(\tilde{b})$ if $\rho_{(\tilde{a}, \tilde{b})} < 0$. So, a decrease in $Var(\tilde{b})$ and consequently, a decrease in $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ results, *ceteris paribus*, in a higher total risk $Var(\tilde{a} + \tilde{b})$, which makes equity financing more preferable.

To sum up, we observe that *musharaka* contracting is more advantageous for the entrepreneur if the risk $Var(\tilde{b})$ from intangible assets or growth opportunities is small relative to the risk $Var(\tilde{a})$ in tangible assets. The higher the cross derivative γ of the entrepreneur's effort costs, the smaller the range of $\frac{Var(\tilde{b})}{Var(\tilde{a})}$ for which *musharaka* contracting is more advantageous. Furthermore, the correlation $\rho_{(\tilde{a}, \tilde{b})}$ between the risks has an impact on the relative advantage of *musharaka* over equity. For given individual risks $Var(\tilde{a})$ and $Var(\tilde{b})$, total risk $Var(\tilde{a} + \tilde{b})$ increases in $\rho_{(\tilde{a}, \tilde{b})}$. And a higher total risk implies a higher importance of comprehensive risk sharing. Hence, equity becomes more favorable, *ceteris paribus*.

8. Conclusion

The principal-agent relationship between an entrepreneur (the agent) and a financier (the principal) is explored for different financial contracts. The type of financial contract used

has an impact on the extent of risk sharing, and the benefits thereof, and on managerial incentives.

We considered a model where all the parties involved are assumed to be risk-averse, and the entrepreneur can influence the outcome of her business by extending managerial effort at a private cost. An entrepreneur seeking funding for a business opportunity is faced with three alternatives: (i) taking out a loan; (ii) financing a project via *musharaka*, an Islamic profit-and-loss-sharing (PLS) contract; (iii) raising regular equity from a financier who subsequently becomes a co-owner in the firm.

A *musharaka* contract can be used to finance a specific project or investment, the acquisition of some real estate, commodities, working capital, or other tangible assets to be used in production or trade. In that case, only the profits and the risk of this particular project or investment are shared by the parties to the contract. In this paper, the above-mentioned investment is assumed to be in tangible assets. Financing with regular equity results in co-ownership of the whole firm instead. Therefore, all profits and losses from all tangible and intangible assets are shared, and so is the total business risk.

The theoretical analyses in this paper show that the entrepreneur can achieve a higher utility if she uses equity or *musharaka* financing instead of debt only. That is, both *musharaka* and equity financing are superior to debt financing for the entrepreneur. The benefits from risk sharing with an equity partner or a *musharaka* investor compensate for the losses from lower managerial incentives compared to debt financing.

Comparison of *musharaka* and equity financing shows no definitive ranking. Both alternative financial contracts result in profit and loss and risk sharing, even though in various modes. The entrepreneur shares either total business risk with an equity partner or some particular risk with a *musharaka* investor. In both cases, the entrepreneur benefits from risk sharing, yet to varying degrees. On the other hand, both considered alternative financial contracts negatively affect the entrepreneur's incentives to work, but also differently. We analyzed the trade-offs between risk sharing and incentives for both financial contracts. As the main determinants, we identified the risks of the firm's tangible and intangible assets or growth opportunities, the correlation between those risks, and the entrepreneur's effort cost function.

Equity financing is preferable with respect to risk sharing, considering the entrepreneur's utility. *Musharaka* also results in risk sharing, although to a lesser extent. This disadvantage is compensated for by better incentives for the entrepreneur in case of *musharaka*

financing, albeit only under certain conditions. Particularly, the risk in the firm's intangible assets or growth opportunities must not be high compared to the risk in tangible assets. Furthermore, the smaller the correlation between those risks, the better the alternative of *musharaka* financing. And the entrepreneur's marginal costs of spending effort aiming to raise the value of tangible (intangible) assets must not increase much in the cost of effort for intangible (tangible) assets for *musharaka* financing to be more beneficial (i.e., the cross derivative of the cost function must not be large).

However, raising capital by means of regular equity is definitely more advantageous if the risk in the firm's tangible assets is smaller than the risk in intangible assets or growth opportunities. If the latter risk is not shared with an outside investor as with, e.g., the *musharaka* financing considered in our model, this disadvantage in risk sharing cannot be compensated for by better incentives.

The considerations in this paper can be extended in various ways. Since a *musharaka* investor, as well as an equity partner, is entitled to also take part in the management process, the effects of them also extending effort on managing either one particular project or the firm in general, respectively, could also be explored within the theoretical framework considered in this paper. Another possible extension would be applying the same assumptions and theoretical model in case the entrepreneur has the opportunity to raise capital from both an equity partner and a *musharaka* investor. In this case, the entrepreneur could, for example, finance tangible assets with *musharaka* besides external equity financing of the firm in total. Furthermore, in a more general setting, a scenario in which the entrepreneur finances a project with funding from two or more investors with different levels of risk aversion could also be analyzed within the theoretical framework applied in this paper. In such a scenario, the extent of profit and loss and risk sharing with different investors would depend on their risk aversion. In conclusion, the theoretical approach of this paper can be applied to analyze the use of multiple external sources of funding that are not limited to Islamic financial instruments. Other hybrid financial instruments could be considered in a similar manner.

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Appendix A

Here we illustrate the reasoning behind restricting γ to the interval $]-1;1[$. When the entrepreneur's efforts are e_a and e_b , the net return from spending effort can be expressed as

$e_a + e_b - \frac{e_a^2}{2} - \frac{e_b^2}{2} - \gamma e_a e_b$. From the first-order condition, we find the first-best levels of effort

that maximize this return to be $e_a = e_b = \frac{1}{1+\gamma}$, which leads to a net return of $\frac{1}{1+\gamma}$.

If $\gamma = -1$, the net return from extending effort is $e_a + e_b - \frac{e_a^2}{2} - \frac{e_b^2}{2} + e_a e_b$, and the optimal levels of e_a and e_b do not exist simultaneously. Only after arbitrarily deciding on e_a (or e_b) can the entrepreneur determine the optimal effort e_b (or e_a).

For $\gamma = 1$, the net return from extending effort is $(e_a + e_b) - \frac{1}{2}(e_a + e_b)^2$. The entrepreneur can choose any efforts e_a and e_b that add up to 1 in order to maximize the net return.

If $|\gamma| > 1$, the entrepreneur could instead choose arbitrarily to spend either no effort e_a or no effort e_b and yet earn a higher (or equal) net return in optimum than if the efforts $e_a = e_b = \frac{1}{1+\gamma}$ are chosen. With $|\gamma| > 1$, the optimal efforts are $e_a = 1$ and $e_b = 0$ or $e_b = 1$ and $e_a = 0$. Then the net return is $\frac{1}{2}$.

Only for $-1 < \gamma < 1$, we find that the net return in optimum, $\frac{1}{1+\gamma}$, exceeds the net return under the arbitrary constraint $e_b = 0$ (or $e_a = 0$), i.e., $\frac{1}{1+\gamma} > \frac{1}{2}$. Therefore, we restrict γ to the interval $]-1;1[$ to rule out nonsensical results.

Appendix B

Here we prove that if the financial structure does not affect the entrepreneur's effort, and the first-best efforts are implemented, risk sharing with equity is superior to risk sharing under *musharaka* financing.

In the absence of agency problems, that is, with the first-best efforts, the certainty equivalent of the entrepreneur is

$$CE_E(\text{equity}) = x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2} [\theta_E(1-\alpha_P)^2 + \theta_P\alpha_P^2] \text{Var}(\tilde{a} + \tilde{b}) \quad (44)$$

and

$$CE_E(\text{musharaka}) = x + y + \frac{1}{1+\gamma} - (1+r_f)(C - W_{0,E}) - \frac{1}{2} [\theta_E(1-\alpha_I)^2 + \theta_I\alpha_I^2] \text{Var}(\tilde{a}) - \frac{1}{2} \theta_E \text{Var}(\tilde{b}) - \theta_E(1-\alpha_I) \text{Cov}(\tilde{a}, \tilde{b}) \quad (45)$$

under equity and *musharaka* financing, respectively.

With given (first-best) efforts, the optimal shares are $\alpha_P^* = \frac{\theta_E}{\theta_E + \theta_P}$ for equity financing

and $\alpha_I^* = \frac{\theta_E [\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b})]}{(\theta_E + \theta_I) \text{Var}(\tilde{a})}$ with *musharaka*.

Inserting these shares in (44) and (45), respectively, and applying $\theta_P = \theta_I = \theta$ (Assumption 9), we find

$$\begin{aligned} CE_E(\text{equity}) &> CE_E(\text{musharaka}) \\ \Leftrightarrow \frac{1}{2} \frac{\theta_E^2}{\theta_E + \theta} \text{Var}(\tilde{a} + \tilde{b}) &> \frac{1}{2} \frac{\theta_E^2 [\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b})]^2}{(\theta_E + \theta) \text{Var}(\tilde{a})} \\ \Leftrightarrow \text{Var}(\tilde{a} + \tilde{b}) \text{Var}(\tilde{a}) &> [\text{Var}(\tilde{a}) + \text{Cov}(\tilde{a}, \tilde{b})]^2 \\ \Leftrightarrow (\text{Cov}(\tilde{a}, \tilde{b}))^2 &< \text{Var}(\tilde{a}) \text{Var}(\tilde{b}) \\ \Leftrightarrow \rho_{(\tilde{a}, \tilde{b})}^2 < 1 &\Leftrightarrow |\rho_{(\tilde{a}, \tilde{b})}| < 1 \end{aligned} \quad (46)$$

Q.e.d.

Appendix C

Here we show that the higher $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$ is, the more important it is for the entrepreneur to share total risk instead of only $Var(\tilde{a})$.

The risk premium of the entrepreneur is

$$RPM = \frac{1}{2}\theta_E Var(\tilde{a} + \tilde{b}) - \frac{1}{2}\theta_E [Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b})] \frac{\theta_E [Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b})]}{\frac{1}{1-\gamma^2} + (\theta_E + \theta)Var(\tilde{a})} \quad (47)$$

and

$$RPE = \frac{1}{2}\theta_E Var(\tilde{a} + \tilde{b}) - \frac{1}{2}\theta_E Var(\tilde{a} + \tilde{b}) \frac{\theta_E Var(\tilde{a} + \tilde{b})}{\frac{2}{1+\gamma} + (\theta_E + \theta)Var(\tilde{a} + \tilde{b})} \quad (48)$$

under *musharaka* and equity financing, respectively.

The relative advantage of risk sharing under equity financing can be represented by the difference between these risk premia:

$$RPM - RPE = \frac{1}{2} \frac{\theta_E^2 (Var(\tilde{a} + \tilde{b}))^2}{\frac{2}{1+\gamma} + (\theta_E + \theta)Var(\tilde{a} + \tilde{b})} - \frac{1}{2} \frac{\theta_E^2 [Var(\tilde{a}) + Cov(\tilde{a}, \tilde{b})]^2}{\frac{1}{1-\gamma^2} + (\theta_E + \theta)Var(\tilde{a})} > 0. \quad (49)$$

With given levels of $Var(\tilde{a})$ and $\rho_{(\tilde{a}, \tilde{b})}$, the relationship between the risks being shared

under the two alternatives, $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{a})}$, depends on $Var(\tilde{b})$. We can show that $(RPM - RPE)$

increases in $Var(\tilde{b})$ and hence in $\frac{Var(\tilde{a} + \tilde{b})}{Var(\tilde{b})}$ for any given values of $Var(\tilde{a})$ and $\rho_{(\tilde{a}, \tilde{b})}$:

$$\frac{d(RPM - RPE)}{dVar(\tilde{b})} = \frac{(1+\gamma)\theta_E^2 \{4 + (1+\gamma)(\theta_E + \theta)Var(\tilde{a} + \tilde{b})\}}{2\{2 + (1+\gamma)(\theta_E + \theta)Var(\tilde{a} + \tilde{b})\}^2} > 0. \quad (50)$$

Nonlinear Profit and Loss Sharing versus Equity Financing in Entrepreneurial Finance: Risk Sharing and Managerial Incentives in a Principal-Agent Relationship When the Agent Is Risk-Neutral

Nonlinear PLS versus Equity: Risk Sharing and Managerial Incentives – Part I

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Published in *New Challenges in Accounting and Finance*

Link: <https://doi.org/10.32038/NCAF.2025.13.02>

Abstract

An entrepreneur raises capital to finance her business and extends effort to facilitate its success. If financing is possible solely with risk-free debt capital, the entrepreneur's effort will be first-best since she will not share the outcome of her business with an outside financier. However, risk-free debt is usually unavailable due to the significant risk involved in starting any business. Therefore, other financing arrangements involving risk sharing have to be considered. Such arrangements will result in a moral hazard problem and dilute the managerial incentives of the entrepreneur to extend effort. Financing with equity capital from an outside partner will result in second-best level of effort, and there will be no advantage from risk sharing if the entrepreneur is risk-neutral. Since risk-free debt financing is not always possible and financing with equity capital is not optimal for a risk-neutral entrepreneur, a third alternative is considered in this paper. It is shown that a nonlinear profit-and-loss-sharing arrangement can be structured so that the incentives of the risk-neutral entrepreneur to extend effort are not diminished. As such, the nonlinear PLS contract considered in this paper, which is similar to Islamic profit-and-loss-sharing contracts, is a feasible alternative to risk-free debt financing and superior to equity financing.

Keywords: entrepreneurial finance, equity, nonlinear contracts, profit and loss sharing, risk sharing, managerial incentives, hidden effort, moral hazard problem

1. Introduction

An entrepreneur with insufficient funds to finance an investment opportunity (i.e., business, project, start-up, etc.) needs outside capital in the form of debt or equity. Hybrid financing combining the features of debt and equity capital could also be considered. Financing with equity or hybrid capital provided by a financier involves risk sharing. These methods of financing lead to a principal-agent relationship, with the entrepreneur as the agent and the financier as the principal, resulting in diminished managerial incentives of the entrepreneur to provide effort to facilitate the success of the business since she has to share the outcome with the outside financier. If the entrepreneur can finance her business with risk-free debt capital, which is usually not available due to the risk involved in starting a business, her effort is first-best. Risk sharing with an equity partner, on the other hand, results in the second-best effort of the entrepreneur, and without the direct benefits of risk sharing, financing with equity is inferior to risk-free debt financing. The focus of this paper is hybrid financing with the aim of finding an optimal contractual structure that can eliminate (or at least alleviate to some extent) the adverse effects of the moral hazard problem.

Hybrid financial instruments exhibit characteristics of both debt and equity capital. Preferred stock paying dividends at a fixed rate can be considered an example of hybrid capital. Participating preferred stock might also pay out participating dividends linked to the success of the company on top of the fixed dividends. Participating loans, on the other hand, entail payments dependent on the success of the company as opposed to fixed interest payments. The creditors of participation loans are entitled to a share of the sales revenue or profit; they do not, however, share the losses the company might incur. In case of financing with a participation loan, a fixed payment can also be agreed upon (OECD, 2015). Closer to equity financing are silent partnerships, two forms of which can be differentiated. A typical silent partner who invests in a project participates in profit sharing while an atypical silent partner shares both the profits and the losses with the entrepreneur. The atypical silent partner has then more rights as a co-entrepreneur than a typical silent partner for taking on a higher level of risk (OECD, 2015). The remuneration resulting from hybrid financial instruments usually involves linear sharing of profits. Some might also entail loss sharing or a fixed payment.

Hybrid financing is especially relevant in project financing and for SMEs. Examples of hybrid financial instruments can also be found in Islamic finance in the form of near-equity methods of financing entailing nonlinear profit and loss sharing, where the profits and losses

resulting from the project are shared between the parties based on different shares.¹ These so-called *musharaka* and *mudaraba* contracts result in arrangements where the profits are shared based on a pre-agreed ratio while the liability for losses is in proportion to the individual capital contributions to the project (Hussain et al., 2015). In a *musharaka* transaction, both the entrepreneur and an investor provide capital to finance a project or an investment and share ownership based on their share of capital. Both parties are entitled to take part in the management process. The (net) profits generated from the project are shared between the entrepreneur and the investor based on a pre-agreed ratio while their liability for any (net) losses is in proportion to their initial capital contributions (Jobst, 2007). A *mudaraba* contract is similar to *musharaka*, but only the investor provides financing for the project, and only the entrepreneur can participate in management. The profits and losses are shared in the same way as with *musharaka* financing, and only the investor is liable for losses unless the manager is at fault (Jobst, 2007). Therefore, the main difference between a *musharaka* and a *mudaraba* transaction is in the level of financing provided by the investor.

In this paper, in addition to debt and equity financing, nonlinear profit and loss sharing with a fixed payment component, which can either be a wage for the entrepreneur or a compensation for the investor, is considered. The way such an arrangement is structured is inspired by the Islamic profit-and-loss-sharing contracts. Similarly to a *musharaka* or *mudaraba* contract, the nonlinear PLS arrangement explored in this paper involves sharing of the profits based on a pre-agreed ratio and of the losses in proportion to initial capital contributions. However, the profits and losses are defined differently. While Islamic PLS contracts involve sharing of net profits and losses resulting from a project, the nonlinear PLS contract considered here entails sharing of the future value of the project, in other words, gross profits and losses. Another difference to a *musharaka* or *mudaraba* contract is the fixed payment component that can be agreed upon between the parties. These differences are illustrated with a simple numerical example in Appendix A.

The financial contract between the entrepreneur and the financier determines how the outcome, that is, profits and losses from the project are shared between the parties to the contract. Conventional equity financing results in a linear profit-and-loss-sharing contract in which the share of the profits the financier is entitled to is same as the share of the losses he is

¹ Islamic financial contracts should adhere to the principles underlying Islamic banking and finance. As such, financial transactions involving activities such as charging interest, speculation, and short selling are prohibited (Iqbal & Mirakhor, 2011).

liable for. The profits and losses in this context refer to the increase and decrease in the market value of the project. Three alternative forms of financing are compared: (1) (risk-free) debt financing, (2) equity financing, and (3) nonlinear profit-and-loss-sharing (PLS) contract with a fixed payment component. In essence, this is a form of hybrid financing that involves outcome-dependent profit and loss sharing and a fixed payment. Such a contract can be arranged by slightly modifying Islamic PLS contracts (see Appendix A).

In the agency-theoretic framework applied in this paper, it is shown that both (risk-free) debt financing and the nonlinear PLS contract result in the first-best effort of the entrepreneur when both the entrepreneur and the financier are risk-neutral. Financing with equity capital does not offer an advantage in terms of risk sharing for the risk-neutral entrepreneur and dilutes her incentive to extend effort. Since risk-free debt financing is not always possible due to the non-negligible risk involved in starting a business, the nonlinear PLS contracting is the superior method of financing.

The rest of this paper is structured as follows. A review of selected literature on principal-agent relationships is provided in section 2 to touch upon the issues regarding risk sharing, diminished managerial incentives, and the use of monitoring in such relationships. Section 3 outlines the theoretical model, where the underlying assumptions are detailed, and the alternative contracts under consideration are specified. Section 4 presents theoretical analyses and results assuming that the entrepreneur is risk-neutral. It is shown that a nonlinear incentive scheme can be arranged so that even though the entrepreneur has to share the outcome of her business with the outside investor, her incentive to extend effort is not diminished. Section 5 offers concluding remarks.

2. Moral Hazard Problem in Principal-Agent Relationships

Both financing with equity and nonlinear profit-and-loss-sharing arrangement as considered in this paper result in a partnership that involves risk sharing. Such a partnership is a principal-agent relationship between the entrepreneur (the agent) attempting to start a (small or medium-sized) business and the outside financier (the principal) who provides capital. A moral hazard problem arises when the agent can influence the outcome with her actions (i.e., effort) that cannot be observed by the principal. In this case, the Pareto-optimal level of risk sharing cannot be achieved since it would dilute the managerial incentives of the entrepreneur to exert a sufficient level of effort. On one hand, risk sharing is beneficial for the (risk-averse) entrepreneur; however, she would also have to share the outcome with the principal. Therefore,

the informational asymmetries in principal-agent relationships result in a trade-off between the benefits of risk sharing and the diminished managerial incentives. This trade-off was first explored by Zeckhauser (1970) in the context of medical insurance. In this setting, optimal risk sharing induces excessive spending (e.g., with unnecessary visits to the emergency room) by insured individuals since they do not have to pay the full amount of their medical expenses. Therefore, the optimal insurance plan has to trade off risk sharing and incentives. In classical agency-theoretic literature, this trade-off has been extensively inspected.

Ross (1973) explores the principal's problem, which is to find an optimal fee schedule to maximize his von Neumann-Morgenstern utility while the agent chooses an action to maximize her utility. He finds that the solution to the principal's problem cannot be Pareto-optimal, but also claims that Pareto-efficiency can be achieved if the principal was aware of the action the agent would choose based on the fee schedule; then, he could just require the agent to act a certain way. However, this would also require monitoring of her actions. Holmström (1979) similarly suggests that the principal can force the agent to act a certain way if complete monitoring is possible. Even if the complete observation of the agent's effort is either impossible or too costly, any amount of information about her actions can alleviate the moral hazard problem. According to Shavell (1979), this information would only be valuable if the agent is risk-averse because a risk-neutral agent will absorb the risk and choose a level of effort to maximize the expected outcome since her payment will depend only on this outcome. If the agent is risk-averse, her fee will depend also on the principal's knowledge about her action. Holmström (1979) also considers that the principal has to bear the costs of monitoring; the value of the information about the agent's action depends on how costly it is to obtain it. He suggests that the outcome must be sufficiently bad in order for the benefits of monitoring to outweigh the costs. Spence and Zeckhauser (1971) explore monitoring costs in case of medical insurance. The cost of information (i.e., monitoring) is also considered by Liu (1982) and Jost (1991).

Harris and Raviv (1979) conclude that it is optimal to combine imperfect monitoring with a nonlinear payment scheme, that is, if the principal decides to (imperfectly) monitor the agent's action, the optimal payment scheme to the agent depends on what the principal discovers regarding the agent's action. If the agent's action is discovered to be acceptable, she is paid based on a pre-determined outcome-dependent schedule. If not, she receives a lower fixed payment. Stiglitz (1974) also considers contracts between a landlord (principal) and a worker (agent) consisting of both a basic wage (i.e., a fixed payment) and an incentive pay in

a simple agricultural economy. In his model, the incentive payment is linear in output. Spremann (1988) explores a principal-agent relationship resulting from such a profit-sharing arrangement that even though the principal can observe only the outcome (but not the agent's effort), he can choose an incentive scheme that induces the agent to (selfishly) decide for a level of effort acceptable for the principal. The agent can influence the outcome of the project by extending effort, which also results in private costs for her. In such a setting, the contract between the entrepreneur and the financier determines how the risk is allocated between them and to what extent the entrepreneur's incentives are diminished as a result of having to share the outcome with an outside financier. By assuming that the agent and principal are both risk-neutral and that the agent can influence the probability of "good" and "bad" outcomes, Breuer (1995) shows that nonlinear incentive contracts, where the agent receives an outcome-dependent and a fixed payment when the outcome is above a certain threshold and no payment at all otherwise, are optimal. The model employed by Breuer (1995) is similar to the one in this paper, where the agent has an influence on not only the expected outcome, but also the risk involved since the agent's actions affect not the expected outcome directly, but the probability of positive and negative outcomes. This approach is different from the LEN (Linear-Exponential-Normal) model of Spremann (1988), where the agent can affect only the expected outcome by shifting its probability distribution to the right.

Nonlinear profit and loss sharing is considered in addition to debt and equity financing in this paper. In an agency-theoretic framework, nonlinear incentive schemes are hardly explored. Even in cases considering nonlinear incentive payments, the nonlinearity stems only from the fixed payment up to or above a certain level of outcome. In the setting explored in this paper, however, the nonlinearity refers to the profits and losses being shared based on different ratios. The motivation for considering such an intricate arrangement stems from how Islamic PLS contracts are structured. The nonlinear PLS contracting is compared to conventional methods of debt and equity financing with respect to the diminished managerial incentives to extend effort in order to find out if such an incentive scheme can alleviate the moral hazard problem in a principal-agent relationship when both the principal and the agent are risk-neutral, where risk sharing does not entail any benefits for the entrepreneur and does not play a role in her effort decision.

3. The Model

A one-period model of an investment opportunity for the entrepreneur, which requires an initial capital in the amount of A_0 in $t = 0$, is considered. The future value of this project in $t = 1$ is defined as $\tilde{A}_1 = (1 + \tilde{r})A_0$ with \tilde{r} as the uncertain rate of return on the investment. A profit on the investment is realized when the value of the project increases, that is, when the rate of return is positive. A negative rate of return results in a decrease in the value of the project and therefore, in a loss.

Assumption 1.

The rate of return on the investment is defined as

$$\tilde{r} = y|\tilde{x}|, \quad (1)$$

where \tilde{x} is a normally distributed random variable, $\tilde{x} \sim N(0, \sigma^2)$, and $y = \pm 1$ is an indicator of a positive or negative rate of return.

Assumption 2.

The entrepreneur can increase the probability of a positive return (simultaneously decreasing the probability of a loss) with her effort, $e \in (0; 1)$. The two-point distribution of y is then defined by the probability mass function

$$f(y; e) = \frac{1}{2}(1 + y \cdot e) = \begin{cases} \frac{1}{2}(1 - e) & \text{if } y = -1 \\ \frac{1}{2}(1 + e) & \text{if } y = 1 \end{cases}. \quad (2)$$

The entrepreneur's effort is a decision variable and a measure of probability and results in private costs, $c(e)$, for the entrepreneur.

Assumption 3.

The entrepreneur's cost of extending effort to facilitate the success of her business is

$$c(e) = \frac{1}{2}e^2 A_0, \quad (3)$$

which also depends on the initial capital A_0 required since the bigger the project, the more it costs (in monetary terms) the entrepreneur to extend effort.

Since the future value of the project is defined as $\tilde{A}_1 = (1 + \tilde{r})A_0 = (1 + y|\tilde{x}|)A_0$, its probability distribution can be represented by the probability density function of \tilde{r} . The rate of return \tilde{r} equals $-|\tilde{x}|$ with $\frac{1}{2}(1-e)$ probability and $|\tilde{x}|$ with $\frac{1}{2}(1+e)$ probability. Both $-|\tilde{x}|$ and $|\tilde{x}|$ have the same probability density function.² Consequently, the probability density function of \tilde{r} can be expressed as

$$f(\tilde{r}) = \frac{1+e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] = \begin{cases} \frac{1-e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } \tilde{r} < 0 \\ \frac{1+e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } \tilde{r} > 0 \end{cases} \quad (4)$$

Therefore, the entrepreneur can, with her effort, influence both the expected value and the standard deviation of the rate of return on the project.

The probability density function of the rate of return \tilde{r} is illustrated in the figures below. Figure 1 represents the normal distribution of \tilde{r} in the absence of effort of the entrepreneur. The probability of both a positive and a negative rate of return is $\frac{1}{2}$ without effort. The entrepreneur's effort influences these probabilities. The probability of a negative return is reduced to $\frac{1}{2}(1-e)$ while the probability of a positive return is increased to $\frac{1}{2}(1+e)$ when the entrepreneur extends effort. Figure 2 shows how the entrepreneur's effort affects (i.e., changes) the distribution of the rate of return.

² $|\tilde{x}|$ and $-|\tilde{x}|$ exhibit a half-normal distribution, i.e., a folded normal distribution with 0 mean and the probability density function $f(|\tilde{x}|) = f(-|\tilde{x}|) = \frac{2}{\sigma\sqrt{2\pi}} \exp\left[-\frac{|\tilde{x}|^2}{2\sigma^2}\right]$ (Tsagris et al., 2014).

³ The probability distribution of the rate of return \tilde{r} , whose properties are detailed in Appendix B, resembles a two-piece normal distribution (see Wallis, 2014).

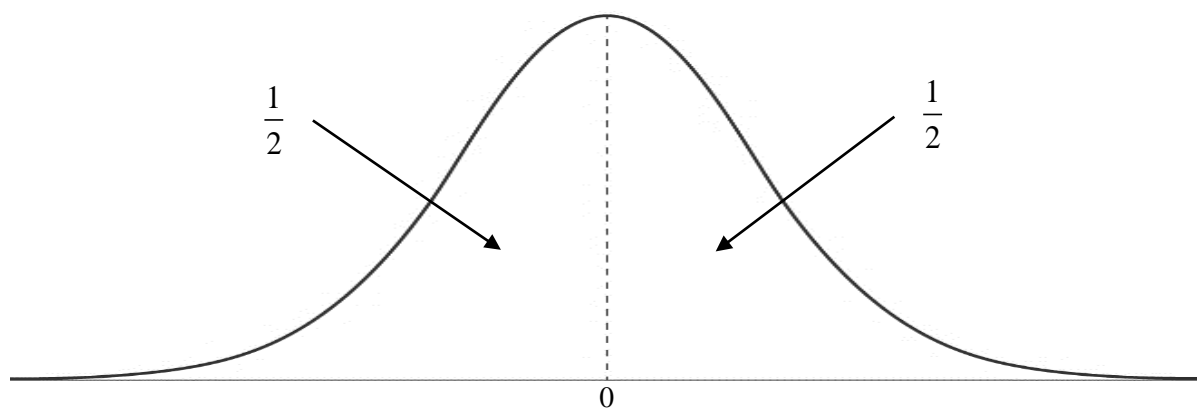


Figure 1. Probability density function: $f(\tilde{r})$, $e = 0$

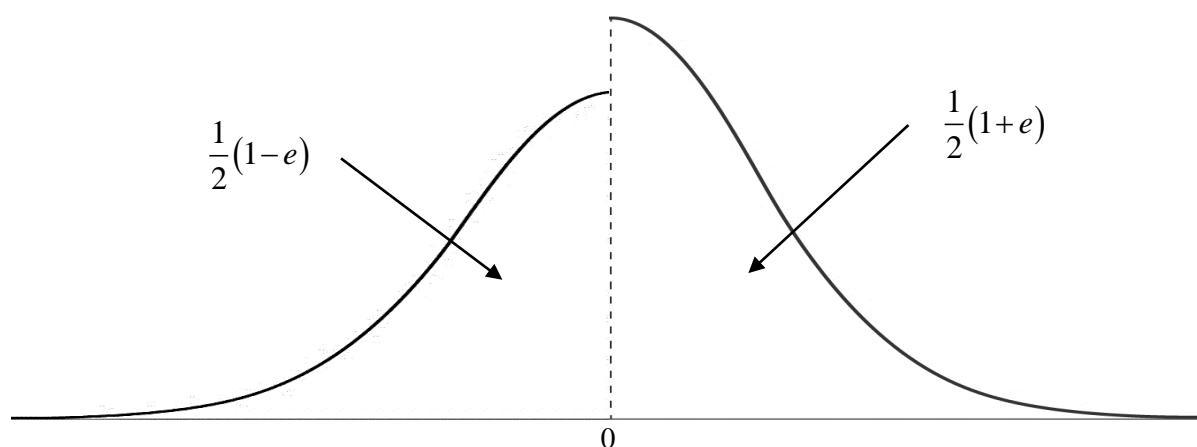


Figure 2. Probability density function: $f(\tilde{r})$, $e > 0$

Assumption 4.

The initial wealth $W_{0,E}$ of the entrepreneur E in $t = 0$ is not sufficient to cover the investment amount A_0 ; therefore, outside capital $C = A_0 - W_{0,E}$ is required for the entrepreneur to be able to start her business. She can either decide to invest her available capital in risk-free assets on the market instead of starting her business or raise the required capital as debt, equity, or with a nonlinear PLS contract.

*Assumption 5.*⁴

The interest rate charged on any debt is same as the rate of return r_f on alternative (risk-free) investments. Debt financing is risk-free (free of default risk), and there are no market imperfections that would result in a difference between lending and borrowing rates.

⁴ This assumption, although unrealistic since risk-free debt is usually unavailable, is made as a point of reference for the sake of simplicity and comparability.

If the entrepreneur decides for a risk-free investment, her future wealth in $t=1$ is $W_{1,E}^{risk-free investment} = (1+r_f)W_{0,E}$. She can also choose to raise the required capital with debt and start her business. In this case, she will have to extend effort as the manager and will have to pay back the principal amount and the interest on the loan. If she can finance her business with risk-free debt capital, her uncertain future wealth will be

$$\begin{aligned}\tilde{W}_{1,E}^{debt} &= \tilde{A}_1 - c(e) - (1+r_f)C \\ &= (1+\tilde{r})A_0 - c(e) - (1+r_f)(A_0 - W_{0,E})\end{aligned}\quad (5)$$

Alternatively, the entrepreneur can finance her project with equity capital from a partner. The equity partner P provides capital in the amount of C_P and becomes a shareholder of the firm. This entitles him to a share α in the outcome of the project.

In case of equity participation from an outside partner, the uncertain future wealth of the entrepreneur is

$$\begin{aligned}\tilde{W}_{1,E}^{equity} &= (1-\alpha)\tilde{A}_1 - c(e) - (1+r_f)(C - C_P) \\ &= (1-\alpha)(1+\tilde{r})A_0 - c(e) + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E})\end{aligned}\quad (6)$$

The amount of capital provided by the partner might be more or less than required. If $C - C_P < 0$, the entrepreneur can invest the difference at the risk-free rate r_f . On the other hand, if $C - C_P > 0$, the entrepreneur will have to raise additional debt capital with the interest rate r_f .

The third option for the entrepreneur is a nonlinear profit-and-loss-sharing (PLS) contract with an investor. The nonlinear PLS investor I provides capital in the amount of $C_I = \lambda A_0$. The investor will then be entitled to a share γ of the outcome in case of a positive return (i.e., profit). The entrepreneur and the investor will share losses in proportion to their initial capital contributions, which means, the investor will be liable for a share λ of any losses. The entrepreneur and the investor will also agree on a fixed payment $B = \beta A_0$ that can be positive or negative. When $B > 0$, the investor additionally receives a fixed compensation from the entrepreneur. Otherwise, i.e., if $B < 0$, the investor agrees to pay the entrepreneur a fixed wage on top of her share of the outcome.

As it is the case with debt financing, the liability of the entrepreneur is not limited when she finances her business via a nonlinear PLS contract. The ramifications of limited liability are not considered in this paper.⁵

Assumption 6.

In case the outcome of the entrepreneur's project is not sufficient for her to be able to fulfill her contractual obligations, she will (and can) cover the agreed upon payments (principal and interest in case of any debt financing and the fixed payment to the investor (if $\beta > 0$) in case of financing with a nonlinear PLS contract) with her own private funds. Furthermore, even if the actual return on the project is too low, the investor will consider it bad luck rather than a result of the entrepreneur shirking her duties and still pay her the agreed upon wage (if $\beta < 0$) with his own private funds (in case his share of the outcome is not sufficient to cover the fixed payment to the entrepreneur).

In case of nonlinear PLS arrangement with an investor, the uncertain future wealth of the entrepreneur is

$$\begin{aligned} \tilde{W}_{1,E}^{PLS} &= \begin{cases} (1-\lambda)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if } \tilde{r} < 0 \\ (1-\gamma)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if } \tilde{r} > 0 \end{cases} \\ &= \begin{cases} [(1-\lambda)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if } \tilde{r} < 0 \\ [(1-\gamma)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if } \tilde{r} > 0 \end{cases} \end{aligned} \quad (7)$$

The amount of capital provided by the investor might be more or less than required. If $C - C_I < 0$, the entrepreneur can invest the difference at the risk-free rate r_f . On the other hand, if $C - C_I > 0$, the entrepreneur will have to raise additional debt capital with the interest rate r_f .

The future wealth of both the entrepreneur and the potential financiers depends on the outcome of the investment and how that outcome is shared between them. The entrepreneur can increase the probability of a positive return, and consequently, the expected return on the investment with her effort, but that comes at a private cost (see (3)).

⁵ Innes (1990) explores the implications of limited liability (for both the risk-neutral principal and the risk-averse agent) for optimal effort and incentives when the agent chooses an effort before the state of nature is known.

Assumption 7.

The entrepreneur and the outside financiers are risk-neutral. The risk-neutral entrepreneur chooses a financial contract that maximizes the expected value of her uncertain future wealth, subsequently deciding on the level of effort. The entrepreneur's effort is known only to her. The potential risk-neutral financiers can observe only the outcome of the project without knowing the extent of the contribution of the entrepreneur's effort to the outcome. The financiers are, however, not ignorant; they are aware of the relevant characteristics of the entrepreneur and the project she wishes to finance. Therefore, the rational financiers can anticipate the effort decision of the rational entrepreneur.

Assumption 8.

The outside financier is willing to provide such an amount of capital that his investment in the entrepreneur's project offers the same expected rate of return as his outside option (i.e., risk-free investment).

4. Comparison: Debt, Equity, and Nonlinear PLS

Three alternative financial contracts are considered: (i) risk-free debt financing, (ii) equity partnership with an outside partner P who provides capital C_p and is entitled to share α in the outcome of the entrepreneur's project, and (iii) nonlinear PLS contract with an investor I who provides capital $C_I = \lambda A_0$ and is entitled to share γ of the value of the project in case it increases while being liable for share λ of the losses. The nonlinear PLS contract also has a fixed component $B = \beta A_0$, meaning, either the entrepreneur or the investor receives a fixed payment (a wage or a compensation, respectively).

Risk-free debt financing involving no profit and loss or risk sharing is considered first. If the entrepreneur finances her business with debt, the expected value of her future wealth from (5) is

$$\begin{aligned} E\{\tilde{W}_{1,E}^{debt}\} &= E\left\{(1+\tilde{r})A_0 - c(e) - (1+r_f)(A_0 - W_{0,E})\right\} \\ &= \left\{1 + e \frac{\sigma\sqrt{2}}{\sqrt{\pi}} - \frac{1}{2}e^2\right\} A_0 - (1+r_f)(A_0 - W_{0,E}), \end{aligned} \quad (8)$$

with the cost of effort function from (3).

The entrepreneur then chooses the level of effort that maximizes this expected value of her future wealth:

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \quad (9)$$

This is the first-best level of effort since the entrepreneur does not share the outcome of her effort and claims the full marginal benefit of extending effort.

Assumption 9.

To make sure $e \in (0;1)$, it is assumed that $\frac{\sigma\sqrt{2}}{\sqrt{\pi}} < 1 \Leftrightarrow \sigma < \sqrt{\frac{\pi}{2}}$.

With the first-best effort from (9), the entrepreneur's expected future wealth is

$$E\{\tilde{W}_{1,E}^{debt}\} = \left\{1 + \frac{\sigma^2}{\pi}\right\} A_0 - (1+r_f)(A_0 - W_{0,E}). \quad (10)$$

She will start her business with debt financing only if it results in a higher expected future wealth than with a risk-free investment:

$$\left\{1 + \frac{\sigma^2}{\pi}\right\} A_0 - (1+r_f)(A_0 - W_{0,E}) > (1+r_f)W_{0,E} \quad (11)$$

$$\Leftrightarrow \frac{\sigma^2}{\pi} > r_f. \quad (12)$$

That is, the expected rate of return $\frac{\sigma^2}{\pi}$ on the investment, given the entrepreneur's effort from (9), should exceed the risk-free rate of return.

Financing with equity capital results in profit and loss and risk sharing. If the entrepreneur finances her business with equity participation from a partner, the expected value of her future wealth from (6) is

$$\begin{aligned} E\{\tilde{W}_{1,E}^{equity}\} &= E\left\{(1-\alpha)(1+\tilde{r})A_0 - c(e) + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E})\right\} \\ &= \left\{(1-\alpha)\left(1 + e\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) - \frac{1}{2}e^2\right\} A_0 + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}), \end{aligned} \quad (13)$$

with the cost of effort function from (3).

After the contract has been signed, that is, with given α and C_p , the entrepreneur chooses a level of effort that maximizes the expected value of her future wealth:

$$e = (1 - \alpha) \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \quad (14)$$

This is lower than the first-best effort from (9) since the entrepreneur has to share the benefits of her effort with the equity partner.

Lemma 1.

The higher the share of the outcome the entrepreneur is to share with the outside partner, the lower the level of effort the entrepreneur is, *ceteris paribus*, willing to extend:

$$\frac{de}{d\alpha} = -\frac{\sigma\sqrt{2}}{\sqrt{\pi}} < 0. \quad (15)$$

The participation constraint (which follows from Assumption 8) of the equity partner P with initial wealth $W_{0,P}$, who correctly anticipates the effort decision of the entrepreneur, determines the level of capital C_p he is willing to provide:

$$\begin{aligned} E\{\tilde{W}_{1,P}\} &= E\left\{\alpha(1+\tilde{r})A_0 + (1+r_f)(W_{0,P} - C_p)\right\} \\ &= (1+r_f)W_{0,P}. \end{aligned} \quad (16)$$

$$\begin{aligned} \Leftrightarrow \alpha \left\{1 + e \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} A_0 + (1+r_f)(W_{0,P} - C_p) \\ = (1+r_f)W_{0,P} \end{aligned} \quad (17)$$

$$\Leftrightarrow C_p = \frac{1}{1+r_f} \alpha \left\{1 + e \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} A_0. \quad (18)$$

With the second-best effort from (14) and the amount of capital provided by the partner from (18), the expected value of the entrepreneur's future wealth is

$$E\{\tilde{W}_{1,E}^{equity}\} = \left\{1 + (1 - \alpha^2) \frac{\sigma^2}{\pi}\right\} A_0 - (1+r_f)(A_0 - W_{0,E}). \quad (19)$$

Considering $\alpha \in (0;1]$, financing with debt results in a higher expected future wealth for the entrepreneur than financing with equity (compare (10) and (19)) because there are no benefits

from risk sharing for a risk-neutral entrepreneur, and the profit and loss sharing results in diluted managerial incentives to extend effort.

The entrepreneur will start her business with equity financing only if it results in a higher expected future wealth than with a risk-free investment:

$$\left\{1 + (1 - \alpha^2) \frac{\sigma^2}{\pi}\right\} A_0 - (1 + r_f)(A_0 - W_{0,E}) > (1 + r_f)W_{0,E} \quad (20)$$

$$\Leftrightarrow (1 - \alpha^2) \frac{\sigma^2}{\pi} > r_f. \quad (21)$$

That is, the expected rate of return $(1 - \alpha^2) \frac{\sigma^2}{\pi}$ on the investment (which is lower than with debt financing (see (12))) given the entrepreneur's effort from (14) should exceed the risk-free rate of return.

Another option for the entrepreneur to raise capital is a nonlinear PLS contract with an investor. If the entrepreneur finances her business via a nonlinear PLS arrangement with an investor, the expected value of her future wealth from (7)

$$\begin{aligned} E\{\tilde{W}_{1,E}^{PLS}\} &= E\left\{\left[(1 - \lambda)(1 + \tilde{r}) - \beta\right]A_0 - c(e) + (1 + r_f)C_I - (1 + r_f)(A_0 - W_{0,E})\right\} \cdot \chi_{\{\tilde{r} < 0\}} \\ &\quad + E\left\{\left[(1 - \gamma)(1 + \tilde{r}) - \beta\right]A_0 - c(e) + (1 + r_f)C_I - (1 + r_f)(A_0 - W_{0,E})\right\} \cdot \chi_{\{\tilde{r} > 0\}} \\ &= \left\{\frac{1}{2}(1 - e)(1 - \lambda)\left\{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \frac{1}{2}(1 + e)(1 - \gamma)\left\{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}e^2 - \beta\right\} A_0 \\ &\quad + (1 + r_f)C_I - (1 + r_f)(A_0 - W_{0,E}) \end{aligned} \quad (22)$$

with the cost of effort function from (3).

Here, $\chi_{\{\tilde{r} < 0\}} = \begin{cases} 1 & \text{if } \tilde{r} < 0 \\ 0 & \text{if } \tilde{r} > 0 \end{cases}$ and $\chi_{\{\tilde{r} > 0\}} = \begin{cases} 1 & \text{if } \tilde{r} > 0 \\ 0 & \text{if } \tilde{r} < 0 \end{cases}$ are indicator functions.

The entrepreneur then chooses a level of effort that maximizes this expected value of her future wealth after the contract has been signed, i.e., with given λ (and C_I), γ , and β :

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}\lambda\left\{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}\gamma\left\{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}. \quad (23)$$

Lemma 2.

The higher (lower) the share of the negative (positive) outcome the investor is entitled to, the higher the level of effort the entrepreneur is, *ceteris paribus*, willing to extend:

$$\frac{de}{d\lambda} = \frac{1}{2} \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) > 0 \quad , \quad \frac{de}{d\gamma} = -\frac{1}{2} \left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) < 0. \quad (24)$$

There are both positive and negative effects of the nonlinear profit-and-loss-sharing arrangement on the managerial incentives of the entrepreneur to extend effort. On one hand, being entitled to a lower share of the negative outcome (when λ increases) is a positive incentive for the entrepreneur; as λ increases, she extends a higher level of effort to decrease the probability of the negative outcome. On the other hand, the entrepreneur is less motivated to increase the probability of the positive outcome when she has to give up a higher share of it (when γ increases). Profit sharing can then be arranged by the choice of optimal γ (dependent on λ) in such a way that the positive and negative effects on the incentives of the entrepreneur to extend effort cancel each other out:

$$\frac{de}{d\lambda} + \frac{de}{d\gamma} \cdot \frac{d\gamma}{d\lambda} = \frac{1}{2} \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) - \frac{1}{2} \left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) \cdot \frac{d\gamma}{d\lambda} = 0 \quad \Leftrightarrow \quad \frac{d\gamma}{d\lambda} = \frac{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \quad (25)$$

The participation constraint (which follows from Assumption 8) of the PLS investor I with initial wealth $W_{0,I}$, who correctly anticipates the effort decision of the entrepreneur, determines the fixed payment:

$$E\{\tilde{W}_{1,I}\} = E\left\{ \left[\begin{array}{l} [\lambda(1+\tilde{r}) + \beta] A_0 \\ + (1+r_f)(W_{0,I} - C_I) \end{array} \right] \cdot \mathcal{X}_{\{\tilde{r}<0\}} + E\left\{ \left[\begin{array}{l} [\gamma(1+\tilde{r}) + \beta] A_0 \\ + (1+r_f)(W_{0,I} - C_I) \end{array} \right] \cdot \mathcal{X}_{\{\tilde{r}>0\}} \right\} \right\} \quad (26)$$

$$= (1+r_f)W_{0,I}$$

$$\Leftrightarrow \left[\frac{1}{2}(1-e)\lambda \left\{ 1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} + \frac{1}{2}(1+e)\gamma \left\{ 1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} \right] A_0 + \beta A_0 + (1+r_f)(W_{0,I} - \lambda A_0) \quad (27)$$

$$= (1+r_f)W_{0,I}$$

$$\Leftrightarrow B = \beta A_0 = \left[(1+r_f)\lambda - \frac{1}{2}(1-e)\lambda \left\{ 1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} - \frac{1}{2}(1+e)\gamma \left\{ 1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} \right] A_0. \quad (28)$$

Considering $C_I = \lambda A_0$, with the effort from (23) and the fixed payment from (28), the entrepreneur's expected future wealth is

$$E\{\tilde{W}_{1,E}^{PLS}\} = \left\{ 1 + \frac{\sigma^2}{\pi} - \frac{1}{8} \left[(\lambda + \gamma) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} - (\lambda - \gamma) \right]^2 \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}), \quad (29)$$

to be maximized with the optimal share of the positive outcome the entrepreneur is to give up:

$$\frac{dE\{\tilde{W}_{1,E}^{PLS}\}}{d\gamma} = 0 \Leftrightarrow \gamma = \frac{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \lambda. \quad (30)$$

With the optimal γ from (30), which fulfills (25), the entrepreneur's effort from (23) becomes

$$e = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2} \lambda \left\{ 1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} - \frac{1}{2} \frac{1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}}}{1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \lambda \left\{ 1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \quad (31)$$

This corresponds to the first-best level of effort (see (9)).

Lastly, the fixed payment $B = \beta A_0$ can be determined from (28) with the optimal effort from (23) (or (31)) and the optimal γ from (30):

$$\beta = \lambda \left(r_f + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) \Leftrightarrow B = \lambda \left(r_f + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) A_0. \quad (32)$$

The positive term $\frac{1}{8} \left[(\lambda + \gamma) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} - (\lambda - \gamma) \right]^2$ in (29) represents the entrepreneur's disadvantage of sharing the outcome of her business with the investor. With the optimal γ from (30), this disadvantage is zero. Therefore, the expected future wealth of the entrepreneur is also at the same level as with debt financing:

$$E\{\tilde{W}_{1,E}^{PLS}\} = \left\{ 1 + \frac{\sigma^2}{\pi} \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}), \quad (33)$$

which means, she will start her business via nonlinear PLS transaction if $\frac{\sigma^2}{\pi} > r_f$ (see (11) and (12)).

Financing via a nonlinear PLS transaction also results in the first-best level of effort and the same level of expected future wealth for the risk-neutral entrepreneur as risk-free debt financing. Risk sharing with the PLS investor does not have any benefits for the risk-neutral entrepreneur. However, her incentives to extend effort are also not diluted; with the first-best effort and the optimal share of the positive outcome the entrepreneur is willing to give up, there is no disadvantage from sharing the outcome with an outside investor. Therefore, when risk-free debt is not readily available for the entrepreneur, she can finance her project via nonlinear PLS transaction and achieve the same expected rate of return.

5. Conclusion

The principal-agent relationship between a risk-neutral entrepreneur (the agent) and a risk-neutral financier (the principal) is analyzed for different financial contracts. The type of financial contract used has an impact on managerial incentives. Three alternatives for an entrepreneur seeking capital to finance her project have been explored: i) risk-free debt financing; ii) equity financing with capital provided by a partner; iii) nonlinear profit-and-loss-sharing contracting with an investor. The effects of sharing the outcome of the project with the equity partner and the nonlinear PLS investor on the optimal effort and expected future wealth of the risk-neutral entrepreneur have been analyzed. It is assumed that the entrepreneur can influence the outcome of the project with her effort, which also results in private costs for her. She can, with her effort, increase the probability of a profit, subsequently decreasing the probability of a loss. The nonlinear PLS contracting considered in this paper involves sharing of profits (an increase in the value of the project) and losses (a decrease in the value of the project) based on different ratios between the entrepreneur and the investor and a fixed payment either the entrepreneur or the investor is to receive. In that case, the negative outcome is shared between the parties based on their initial capital contributions to the project. The entrepreneur and investor also agree on their shares of the positive outcome and the fixed payment.

Financing with (risk-free) debt capital does not involve any profit and loss or risk sharing. In that case, the optimal effort of the entrepreneur is first-best since she does not have to share the outcome with an outside financier. However, risk-free debt financing is usually not possible. With equity or nonlinear PLS financing, the entrepreneur shares both the outcome and the risk with the outside financier. For the risk-neutral entrepreneur, risk sharing does not entail an advantage. With equity financing, the incentives of the entrepreneur to extend effort are diminished, resulting in a lower level of effort. Therefore, financing with equity capital is

inferior compared to risk-free debt financing and leads to a lower expected future wealth of the entrepreneur. The nonlinear PLS transaction, on the other hand, can be arranged in such a way that the incentives of the entrepreneur to extend effort are not diluted. In that case, the entrepreneur's effort is also first-best, and the nonlinear PLS contracting is an alternative to risk-free debt financing, that is, it results in the same expected future wealth of the entrepreneur. Considering the entrepreneur might not have access to risk-free debt capital, the nonlinear PLS arrangement is then optimal for the risk-neutral entrepreneur.

The analysis in this paper has demonstrated the advantages of a nonlinear profit-and-loss-sharing contract. Generally speaking, outcome dependent payments to the agent in a principal-agent relationship can be structured to mitigate the dilution of the managerial incentives of the agent to extend effort. An extension would combine the ramifications of risk sharing and diluted managerial incentives for a risk-averse entrepreneur. Financing with risk-free debt capital would again result in the first-best effort of the entrepreneur. For the risk-averse entrepreneur, however, this would not indicate that debt financing is optimal since the benefits of risk sharing with an outside financier have to be also considered. When financing with equity or via a nonlinear PLS arrangement, the optimal contract structure has to trade off the benefits of risk sharing and diminished managerial incentives of the entrepreneur to extend effort. Financing with equity will dilute the incentives of the entrepreneur and result in second-best effort. It can, *a priori*, be claimed that the risk-averse entrepreneur's effort in case of nonlinear PLS contracting will be lower than the first-best effort, but higher than the second-best effort since it is a hybrid financial instrument combining elements of both debt and equity financing. Which financing alternative is more advantageous for the risk-averse entrepreneur will then depend on the trade-off between the benefits of risk sharing and the adverse effects of the moral hazard problem resulting from financing with equity or via the nonlinear PLS contract.

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Appendix A. Difference between a *musharaka* contract and the nonlinear PLS

The nonlinear PLS arrangement considered in this paper was inspired by profit-and-loss-sharing *musharaka* contracts in Islamic finance; however, there are differences between them. Based on a *musharaka* contract, the net losses and profits are shared based on (possibly) different shares. According to the nonlinear PLS contract considered here, on the other hand, the different profit and loss shares apply to the future value of the project. In essence, gross profits and losses are shared based on different ratios. This arrangement also entails a fixed payment to one of the parties to the contract, which does not apply to *musharaka* financing. These differences are illustrated below with a simple numerical example.

Suppose the entrepreneur has an investment opportunity that requires an initial investment amount of 100 in $t = 0$. The future value of the project can be either 80 (20% decrease in value, i.e., loss) or 120 (20% increase in value, i.e., profit) in $t = 1$. If the entrepreneur finances the project with an investor through a *musharaka* contract, where the investor provides 60% of the capital, the entrepreneur will be liable for only 40% of the losses. Assuming the parties agree to share the net profits fifty-fifty, the cash flows and the returns for the investor and entrepreneur are calculated as follows:

Table 1. Example: Cash flows and returns resulting from a *musharaka* contract

Investment		Investor			Entrepreneur		
$t = 0$	$t = 1$	$t = 0$	$t = 1$		$t = 0$	$t = 1$	
-100	80	-60	$60 - 0.60 \cdot 20 = 48$	-20.00%	-40	$40 - 0.40 \cdot 20 = 32$	-20.00%
	120		$60 + 0.50 \cdot 20 = 70$	16.67%		$40 + 0.50 \cdot 20 = 50$	25.00%

The cash flow to the investor in case the value of the project in $t = 1$ turns out to be 80 is his initial capital contribution less 60% of the decrease in value. His rate of return is then calculated as $\frac{60 - 0.60 \cdot 20}{60} - 1 = \frac{48}{60} - 1 = -0.20$. The entrepreneur, on the other hand, is liable for 40% of the decrease in the value of the project. Consequently, her rate of return will be $\frac{40 - 0.40 \cdot 20}{40} - 1 = \frac{32}{40} - 1 = -0.20$, which is the same rate of return as the investor's since the losses are shared in proportion to initial capital contributions. When the value of the project increases and turns out to be 120 in $t = 1$, the investor is entitled to 50% of this increase based

on the *musharaka* contract. Therefore, his rate of return on the investment will be $\frac{60+0.50 \cdot 20}{60} - 1 = \frac{70}{60} - 1 \approx 0.17$. The entrepreneur is also entitled to 50% of the 20 increase

in the value of the project, resulting in a rate of return higher than the investor's:

$$\frac{40+0.50 \cdot 20}{40} - 1 = \frac{50}{40} - 1 = 0.25.$$

Based on the nonlinear PLS arrangement considered in this paper, the future value of the project instead of net profits (or losses) from it are shared between the parties to the contract, which also includes a fixed payment. If the investor provides 60% of the capital, he is liable for 60% of the losses. Assuming that the investor agrees to the 50% share of the value of the project in case the value increases, and that he is also promised a fixed payment amounting to 5% of the initial investment amount, the cash flows and returns for the investor and the entrepreneur can be calculated as follows:

Table 2. Example: Cash flows and returns resulting from the nonlinear PLS contract

Investment		Investor		Entrepreneur			
$t = 0$	$t = 1$	$t = 0$	$t = 1$	$t = 0$	$t = 1$		
-100	80	-60	$0.60 \cdot 80 + 5 = 53$	-11.67%	-40	$0.40 \cdot 80 - 5 = 27$	-32.50%
	120		$0.50 \cdot 120 + 5 = 65$	8.33%		$0.50 \cdot 120 - 5 = 55$	37.50%

As agreed upon in the nonlinear PLS contract, in case the value of the project decreases, the investor is entitled to 60% of the value of the project in $t = 1$ and a fixed payment that corresponds to 5% of the initial investment amount 100. His rate of return from the investment

is then $\frac{0.60 \cdot 80 + 0.05 \cdot 100}{60} - 1 = \frac{53}{60} - 1 \approx -0.12$. The entrepreneur, on the other hand, is entitled

to 40% of the value of the project and has to provide the agreed upon fixed payment to the

investor, resulting in a rate of return of $\frac{0.40 \cdot 80 - 0.05 \cdot 100}{40} - 1 = \frac{27}{40} - 1 \approx -0.33$. If the project

is successful and is valued at 120 in $t = 1$, only 50% of it belongs to the investor who also receives the fixed payment (5% of the initial investment amount 100). The rate of return is

then $\frac{0.50 \cdot 120 + 0.05 \cdot 100}{60} - 1 = \frac{65}{60} - 1 \approx 0.08$ for the investor. The entrepreneur's share of the

value of the project in this case is also 50% , and she also has to pay the fixed amount to the investor. Her rate of return can then be calculated as $\frac{0.50 \cdot 120 - 0.05 \cdot 100}{40} - 1 = \frac{55}{40} - 1 \approx 0.38$.

When the entrepreneur finances the project by entering into a *musharaka* contract with an investor, the rate of return for both parties is the same as the rate of return on the project in case the value of the project decreases since they share net losses in accordance with their initial capital contributions. The sharing rule for the net profits can, however, be arranged so that the entrepreneur receives a higher return than the investor in case the value of the project increases since she is the one managing the project. Here, the entrepreneur gets rewarded when the outcome is good, but does not get punished for a bad outcome. In a nonlinear PLS arrangement considered in this paper, however, the rules for sharing profits and losses are slightly different than in a *musharaka* contract, and there is also a fixed payment component. The simple numerical example presented here shows that the rate of return for the investor and the entrepreneur are higher and lower (compared to *musharaka*), respectively, if the value of the project decreases. In essence, the entrepreneur is punished for the bad outcome. On the other hand, the investor and the entrepreneur receive lower and higher returns (compared to *musharaka*), respectively, when the project value increases; the entrepreneur is, in a way, rewarded for the good outcome.

The nonlinear profit-and-loss-sharing arrangement analyzed in this paper can be constructed using a combination of a slightly modified Islamic profit-and-loss-sharing *musharaka* (or *mudaraba*) contract, where the gross profits instead of net profits are shared between the parties based on a pre-agreed ratio, and a *murabaha* transaction, which would be used to turn the fixed payment component into a markup sale to be paid off in installments.⁶

The fixed payment (see (32)) from the nonlinear PLS transaction is positive, that is, the investor is to receive this payment. The total payment to the investor, including his share of the outcome of the project, can be constructed as a *murabaha* transaction, where the entrepreneur essentially buys out the investor at the end of the project at a markup. Then, instead of receiving his share of the outcome and the fixed payment immediately at the end of the project, the investor would agree to sell his share to the entrepreneur at a markup to be paid in installments over a period of time, which constitutes a *murabaha* sale. The only caveat is that the installment

⁶ A *murabaha* contract is a markup sale of assets. In such a transaction, the lender purchases an asset (or already owns it) and sells it to the borrower at a higher price to be paid in installments in the future (Jobst, 2007).

payment in a *murabaha* transaction has to be determined from the beginning, which is not possible since the outcome of the project is uncertain. However, the parties can beforehand agree on a markup payment that will have to be paid as part of every installment after the project is concluded. The present value of the expected installment payments has to then equal the initial capital contribution of the investor.

The expected value of the investor's share at the end of the project is

$$\begin{aligned} & E\{\lambda(1+\tilde{r})A_0\} \cdot \chi_{\{\tilde{r}<0\}} + E\{\gamma(1+\tilde{r})A_0\} \cdot \chi_{\{\tilde{r}>0\}} \\ &= \left[\frac{1}{2}(1-e)\lambda \left\{ 1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} + \frac{1}{2}(1+e)\gamma \left\{ 1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right\} \right] A_0 = \lambda \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) A_0 \end{aligned} \quad (34)$$

with effort from (23) (or (31)) and the profit share γ from (30). Instead of receiving the fixed payment B from (32) immediately at the end of the project, the investor would agree to sell his share to the entrepreneur at a markup $M = mA_0$ to be paid in n installments after the project

is concluded. Then, an expected installment payment would be $\frac{\lambda \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) + m}{n} A_0$, which can be considered an annuity. The present value of this annuity should equal the initial capital contribution of the investor:

$$C_I = \lambda A_0 = \frac{1}{1+r_f} \cdot \frac{(1+r_f)^n - 1}{(1+r_f)^n r_f} \cdot \frac{\lambda \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) + m}{n} A_0. \quad (35)$$

The markup from the *murabaha* transaction can then be determined as

$$M = mA_0 = \lambda \left[\frac{n(1+r_f)^{n+1} r_f}{(1+r_f)^n - 1} - \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) \right] A_0. \quad (36)$$

Appendix B. Effort-dependent two-piece normal distribution of the rate of return

The uncertain rate of return on the investment is defined by the probability density function from (4). The probabilities of a negative and a positive return (i.e., the probabilities of losses and profits) are then

$$\Pr(\tilde{r} < 0) = \int_{-\infty}^0 f(\tilde{r}) d\tilde{r} = \frac{1}{2}(1-e), \quad (37)$$

and

$$\Pr(\tilde{r} > 0) = \int_0^{+\infty} f(\tilde{r}) d\tilde{r} = \frac{1}{2}(1+e), \quad (38)$$

respectively.

The expected negative return, positive return, and total return are derived using the probability density function (4):

$$E(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} < 0\}} = \Pr(\tilde{r} < 0) \cdot E(\tilde{r} | \tilde{r} < 0) = \int_{-\infty}^0 \tilde{r} f(\tilde{r}) d\tilde{r} = -\frac{1}{2}(1-e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}}, \quad (39)$$

$$E(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} > 0\}} = \Pr(\tilde{r} > 0) \cdot E(\tilde{r} | \tilde{r} > 0) = \int_0^{+\infty} \tilde{r} f(\tilde{r}) d\tilde{r} = \frac{1}{2}(1+e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}}, \quad (40)$$

and

$$E(\tilde{r}) = E(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} < 0\}} + E(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} > 0\}} = -\frac{1}{2}(1-e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}(1+e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} = e \frac{\sigma\sqrt{2}}{\sqrt{\pi}}. \quad (41)$$

The corresponding risks, that is, the lower semi-variance, upper semi-variance, and total variance are

$$\begin{aligned} \text{Var}(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} < 0\}} &= E(\tilde{r}^2) \cdot \mathcal{X}_{\{\tilde{r} < 0\}} - \left[E(\tilde{r}) \cdot \mathcal{X}_{\{\tilde{r} < 0\}} \right]^2 \\ &= \Pr(\tilde{r} < 0) \cdot E(\tilde{r}^2 | \tilde{r} < 0) - \Pr(\tilde{r} < 0) \cdot \left[E(\tilde{r} | \tilde{r} < 0) \right]^2 \\ &= \int_{-\infty}^0 \tilde{r}^2 f(\tilde{r}) d\tilde{r} - \frac{\left[\int_{-\infty}^0 \tilde{r} f(\tilde{r}) d\tilde{r} \right]^2}{\Pr(\tilde{r} < 0)} = \frac{1}{2}(1-e) \sigma^2 - \frac{\left[-\frac{1}{2}(1-e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2}{\frac{1}{2}(1-e)}, \quad (42) \\ &= \frac{1}{2}(1-e) \sigma^2 \left(1 - \frac{2}{\pi} \right) \end{aligned}$$

$$\begin{aligned}
\text{Var}(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}} &= E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}>0\}} - [E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}}]^2 \\
&= \Pr(\tilde{r} > 0) \cdot E(\tilde{r}^2 | \tilde{r} > 0) - \Pr(\tilde{r} > 0) \cdot [E(\tilde{r} | \tilde{r} > 0)]^2 \\
&= \int_0^{+\infty} \tilde{r}^2 f(\tilde{r}) d\tilde{r} - \frac{\left[\int_0^{+\infty} \tilde{r} f(\tilde{r}) d\tilde{r} \right]^2}{\Pr(\tilde{r} > 0)} = \frac{1}{2}(1+e)\sigma^2 - \frac{\left[\frac{1}{2}(1+e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2}{\frac{1}{2}(1+e)}, \quad (43) \\
&= \frac{1}{2}(1+e)\sigma^2 \left(1 - \frac{2}{\pi} \right)
\end{aligned}$$

and

$$\begin{aligned}
\text{Var}(\tilde{r}) &= E(\tilde{r}^2) - [E(\tilde{r})]^2 \\
&= \left\{ E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}<0\}} + E(\tilde{r}^2) \cdot \chi_{\{\tilde{r}>0\}} \right\} - \left[E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}} + E(\tilde{r}) \cdot \chi_{\{\tilde{r}>0\}} \right]^2 \\
&= \left\{ \frac{1}{2}(1-e)\sigma^2 + \frac{1}{2}(1+e)\sigma^2 \right\} - \left[-\frac{1}{2}(1-e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} + \frac{1}{2}(1+e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2. \quad (44) \\
&= \sigma^2 - \left[e \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]^2 = \sigma^2 \left(1 - \frac{2e^2}{\pi} \right)
\end{aligned}$$

The entrepreneur's optimal effort increases in standard deviation (see (9) as reference) for two main reasons:

- I. As standard deviation σ increases, the expected negative return decreases while the expected positive return and expected total return increase for a given level of effort $e \in (0;1)$. An increase in effort can then limit the decrease in the expected negative return while boosting the increase in the expected positive return and expected total return resulting from the increase in standard deviation since an increase in effort leads to an increase in expected negative return, expected positive return, and expected total return:

$$\begin{aligned}
\frac{d^2 E(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}}}{d\sigma de} &= \frac{d \left\{ \frac{dE(\tilde{r}) \cdot \chi_{\{\tilde{r}<0\}}}{d\sigma} \right\}}{de} = \frac{d \left\{ \frac{d \left[-\frac{1}{2}(1-e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]}{d\sigma} \right\}}{de}, \quad (45) \\
&= \frac{d \left\{ -\frac{1-e}{\sqrt{2\pi}} \right\}}{de} = \frac{1}{\sqrt{2\pi}} > 0
\end{aligned}$$

$$\begin{aligned}
\frac{d^2 E(\tilde{r}) \cdot \chi_{\{\tilde{r} > 0\}}}{d\sigma de} &= \frac{d \left\{ \frac{dE(\tilde{r}) \cdot \chi_{\{\tilde{r} > 0\}}}{d\sigma} \right\}}{de} = \frac{d \left\{ \frac{d \left[\frac{1}{2}(1+e) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]}{d\sigma} \right\}}{de}, \\
&= \frac{d \left\{ \frac{1+e}{\sqrt{2\pi}} \right\}}{de} = \frac{1}{\sqrt{2\pi}} > 0
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
\frac{d^2 E(\tilde{r})}{d\sigma de} &= \frac{d \left\{ \frac{dE(\tilde{r})}{d\sigma} \right\}}{de} = \frac{d \left\{ \frac{d \left[e \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]}{d\sigma} \right\}}{de}. \\
&= \frac{d \left\{ e \frac{\sqrt{2}}{\sqrt{\pi}} \right\}}{de} = \frac{2}{\sqrt{2\pi}} > 0
\end{aligned} \tag{47}$$

- II. As standard deviation σ increases, the lower semi-variance, the upper semi-variance, and total variance increase for a given level of effort $e \in (0;1)$. An increase in effort can then limit the increase in the lower semi-variance and total variance while boosting the increase in upper semi-variance resulting from the increase in standard deviation since an increase in effort leads to an increase in the upper semi-variance and a decrease in the lower semi-variance and total variance:

$$\begin{aligned}
\frac{d^2 \text{Var}(\tilde{r}) \cdot \chi_{\{\tilde{r} < 0\}}}{d\sigma de} &= \frac{d \left\{ \frac{d\text{Var}(\tilde{r}) \cdot \chi_{\{\tilde{r} < 0\}}}{d\sigma} \right\}}{de} \\
&= \frac{d \left\{ \frac{d \left[\frac{1}{2}(1-e) \sigma^2 \left(1 - \frac{2}{\pi} \right) \right]}{d\sigma} \right\}}{de}, \\
&= \frac{d \left\{ (1-e) \sigma \left(1 - \frac{2}{\pi} \right) \right\}}{de} = -\sigma \left(1 - \frac{2}{\pi} \right) < 0
\end{aligned} \tag{48}$$

$$\begin{aligned}
\frac{d^2 \text{Var}(\tilde{r}) \cdot \chi_{\{\tilde{r} > 0\}}}{d\sigma de} &= \frac{d \left\{ \frac{d \text{Var}(\tilde{r}) \cdot \chi_{\{\tilde{r} > 0\}}}{d\sigma} \right\}}{de} \\
&= \frac{d \left\{ \frac{d \left[\frac{1}{2} (1+e) \sigma^2 \left(1 - \frac{2}{\pi} \right) \right]}{d\sigma} \right\}}{de}, \tag{49} \\
&= \frac{d \left\{ (1+e) \sigma \left(1 - \frac{2}{\pi} \right) \right\}}{de} = \sigma \left(1 - \frac{2}{\pi} \right) > 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^2 \text{Var}(\tilde{r})}{d\sigma de} &= \frac{d \left\{ \frac{d \text{Var}(\tilde{r})}{d\sigma} \right\}}{de} \\
&= \frac{d \left\{ \frac{d \left[\sigma^2 \left(1 - \frac{2e^2}{\pi} \right) \right]}{d\sigma} \right\}}{de} \tag{50} \\
&= \frac{d \left\{ 2\sigma \left(1 - \frac{2e^2}{\pi} \right) \right\}}{de} = -8e \frac{\sigma}{\pi} < 0
\end{aligned}$$

Here, $\chi_{\{\tilde{r} < 0\}} = \begin{cases} 1 & \text{if } \tilde{r} < 0 \\ 0 & \text{if } \tilde{r} > 0 \end{cases}$ and $\chi_{\{\tilde{r} > 0\}} = \begin{cases} 1 & \text{if } \tilde{r} > 0 \\ 0 & \text{if } \tilde{r} < 0 \end{cases}$ are indicator functions.

Nonlinear Profit and Loss Sharing versus Equity Financing in Entrepreneurial Finance: Risk Sharing and Managerial Incentives in a Principal-Agent Relationship When the Agent Is Risk-Averse

Nonlinear PLS versus Equity: Risk Sharing and Managerial Incentives – Part II

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Posted on SSRN: <https://ssrn.com/abstract=5815443>

Abstract

An entrepreneur shares business risk with an outside financier providing capital for her project. Risk sharing is, for a risk-averse entrepreneur, beneficial; however, it also results in diminished incentives of the entrepreneur to extend effort since she has to share the outcome of the project with the financier. The ramifications of this trade-off depend on the financial contracting between the entrepreneur and the financier. Besides debt and equity financing, nonlinear profit-and-loss-sharing contracts are considered in this paper. Risk-free debt financing involves no risk or profit sharing and is found to be inferior to equity financing and nonlinear PLS contracting despite resulting in the first-best level of effort of the entrepreneur. Even though the entrepreneur's incentives to extend effort are diluted in case of equity financing and nonlinear PLS, the benefits of sharing risk with an outside financier turn out to outweigh the adverse effects of diminished incentives. That is, financing with equity or via nonlinear PLS contracting results in a higher expected utility for the risk-averse entrepreneur than debt financing. It is also discovered that the nonlinear PLS contracting considered in this paper results in a higher level of effort of the entrepreneur than equity financing, alleviating the adverse effect of the moral hazard problem in the principal-agent relationship to some extent. The entrepreneur's expected utility depends on various factors such as the riskiness of the project, the entrepreneur's (and the financier's) level of risk aversion, the entrepreneur's ability to manage the project, and the structure of the financial contract between the entrepreneur and the financier. The numerical analyses suggest that the optimal nonlinear PLS contract leads to a higher expected utility for the entrepreneur compared to financing with equity.

Keywords: entrepreneurial finance, equity, nonlinear contracts, profit and loss sharing, risk sharing, managerial incentives, hidden effort, moral hazard problem

1. Introduction

An entrepreneur with insufficient funds to finance an investment opportunity (i.e., business, project, start-up, etc.) needs outside capital in the form of debt or equity. Hybrid financing combining the features of debt and equity capital could also be considered. Financing with equity or hybrid capital provided by a financier involves risk sharing, which is *per se* beneficial for the (risk-averse) entrepreneur, however, these methods of financing also result in diminished incentives for the entrepreneur to provide effort to facilitate the success of the business since she has to share the outcome with the outside financier. Such a principal-agent relationship, with the entrepreneur as the agent and the financier as the principal, results in a trade-off between the benefits of risk sharing and the diminished managerial incentives of the entrepreneur.

Hybrid financial instruments exhibit characteristics of both debt and equity capital. Preferred stock paying dividends at a fixed rate can be considered an example of hybrid capital. Participating preferred stock might also pay out participating dividends linked to the success of the company on top of the fixed dividends. Participating loans, on the other hand, entail payments dependent on the success of the company as opposed to fixed interest payments. The creditors of participation loans are entitled to a share of the sales revenue or profit; they do not, however, share the losses the company might incur. In case of financing with a participation loan, a fixed payment can also be agreed upon (OECD, 2015). Closer to equity financing are silent partnerships, two forms of which can be differentiated. A typical silent partner who invests in a project participates in profit sharing while an atypical silent partner shares both the profits and the losses with the entrepreneur. The atypical silent partner has then more rights as a co-entrepreneur than a typical silent partner for taking on a higher level of risk (OECD, 2015). The remuneration resulting from hybrid financial instruments usually involves linear sharing of profits. Some might also entail loss sharing or a fixed payment.

Hybrid financing is especially relevant in project financing and for small and medium-sized enterprises (SMEs). Examples of hybrid financial instruments can also be found in Islamic finance¹ in the form of near-equity methods of financing entailing nonlinear profit and loss sharing, where the profits and losses resulting from the project are shared between the

¹ Islamic financial contracts should adhere to the principles underlying Islamic banking and finance. As such, financial transactions involving activities such as charging interest, speculation, and short selling are prohibited (Iqbal & Mirakhor, 2011).

parties based on different shares. These so-called *musharaka* and *mudaraba* contracts result in arrangements in which the profits are shared based on a pre-agreed ratio while the liability for losses is in proportion to the individual capital contributions to the project (Hussain et al., 2015). In a *musharaka* transaction, both the entrepreneur and an investor provide capital to finance a project or an investment and share ownership based on their share of capital. Both parties are entitled to take part in the management process. The (net) profits generated from the project are shared between the entrepreneur and the investor based on a pre-agreed ratio while their liability for any (net) losses is in proportion to their initial capital contributions (Jobst, 2007). A *mudaraba* contract is similar to *musharaka*, but only the investor provides financing for the project, and only the entrepreneur can participate in management. The profits and losses are shared in the same way as with *musharaka* financing, and only the investor is liable for losses unless the manager is at fault (Jobst, 2007). Therefore, the main difference between a *musharaka* and a *mudaraba* transaction is in the level of financing provided by the investor.

In this paper, in addition to debt and equity financing, nonlinear profit and loss sharing with a fixed payment component, which can either be a wage for the entrepreneur or a compensation for the investor, is considered. The way such an arrangement is structured is inspired by the Islamic profit-and-loss-sharing contracts. Similarly to a *musharaka* or *mudaraba* contract, the nonlinear PLS arrangement explored in this paper involves sharing of the profits based on a pre-agreed ratio and of the losses in proportion to initial capital contributions. However, the profits and losses are defined differently. While Islamic PLS contracts involve sharing of net profits and losses resulting from a project, the nonlinear PLS contract considered here entails sharing of the future value of the project, in other words, gross profits and losses. Another difference to a *musharaka* or *mudaraba* contract is the fixed payment component that can be agreed upon between the parties.

The financial contract between the entrepreneur and the financier determines how the outcome, that is, profits and losses from the project are shared between the parties to the contract. Conventional equity financing results in a linear profit-and-loss-sharing contract in which the share of the profits the financier is entitled to is same as the share of the losses he is liable for. The profits and losses in this context refer to the increase and decrease in the market value of the project. Risk sharing resulting from the type of nonlinear profit and loss sharing considered in this paper will be different compared to equity financing. Furthermore, it can be *a priori* claimed that the incentives of the entrepreneur to provide effort will differ as well. The benefits of risk sharing and managerial incentives determine the entrepreneur's expected utility

in the agency-theoretic model employed in this paper. Three alternative forms of financing are compared: (1) (risk-free) debt financing, (2) equity financing, and (3) nonlinear profit-and-loss-sharing (PLS) contract with a fixed payment component. In essence, this is a form of hybrid financing that involves outcome-dependent profit and loss sharing and a fixed payment.

In the agency-theoretic framework applied in this paper, it is shown that (risk-free) debt financing results in the first-best effort of the entrepreneur, however, financing with equity and nonlinear profit and loss sharing turn out to be superior, that is, more advantageous for the entrepreneur since she foregoes the benefits of risk sharing when financing with debt. Moreover, the nonlinear profit and loss sharing results in a higher optimal effort than in case of equity financing. The optimal nonlinear PLS contract leads to a higher risk-adjusted return (which depends on various factors such as the level of risk, how risk-averse the entrepreneur is, her ability to manage the business, and how the profits and losses are shared), and consequently, a higher expected utility for the entrepreneur compared to equity financing.

The rest of this paper is structured as follows. A review of selected literature on principal-agent relationships is provided in section 2 to touch upon the issues regarding risk sharing, diminished managerial incentives, and the use of monitoring in such relationships. Section 3 outlines the theoretical model, where the underlying assumptions are detailed and the alternative contracts under consideration are specified. Section 4 compares debt, equity, and nonlinear PLS financing assuming the entrepreneur is risk-averse while the outside financiers are risk-neutral. Section 5 presents numerical analyses, which include cases where the outside financiers are also risk-averse. Section 6 offers concluding remarks.

2. Moral Hazard Problem in Principal-Agent Relationships

Both financing with equity and nonlinear profit-and-loss-sharing arrangement as considered in this paper result in a partnership that involves risk sharing. Such a partnership is a principal-agent relationship between the entrepreneur (the agent) attempting to start a (small or medium-sized) business and the outside financier (the principal) who provides capital. A moral hazard problem arises when the agent can influence the outcome with her actions (i.e., effort) that cannot be observed by the principal. In this case, the Pareto-optimal level of risk sharing cannot be achieved since it would dilute the managerial incentives of the entrepreneur to exert a sufficient level of effort. On one hand, risk sharing is beneficial for the (risk-averse) entrepreneur; however, she would also have to share the outcome with the principal. Therefore, the informational asymmetries in principal-agent relationships result in a trade-off between the

benefits of risk sharing and the diminished managerial incentives. This trade-off was first explored by Zeckhauser (1970) in the context of medical insurance. In this setting, optimal risk sharing induces excessive spending (e.g., with unnecessary visits to the emergency room) by insured individuals since they do not have to pay the full amount of their medical expenses. Therefore, the optimal insurance plan has to trade off risk sharing and incentives. In classical agency-theoretic literature, this trade-off in principal-agent relationships has been extensively inspected.

Ross (1973) explores the principal's problem, which is to find an optimal fee schedule to maximize his von Neumann-Morgenstern utility while the agent chooses an action to maximize her utility. He finds that the solution to the principal's problem cannot be Pareto-optimal, but also claims that Pareto-efficiency can be achieved if the principal was aware of the action the agent would choose based on the fee schedule; then, he could just require the agent to act a certain way. However, this would also require monitoring of her actions. Holmström (1979) similarly suggests that the principal can force the agent to act a certain way if complete monitoring is possible. Even if the complete observation of the agent's effort is either impossible or too costly, any amount of information about her actions can alleviate the moral hazard problem. According to Shavell (1979), this information would only be valuable if the agent is risk-averse because a risk-neutral agent will absorb the risk and choose a level of effort to maximize the expected outcome since her payment will depend only on this outcome. If the agent is risk-averse, her fee will depend also on the principal's knowledge about her action. Holmström (1979) also considers that the principal has to bear the costs of monitoring; the value of the information about the agent's action depends on how costly it is to obtain it. He suggests that the outcome must be sufficiently bad in order for the benefits of monitoring to outweigh the costs. Spence and Zeckhauser (1971) explore monitoring costs in case of medical insurance. The cost of information (i.e., monitoring) is also considered by Liu (1982) and Jost (1991).

Harris and Raviv (1979) conclude that it is optimal to combine imperfect monitoring with a nonlinear payment scheme, that is, if the principal decides to (imperfectly) monitor the agent's action, the optimal payment scheme to the agent depends on what the principal discovers regarding the agent's action. If the agent's action is discovered to be acceptable, she is paid based on a pre-determined outcome-dependent schedule. If not, she receives a lower fixed payment. Stiglitz (1974) also considers contracts between a landlord (principal) and a worker (agent) consisting of both a basic wage (i.e., a fixed payment) and an incentive pay in

a simple agricultural economy. In his model, the incentive payment is linear in output. By assuming that the agent and principal are both risk-neutral and that the agent can influence the probability of “good” and “bad” outcomes, Breuer (1995) shows that nonlinear incentive contracts, where the agent receives an outcome-dependent and a fixed payment when the outcome is above a certain threshold and no payment at all otherwise, are optimal. The model employed by Breuer (1995) is similar to the one in this paper, where the agent has an influence on not only the expected outcome, but also the risk involved since the agent’s actions affect not the expected outcome directly, but the probability of positive and negative outcomes.

This approach is different from the LEN (Linear-Exponential-Normal) model of Spremann (1988), where the agent can only affect the expected outcome by shifting its probability distribution to the right. Spremann (1988) explores a principal-agent relationship resulting from such a profit-sharing arrangement that even though the principal can observe only the outcome (but not the agent’s effort), he can choose an incentive scheme that induces the agent to (selfishly) decide for a level of effort acceptable for the principal. A similar approach is used for the analyses in this paper. In such a setting, similar to the one explored by Grossman and Hart (1983), the contract between the entrepreneur and the financier determines how the risk is allocated between them and to what extent the entrepreneur’s incentives are diminished as a result of having to share the outcome with an outside financier.

Nonlinear profit and loss sharing is considered in addition to debt and equity financing in this paper. In an agency-theoretic framework, nonlinear incentive schemes are hardly explored. Even in cases considering nonlinear incentive payments, the nonlinearity stems only from the fixed payment up to or above a certain level of outcome. In the setting explored in this paper, however, the nonlinearity refers to the profits and losses being shared based on different ratios. The motivation for considering such an intricate arrangement stems from how Islamic PLS contracts are structured. The nonlinear profit-and-loss-sharing contracting is compared to conventional methods of debt and equity financing with respect to the trade-off between the benefits of risk sharing and diminished managerial incentives to extend effort in order to find out if such an incentive scheme can alleviate the moral hazard problem in a principal-agent relationship.

3. The Model

A one-period model of an investment opportunity for the entrepreneur, which requires an initial capital in the amount of A_0 in $t = 0$, is considered. The future value of this project in

$t = 1$ is defined as $\tilde{A}_1 = (1 + \tilde{r})A_0$ with \tilde{r} as the uncertain rate of return on the investment. A profit on the investment is realized when the value of the project increases, that is, when the rate of return is positive. A negative rate of return results in a decrease in the value of the project and therefore, in a loss.

Assumption 1.

The rate of return on the investment is defined as

$$\tilde{r} = y|\tilde{x}|, \quad (1)$$

where \tilde{x} is a normally distributed random variable, $\tilde{x} \sim N(0, \sigma^2)$, and $y = \pm 1$ is an indicator of a positive or negative rate of return.

Assumption 2.

The entrepreneur can increase the probability of a positive return (simultaneously decreasing the probability of a loss) with her effort, $e \in (0; 1)$. The two-point distribution of y is then defined by the probability mass function

$$f(y; e) = \frac{1}{2}(1 + y \cdot e) = \begin{cases} \frac{1}{2}(1 - e) & \text{if } y = -1 \\ \frac{1}{2}(1 + e) & \text{if } y = 1 \end{cases}. \quad (2)$$

The entrepreneur's effort is a decision variable and a measure of probability and results in private costs, $c(e)$, for the entrepreneur.

Since the future value of the project is defined as $\tilde{A}_1 = (1 + \tilde{r})A_0 = (1 + y|\tilde{x}|)A_0$, its probability distribution can be represented by the probability density function of \tilde{r} . The rate of return \tilde{r} equals $-|\tilde{x}|$ with $\frac{1}{2}(1 - e)$ probability and $|\tilde{x}|$ with $\frac{1}{2}(1 + e)$ probability. Both $-|\tilde{x}|$ and $|\tilde{x}|$ have the same probability density function.² Consequently, the probability density function of \tilde{r} can be expressed as

² $|\tilde{x}|$ and $-|\tilde{x}|$ exhibit a half-normal distribution, i.e., a folded normal distribution with 0 mean and the probability density function $f(|\tilde{x}|) = f(-|\tilde{x}|) = \frac{2}{\sigma\sqrt{2\pi}} \exp\left[-\frac{|\tilde{x}|^2}{2\sigma^2}\right]$ (Tsagris et al., 2014).

$$f(\tilde{r}) = \frac{1+e\frac{\tilde{r}}{|\tilde{r}|}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] = \begin{cases} \frac{1-e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } \tilde{r} < 0 \\ \frac{1+e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } \tilde{r} > 0 \end{cases} \quad (3)$$

Therefore, the entrepreneur can, with her effort, influence both the expected value and the standard deviation of the rate of return on the project.⁴

The probability density function of the rate of return \tilde{r} is illustrated in the figures below. Figure 1 represents the normal distribution of \tilde{r} in the absence of effort of the entrepreneur. The probability of both a positive and a negative rate of return is $\frac{1}{2}$ without effort. The entrepreneur's effort influences these probabilities. The probability of a negative return is reduced to $\frac{1}{2}(1-e)$ while the probability of a positive return is increased to $\frac{1}{2}(1+e)$ when the entrepreneur extends effort. Figure 2 shows how the entrepreneur's effort affects (i.e., changes) the distribution of the rate of return.

³ The probability distribution of the rate of return \tilde{r} resembles a two-piece normal distribution (see Wallis, 2014).

⁴ The rate of return on the investment, \tilde{r} , cannot be below -1 . A rate of return below -100% is, however, not precluded when \tilde{r} is assumed to be normally distributed. In order to remain realistic, the distribution can be truncated from below at -1 . It would be practical to truncate the distribution also from above at 1 ($-1 \leq \tilde{r} \leq 1$), so that the distribution remains symmetrical. The future value of the project is then represented by the probability density function

$$f(\tilde{r}) = \begin{cases} \frac{1+e\frac{\tilde{r}}{|\tilde{r}|}}{\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } -1 \leq \tilde{r} \leq 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } \tilde{r} < -1 \\ \frac{1-e}{\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } -1 \leq \tilde{r} < 0 \\ \frac{1+e}{\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } 0 < \tilde{r} \leq 1 \\ 0 & \text{if } \tilde{r} > 1 \end{cases},$$

where $\operatorname{erf}(s) = \frac{2}{\sqrt{\pi}} \int_0^s \exp(-t^2) dt$ is the (Gauss) error function (Andrews, 1998). It is an increasing, odd function, and its image is $(-1;1)$. For a normally distributed random variable with mean 0 and standard deviation σ , $\operatorname{erf}(s)$ is the probability that the random variable falls in the range $[-s\sigma\sqrt{2}; s\sigma\sqrt{2}]$. Using a truncated normal distribution results in complicated mathematical expressions for the expected values of profits and losses. Deriving the moment generating function, however, is mathematically more tractable (Norgaard & Killeen, 1980). Therefore, when the utility function is exponential, the expected utility can be determined (see Appendix C).

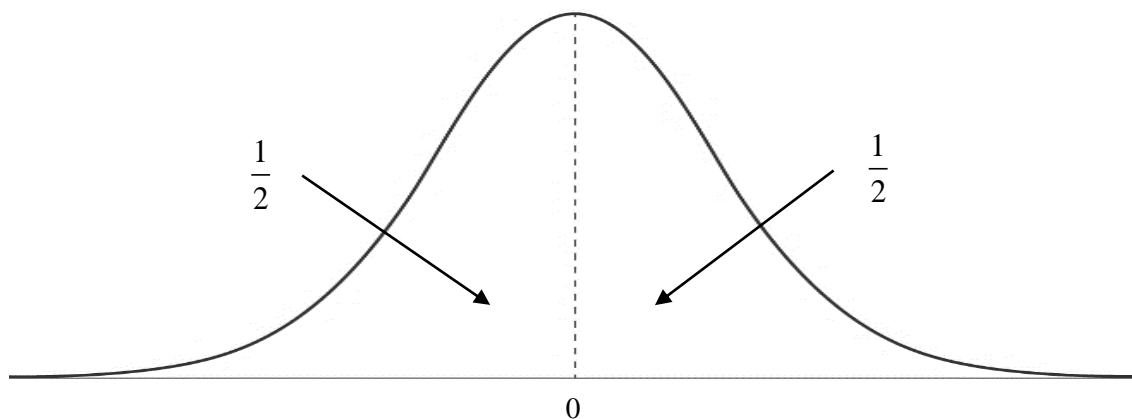


Figure 1. Probability density function: $f(\tilde{r}), e = 0$

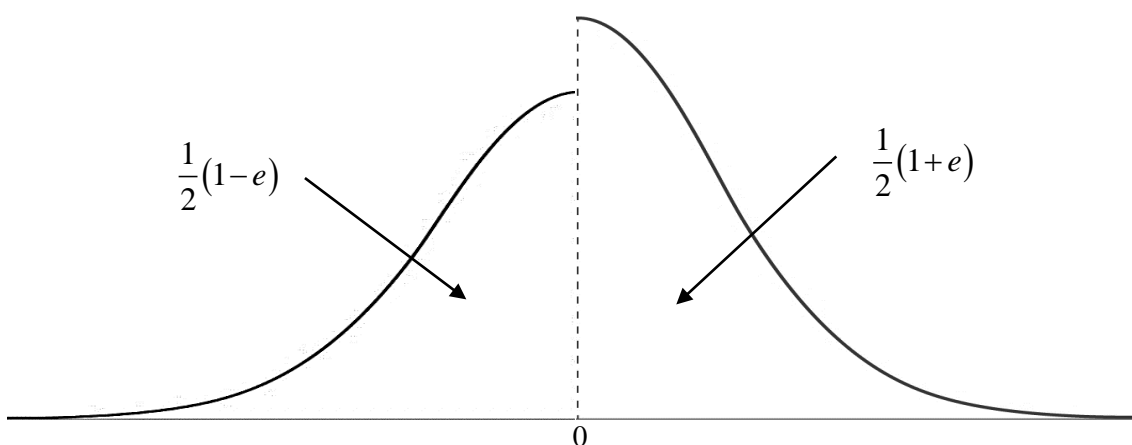


Figure 2. Probability density function: $f(\tilde{r}), e > 0$

Assumption 3.

The initial wealth $W_{0,E}$ of the entrepreneur E in $t = 0$ is not sufficient to cover the investment amount A_0 ; therefore, outside capital $C = A_0 - W_{0,E}$ is required for the entrepreneur to be able to start her business. She can either decide to invest her available capital in risk-free assets on the market instead of starting her business or raise the required capital as debt, equity, or with a nonlinear PLS contract.

Assumption 4.

The interest rate charged on any debt is same as the rate of return r_f on alternative (risk-free) investments. Debt financing is risk-free (free of default risk), and there are no market imperfections that would result in a difference between lending and borrowing rates.

If the entrepreneur decides for a risk-free investment, her future wealth in $t = 1$ is $W_{1,E}^{risk-free investment} = (1 + r_f)W_{0,E}$. Alternatively, she can choose to finance the required capital with debt and start her business. In this case, she will have to extend effort as the manager and

will have to pay back the principal amount and the interest on the loan. If she can finance her business with risk-free debt capital, her uncertain future wealth will be

$$\begin{aligned}\tilde{W}_{1,E}^{debt} &= \tilde{A}_1 - c(e) - (1+r_f)C \\ &= (1+\tilde{r})A_0 - c(e) - (1+r_f)(A_0 - W_{0,E})\end{aligned}\quad (4)$$

The entrepreneur can also finance her project with equity capital from a partner. The equity partner P provides capital in the amount of C_p and becomes a shareholder of the firm. This entitles him to a share α in the outcome of the project. The uncertain future wealth of the entrepreneur is

$$\begin{aligned}\tilde{W}_{1,E}^{equity} &= (1-\alpha)\tilde{A}_1 - c(e) - (1+r_f)(C - C_p) \\ &= (1-\alpha)(1+\tilde{r})A_0 - c(e) + (1+r_f)C_p - (1+r_f)(A_0 - W_{0,E})\end{aligned}\quad (5)$$

in case of equity participation from an outside partner. The amount of capital provided by the partner might be more or less than required. If $C - C_p < 0$, the entrepreneur can invest the difference at the risk-free rate r_f . On the other hand, if $C - C_p > 0$, the entrepreneur will have to raise additional debt capital with the interest rate r_f .

The third option for the entrepreneur is a nonlinear profit-and-loss-sharing (PLS) contract with an investor. The nonlinear PLS investor I provides capital in the amount of $C_I = \lambda A_0$. The investor will then be entitled to a share γ of the outcome in case of a positive return (i.e., profit). The entrepreneur and the investor will share losses in proportion to their initial capital contributions, which means the investor will be liable for a share λ of any losses. The entrepreneur and the investor will also agree on a fixed payment $B = \beta A_0$ that can be positive or negative. When $B > 0$, the investor additionally receives a fixed compensation from the entrepreneur. Otherwise, i.e., if $B < 0$, the investor agrees to pay the entrepreneur a fixed wage on top of her share of the outcome.

As it is the case with debt financing, the liability of the entrepreneur is not limited when she finances her business via a nonlinear PLS contract. The ramifications of limited liability are not considered in this paper.⁵

⁵ Innes (1990) explores the implications of limited liability (for both the risk-neutral principal and the risk-averse agent) for optimal effort and incentives when the agent chooses an effort before the state of nature is known.

Assumption 5.

In case the outcome of the entrepreneur's project is not sufficient for her to be able to fulfill her contractual obligations, she will (and can) cover the agreed upon payments (principal and interest in case of any debt financing and the fixed payment to the investor (if $\beta > 0$) in case of financing with a nonlinear PLS contract) with her own private funds. Furthermore, even if the actual return on the project is too low, the investor will consider it bad luck rather than a result of the entrepreneur shirking her duties and still pay her the agreed upon wage (if $\beta < 0$) with his own private funds (in case his share of the outcome is not sufficient to cover the fixed payment to the entrepreneur).

In case of nonlinear PLS arrangement with an investor, the uncertain future wealth of the entrepreneur is

$$\begin{aligned} \tilde{W}_{1,E}^{PLS} &= \begin{cases} (1-\lambda)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if } \tilde{r} < 0 \\ (1-\gamma)\tilde{A}_1 - B - c(e) - (1+r_f)(C-C_I) & \text{if } \tilde{r} > 0 \end{cases} \\ &= \begin{cases} [(1-\lambda)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if } \tilde{r} < 0 \\ [(1-\gamma)(1+\tilde{r}) - \beta]A_0 - c(e) + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) & \text{if } \tilde{r} > 0 \end{cases} \end{aligned} \quad (6)$$

if she finances her business via a nonlinear PLS contract. The amount of capital provided by the investor might be more or less than required. If $C - C_I < 0$, the entrepreneur can invest the difference at the risk-free rate r_f . On the other hand, if $C - C_I > 0$, the entrepreneur will have to raise additional debt capital with the interest rate r_f .

Assumption 6.

The entrepreneur is risk-averse, and her risk aversion is independent of her effort.⁶

The entrepreneur's utility of her uncertain future wealth $\tilde{W}_{1,E}$ is multiplicatively separable in income i , consisting of the entrepreneur's share of the outcome from the project and any proceeds from risk-free investments less any fixed payments the entrepreneur has to make, and in effort e :

$$U(\tilde{W}_{1,E}) = U(i, e) = u(i)d(e), \quad (7)$$

⁶ See Grossman and Hart (1983).

where $u(i)$ is a negative, increasing, and concave utility function of income, and $d(e)$ is a positive, increasing, and convex disutility function of effort.⁷

Assumption 7.

The entrepreneur's utility function of income is exponential:

$$u(i) = -\exp(-\theta i), \quad (8)$$

where $\theta = -\frac{u''(i)}{u'(i)}$ is the constant absolute risk aversion coefficient (Pratt, 1964).

Assumption 8.

The entrepreneur's disutility function of effort is radical:

$$d(e) = \sqrt[2\ell]{\frac{1}{1-e^2}}, \quad (9)$$

where $\ell = (1+a)^2$ is the entrepreneur's cost efficiency coefficient, with $a \in [0;1]$ as her level of managerial ability.⁸

Considering (8) and (9), the entrepreneur's utility of her uncertain future wealth is

$$U(\tilde{W}_{1,E}) = U(i, e) = -\frac{\exp(-\theta i)}{\sqrt[2\ell]{1-e^2}} = -\exp\left[-\theta\left\{i - \frac{1}{2\ell\theta} \ln\left(\frac{1}{1-e^2}\right)\right\}\right] = -\exp(-\theta\tilde{W}_{1,E}). \quad (10)$$

$$c(e) = \frac{1}{2\ell\theta} \ln\left(\frac{1}{1-e^2}\right) \quad (11)$$

can therefore be considered the cost of effort for the risk-averse entrepreneur since the uncertain future wealth of the entrepreneur is her income less cost of her effort. The cost of effort function from (11) represents the monetary equivalent of the entrepreneur's disutility of extending effort and appears as negative income in (10).

⁷ The entrepreneur's utility of income and disutility of effort functions were selected so that the first-order approach to her optimization problem is valid. For the necessary and sufficient conditions for the validity of FOA, see Rogerson (1985), Jewitt (1988), Mirrlees (1999), Kirkegaard (2017).

⁸ Martinez-Gorricho and Sanchez Villalba (2021) take into account that agents can be heterogeneous in ability. They argue, the more capable the agent, the higher the optimal level of her effort since her disutility of effort decreases as ability increases. One could also argue that there is a negative relation between an individual's level of risk aversion and their ability (Dohmen et al., 2007). If that is the case or to be assumed, ability can be modelled as a decreasing function of the risk aversion coefficient.

Assumption 9.

The risk-averse entrepreneur chooses a financial contract that maximizes the expected utility of her uncertain future wealth, subsequently deciding on the level of effort. The entrepreneur's effort is known only to her. The potential outside financiers can only observe the outcome of the project without knowing the extent of the contribution of the entrepreneur's effort to the outcome. The financiers are, however, not ignorant; they are aware of the relevant characteristics of the entrepreneur and the project she wishes to finance. Therefore, the rational financiers can anticipate the effort decision of the rational entrepreneur.

Assumption 10.

The potential outside financiers are risk-neutral, and a risk-neutral financier is willing to provide such an amount of capital that his investment in the entrepreneur's project offers the same expected rate of return as his outside option (i.e., risk-free investment).⁹

Hence, the entrepreneur's aim is, by deciding on the optimal financial contract and subsequently, on the optimal level of effort, to maximize the expected utility of her uncertain future wealth subject to the financier's participation constraint, which can be achieved by maximizing the certainty equivalent of her uncertain future income $CE(i)$ less her cost of effort $c(e)$ since by definition (Laux et al., 2014),

$$E[u(i)] = u[CE(i)] \quad (12)$$

$$\Leftrightarrow CE(i) = -\frac{1}{\theta} \ln[-E\{u(i)\}], \quad (13)$$

considering the exponential utility function of income (see (8)).

The certainty equivalent of the entrepreneur's uncertain future wealth can then be defined as $CE(\tilde{W}_{1,E}) = CE(i) - c(e)$, which depends on the financial contract between her and the outside financier, which directly affects her expected utility by allocating the outcome and the risk of the project. The financial contract also imposes incentive effects on the entrepreneur's effort decision, which is also a determinant of her expected utility.

⁹ Only the theoretical analyses assuming that the potential outside financiers are risk-neutral are presented. For the sake of simplicity, the cases with risk-averse financiers are illustrated only in the numerical analyses. The participation constraints of risk-averse financiers are derived in Appendix C.

4. Debt, Equity, and Nonlinear PLS

Three alternative financial contracts are considered: (i) risk-free debt financing, (ii) equity partnership with an outside partner P who provides capital C_p and is entitled to share α in the outcome of the entrepreneur's project, and (iii) nonlinear PLS contract with an investor I who provides capital $C_I = \lambda A_0$ and is entitled to share γ of the value of the project in case it increases while being liable for share λ of the losses. The nonlinear PLS contract also has a fixed component $B = \beta A_0$, meaning, either the entrepreneur or the investor receives a fixed payment (a wage or a compensation, respectively). In case the entrepreneur opts out of the project and decides to invest her available funds at the risk-free rate of return, the certainty equivalent of her future wealth is

$$CE_E^{risk-free investment} = (1 + r_f)W_{0,E}. \quad (14)$$

4.1. Debt financing

Risk-free debt financing is considered at first. In this case, the entrepreneur's income consists of the outcome of her project less the debt she has to pay back (including the principal amount and the interest):

$$\begin{aligned} i_E^{debt} &= \tilde{A}_1 - (1 + r_f)C \\ &= (1 + \tilde{r})A_0 - (1 + r_f)(A_0 - W_{0,E}). \end{aligned} \quad (15)$$

Her expected utility from this income is then

$$E\{u(i_E^{debt})\} = -\left[1 - e \cdot \operatorname{erf}\left(\frac{\theta A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left(\begin{array}{l} -\theta A_0 \\ +\frac{1}{2}\theta^2 A_0^2 \sigma^2 \\ +\theta(1 + r_f)(A_0 - W_{0,E}) \end{array}\right) \quad (16)$$

(see Appendix A).

Taking (13) into account and with the cost of effort function from (11), the certainty equivalent of the entrepreneur's uncertain future wealth in case of risk-free debt financing is

$$\begin{aligned}
CE_E^{debt} &= CE(i_E^{debt}) - c(e) \\
&= \left\{ 1 - \frac{1}{2} \theta A_0 \sigma^2 \right. \\
&\quad \left. + \frac{1}{\theta A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta A_0 \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell \theta A_0} \ln \left(\frac{1}{1 - e^2} \right) \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}) \quad , \quad (17) \\
&= \left\{ 1 - \frac{1}{2} \omega \sigma^2 + \frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell \omega} \ln \left(\frac{1}{1 - e^2} \right) \right\} A_0 - (1 + r_f)(A_0 - W_{0,E})
\end{aligned}$$

where $\omega = \theta A_0$ is the entrepreneur's modified risk aversion coefficient.

In (17), the second term, $\frac{1}{2} \omega \sigma^2$, is the risk premium of the entrepreneur. The third and

fourth terms, $\frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right]$ and $\frac{1}{2\ell \omega} \ln \left(\frac{1}{1 - e^2} \right)$, indicate the benefit and cost of

effort, respectively. The entrepreneur chooses her effort to maximize her certainty equivalent from (17):

$$\frac{dCE_E^{debt}}{de} = 0 \Leftrightarrow e = \frac{\sqrt{1 + 4\ell(\ell - 1) \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} - 1}{2(\ell - 1) \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}. \quad (18)$$

For simplicity, only the case when $a = 0 \Leftrightarrow \ell = 1$ is analytically considered in this section. In the numerical analyses, cases with $a = 1$ are also illustrated. When $a = 0 \Leftrightarrow \ell = 1$, the optimal effort of the entrepreneur is

$$e^* = \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right) \quad (19)$$

(see Appendix B), which is the first-best effort since the incentives of the entrepreneur to extend effort are not diluted.

Considering $a = 0 \Leftrightarrow \ell = 1$ and with the first-best effort from (19), the certainty equivalent of the entrepreneur in case of risk-free debt financing is

$$CE_E^{debt} = \left\{ 1 - \frac{1}{2} \omega \sigma^2 + \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}). \quad (20)$$

The entrepreneur would start her business with risk-free debt only if her certainty equivalent from (20) is higher than the one resulting from a risk-free investment (see (14)):

$$\left\{ 1 - \frac{1}{2} \omega \sigma^2 + \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}) > (1 + r_f) W_{0,E} \quad (21)$$

$$\Leftrightarrow \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2} \omega \sigma^2 > r_f. \quad (22)$$

This means that the difference between the net benefit from extending effort and her risk premium, expressed as a rate of return on the investment, should exceed the risk-free rate of return. This difference, i.e., the LHS of (22) can be considered the entrepreneur's risk-adjusted rate of return on the investment financed with risk-free debt capital:

$$r_{debt} = \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2} \omega \sigma^2. \quad (23)$$

Since the entrepreneur does not share the outcome of her business with an outside financier, the net benefit from her effort (the first term on the RHS of (23)) depends only on her level of risk aversion and the standard deviation. Similarly, the deduction for risk, i.e., the risk premium (the second term on the RHS of (23)) also depends only on the risk aversion coefficient and the standard deviation because there is no risk sharing involved.

4.2. Equity partnership

The participation constraint (which follows from Assumption 10) of the (risk-neutral) equity partner P with initial wealth $W_{0,P}$, who correctly anticipates the effort decision of the entrepreneur for a given α , determines the level of capital C_P he is willing to provide:

$$\begin{aligned} E\{\tilde{W}_{1,P}\} &= E\left\{\alpha(1+\tilde{r})A_0 + (1+r_f)(W_{0,P} - C_P)\right\} \\ &= (1+r_f)W_{0,P} \end{aligned} \quad (24)$$

$$\begin{aligned} \Leftrightarrow \alpha \left\{1 + e \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} A_0 + (1+r_f)(W_{0,P} - C_P) \\ = (1+r_f)W_{0,P} \end{aligned} \quad (25)$$

$$\Leftrightarrow C_P = \frac{1}{1+r_f} \alpha \left\{1 + e \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} A_0. \quad (26)$$

If the entrepreneur finances her project with equity, her income consists of her share of the outcome of the project and any debt she has to pay back (in case $C - C_P > 0$) or any proceeds from a risk-free investment (in case $C - C_P < 0$):

$$\begin{aligned} i_E^{equity} &= (1-\alpha)\tilde{A}_1 - (1+r_f)(C - C_P) \\ &= (1-\alpha)(1+\tilde{r})A_0 + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}) \end{aligned} \quad (27)$$

Her expected utility from this income is then

$$E\left\{u\left(i_E^{equity}\right)\right\} = -\left[1 - e \cdot \operatorname{erf}\left(\frac{\theta(1-\alpha)A_0\sigma}{\sqrt{2}}\right)\right] \exp\left(\begin{array}{l} -\theta(1-\alpha)A_0 \\ +\frac{1}{2}\theta^2(1-\alpha)^2A_0^2\sigma^2 \\ -\theta(1+r_f)C_P \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{array}\right) \quad (28)$$

(see Appendix A).

Taking (13) into account and with the cost of effort function from (11), the certainty equivalent of the entrepreneur's uncertain future wealth in case of equity financing is

$$\begin{aligned}
CE_E^{equity} &= CE(i_E^{equity}) - c(e) \\
&= \left\{ \begin{aligned} &\left((1-\alpha) - \frac{1}{2} \theta (1-\alpha)^2 A_0 \sigma^2 \right. \\ &\left. + \frac{1}{\theta A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta (1-\alpha) A_0 \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell \theta A_0} \ln \left(\frac{1}{1-e^2} \right) \right\} A_0 \\ &+ (1+r_f) C_P - (1+r_f)(A_0 - W_{0,E}) \end{aligned} \right. , \quad (29) \\
&= \left\{ \begin{aligned} &\left((1-\alpha) - \frac{1}{2} \omega (1-\alpha)^2 \sigma^2 + \frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega (1-\alpha) \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell \omega} \ln \left(\frac{1}{1-e^2} \right) \right\} A_0 \\ &+ (1+r_f) C_P - (1+r_f)(A_0 - W_{0,E}) \end{aligned} \right.
\end{aligned}$$

where $\omega = \theta A_0$ is the entrepreneur's modified risk aversion coefficient.

The entrepreneur decides for a level of effort that maximizes her certainty equivalent from (29) after the contract has been signed, meaning, with given α and C_P :

$$\frac{dCE_E^{equity}}{de} = 0 \Leftrightarrow e = \frac{\sqrt{1 + 4\ell(\ell-1) \operatorname{erf}^2 \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)} - 1}{2(\ell-1) \operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)}. \quad (30)$$

For simplicity, only the case when $a = 0 \Leftrightarrow \ell = 1$ is analytically considered in this section. In the numerical analyses, cases with $a = 1$ are also illustrated. When $a = 0 \Leftrightarrow \ell = 1$, the optimal effort of the entrepreneur is

$$e^* = \operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right) \quad (31)$$

(see Appendix B), which is lower than the first-best effort from (19) for all $\alpha \in (0; 1]$ since the entrepreneur has to share the benefits of her effort with the equity partner. Consequently, the second-best effort from (31), *ceteris paribus*, decreases as α increases.

Considering $a = 0 \Leftrightarrow \ell = 1$, the optimal effort from (31), and the outside capital provided (see (26)), the certainty equivalent of the entrepreneur in case of equity financing is

$$CE_E^{equity} = \left\{ \begin{array}{l} 1 - \frac{1}{2} \omega (1 - \alpha)^2 \sigma^2 + \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right)} \right] \\ + \alpha \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \operatorname{erf} \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right) \end{array} \right\} A_0 - (1 + r_f) (A_0 - W_{0,E}). \quad (32)$$

The entrepreneur's effort decision depends on α . She chooses her effort after deciding on the optimal share that maximizes her certainty equivalent from (32). Solving this optimization problem mathematically is not possible since α is a term of the error function. Instead, numerical analyses were conducted, whose results will be presented in section 5.

The entrepreneur will decide whether to finance her business with equity capital by comparing her certainty equivalent with her outside option (i.e., risk-free investment):

$$\left\{ \begin{array}{l} 1 - \frac{1}{2} \omega (1 - \alpha)^2 \sigma^2 + \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right)} \right] \\ + \alpha \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \operatorname{erf} \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right) \end{array} \right\} A_0 - (1 + r_f) (A_0 - W_{0,E}) > (1 + r_f) W_{0,E} \quad (33)$$

$$\Leftrightarrow \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right)} \right] + \alpha \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \operatorname{erf} \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right) - \frac{1}{2} \omega (1 - \alpha)^2 \sigma^2 > r_f. \quad (34)$$

This means that the difference between the net benefit from extending effort and her risk premium, expressed as a rate of return on the investment, should exceed the risk-free rate of return. This difference, i.e., the LHS of (34) can be considered the entrepreneur's risk-adjusted rate of return on the investment financed with equity capital:

$$r_{equity} = \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right)} \right] + \alpha \frac{\sigma \sqrt{2}}{\sqrt{\pi}} \operatorname{erf} \left(\frac{\omega (1 - \alpha) \sigma}{\sqrt{2}} \right) - \frac{1}{2} \omega (1 - \alpha)^2 \sigma^2. \quad (35)$$

Since with equity financing, the entrepreneur shares both the outcome of the project and the risk with the equity partner, both the benefit from her effort (the sum of the first two terms on the RHS of (35)) and the risk premium (the third term in the RHS of (35)) depend additionally on the sharing rule, i.e., the so far undetermined optimal α .

Lemma 1.

The disadvantage of profit and loss sharing with an equity partner (as a result of diminished managerial incentives of the entrepreneur), that is, the net benefit from effort, decreases in $\alpha \in (0;1]$, *ceteris paribus*:

$$NB_E^{equity} = \frac{1}{2\omega} \ln \left[\frac{1}{1 - \operatorname{erf}^2 \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)} \right] + \alpha \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right). \quad (36)$$

$$\begin{aligned} \frac{dNB_E^{equity}}{d\alpha} &= \left\{ 1 - \frac{\exp \left[-\frac{1}{2} \omega^2 (1-\alpha)^2 \sigma^2 \right]}{1 - \operatorname{erf}^2 \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)} \right\} \operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \\ &\quad - \exp \left[-\frac{1}{2} \omega^2 (1-\alpha)^2 \sigma^2 \right] \omega \alpha \left(\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right)^2 < 0 \end{aligned} \quad (37)$$

Proof: Using the approximation $\operatorname{erf}^2(\zeta) \approx 1 - \exp \left[-\frac{4\zeta^2}{\pi} \right]$ (Howard, 2022),

$$1 - \frac{\exp \left[-\frac{1}{2} \omega^2 (1-\alpha)^2 \sigma^2 \right]}{1 - \operatorname{erf}^2 \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)} = 1 - \exp \left[\left(\frac{4}{\pi} - 1 \right) \frac{\omega^2 (1-\alpha)^2 \sigma^2}{2} \right]. \quad \frac{dNB_E^{equity}}{d\alpha} < 0 \quad \text{since}$$

$$\left(\frac{4}{\pi} - 1 \right) \frac{\omega^2 (1-\alpha)^2 \sigma^2}{2} > 0 \Leftrightarrow 1 - \exp \left[\left(\frac{4}{\pi} - 1 \right) \frac{\omega^2 (1-\alpha)^2 \sigma^2}{2} \right] < 0,$$

$$\operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right) \frac{\sigma\sqrt{2}}{\sqrt{\pi}} > 0, \text{ and } -\exp \left[-\frac{\omega^2 (1-\alpha)^2 \sigma^2}{2} \right] \omega \alpha \left(\frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right)^2 < 0.$$

Lemma 2.

The advantage of risk sharing with an equity partner, *ceteris paribus*, increases as α increases; i.e., the risk premium of the entrepreneur decreases in $\alpha \in (0;1]$ for a given risk aversion coefficient and a given standard deviation:

$$RP_E^{equity} = \frac{1}{2} \omega(1-\alpha)^2 \sigma^2. \quad (38)$$

$$\frac{dRP_E^{equity}}{d\alpha} = -\omega(1-\alpha)\sigma^2 < 0. \quad (39)$$

Therefore, the choice of optimal α has to trade off the benefits of risk sharing and the disadvantage of profit and loss sharing. Comparing the risk-adjusted returns in case of debt and equity financing is not analytically straightforward and unwarranted without having determined the optimal α first. This comparison is hence conducted only numerically in section 5.

4.3. Nonlinear PLS contracting

The participation constraint (which follows from Assumption 10) of the (risk-neutral) PLS investor I with initial wealth $W_{0,I}$, who correctly anticipates the effort decision of the entrepreneur for given λ and γ , determines the fixed payment either he (if $B = \beta A_0 > 0$) or the entrepreneur (if $B = \beta A_0 < 0$) is going to receive:

$$E\{\tilde{W}_{1,I}\} = E\left\{\left[\frac{\lambda(1+\tilde{r})+\beta}{1+(1+r_f)(W_{0,I}-C_I)}\right]A_0\right\} \cdot \mathcal{X}_{\{\tilde{r}<0\}} + E\left\{\left[\frac{\gamma(1+\tilde{r})+\beta}{1+(1+r_f)(W_{0,I}-C_I)}\right]A_0\right\} \cdot \mathcal{X}_{\{\tilde{r}>0\}}. \quad (40)$$

$$= (1+r_f)W_{0,I}$$

$$\Leftrightarrow \left[\frac{1}{2}(1-e)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} + \frac{1}{2}(1+e)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}\right]A_0 + \beta A_0 + (1+r_f)(W_{0,I} - \lambda A_0) \quad (41)$$

$$= (1+r_f)W_{0,I}$$

$$\Leftrightarrow B = \beta A_0 = \left[(1+r_f)\lambda - \frac{1}{2}(1-e)\lambda\left\{1-\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\} - \frac{1}{2}(1+e)\gamma\left\{1+\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right\}\right]A_0. \quad (42)$$

Here, $\mathcal{X}_{\{\tilde{r}<0\}} = \begin{cases} 1 & \text{if } \tilde{r} < 0 \\ 0 & \text{if } \tilde{r} > 0 \end{cases}$ and $\mathcal{X}_{\{\tilde{r}>0\}} = \begin{cases} 1 & \text{if } \tilde{r} > 0 \\ 0 & \text{if } \tilde{r} < 0 \end{cases}$ are indicator functions.

If the entrepreneur finances her project with a nonlinear PLS contract, her income consists of her share of the outcome of the project, any debt she has to pay back (in case $C - C_I > 0$) or any proceeds from a risk-free investment (in case $C - C_I < 0$), and the fixed payment she is either to make (if $B = \beta A_0 > 0$) or to receive (if $B = \beta A_0 < 0$):

$$\begin{aligned}
 i_E^{PLS} &= \begin{cases} (1-\lambda)\tilde{A}_1 - B - (1+r_f)(C - C_I), & \text{if } \tilde{r} < 0 \\ (1-\gamma)\tilde{A}_1 - B - (1+r_f)(C - C_I), & \text{if } \tilde{r} > 0 \end{cases} \\
 &= \begin{cases} [(1-\lambda)(1+\tilde{r}) - \beta]A_0 + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}), & \text{if } \tilde{r} < 0 \\ [(1-\gamma)(1+\tilde{r}) - \beta]A_0 + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}), & \text{if } \tilde{r} > 0 \end{cases}.
 \end{aligned} \tag{43}$$

Her expected utility from this income is then

$$\begin{aligned}
 E\{u(i_E^{PLS})\} &= -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta(1-\lambda)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\lambda)A_0 \\ +\frac{1}{2}\theta^2(1-\lambda)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 \\ -\theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \\
 &\quad -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta(1-\gamma)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\gamma)A_0 \\ +\frac{1}{2}\theta^2(1-\gamma)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 \\ -\theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix}
 \end{aligned} \tag{44}$$

(see Appendix A).

Taking (13) into account and with the cost of effort function from (11), the certainty equivalent of the entrepreneur's uncertain future wealth in case of nonlinear PLS is

$$\begin{aligned}
CE_E^{PLS} &= CE(i_E^{PLS}) - c(e) \\
&= -\frac{1}{\theta} \ln \left\{ \begin{aligned} &\frac{1}{2}(1-e) \left[1 + \operatorname{erf} \left(\frac{\theta(1-\lambda)A_0\sigma}{\sqrt{2}} \right) \right] \exp \left(\begin{aligned} &-\theta(1-\lambda)A_0 \\ &+ \frac{1}{2}\theta^2(1-\lambda)^2 A_0^2\sigma^2 \end{aligned} \right) \\ &+ \frac{1}{2}(1+e) \left[1 - \operatorname{erf} \left(\frac{\theta(1-\gamma)A_0\sigma}{\sqrt{2}} \right) \right] \exp \left(\begin{aligned} &-\theta(1-\gamma)A_0 \\ &+ \frac{1}{2}\theta^2(1-\gamma)^2 A_0^2\sigma^2 \end{aligned} \right) \end{aligned} \right\}^{-B} \\
&\quad - \frac{1}{2\ell\theta} \ln \left(\frac{1}{1-e^2} \right) + (1+r_f)C_I + (1+r_f)(A_0 - W_{0,E}) \\
&= -\frac{1}{\omega} \ln \left\{ \begin{aligned} &\frac{1}{2}(1-e) \left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right] \exp \left(\begin{aligned} &-\omega(1-\lambda) \\ &+ \frac{1}{2}\omega^2(1-\lambda)^2 \sigma^2 \end{aligned} \right) \\ &+ \frac{1}{2}(1+e) \left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right] \exp \left(\begin{aligned} &-\omega(1-\gamma) \\ &+ \frac{1}{2}\omega^2(1-\gamma)^2 \sigma^2 \end{aligned} \right) \end{aligned} \right\}^{A_0 - \beta A_0} \\
&\quad - \frac{1}{2\ell\omega} \ln \left(\frac{1}{1-e^2} \right) A_0 + (1+r_f)C_I + (1+r_f)(A_0 - W_{0,E}) \quad , (45)
\end{aligned}$$

where $\omega = \theta A_0$ is the entrepreneur's modified risk aversion coefficient.

By denoting the conditional utility of the entrepreneur from her share of the outcome of the project in case of a decrease and increase in the value of the project as

$$\begin{aligned}
L &= - \left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right] \exp \left(-\omega(1-\lambda) + \frac{1}{2}\omega^2(1-\lambda)^2 \sigma^2 \right) \\
&= - \exp \left\{ -\omega(1-\lambda) + \frac{1}{2}\omega^2(1-\lambda)^2 \sigma^2 + \ln \left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right] \right\} \quad (46)
\end{aligned}$$

and

$$\begin{aligned}
G &= - \left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right] \exp \left(-\omega(1-\gamma) + \frac{1}{2}\omega^2(1-\gamma)^2 \sigma^2 \right) \\
&= - \exp \left\{ -\omega(1-\gamma) + \frac{1}{2}\omega^2(1-\gamma)^2 \sigma^2 + \ln \left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right] \right\} \quad , (47)
\end{aligned}$$

respectively, her certainty equivalent (45) can be presented as follows:

$$\begin{aligned}
CE_E^{PLS} &= CE(i_E^{PLS}) - c(e) \\
&= -\frac{1}{\omega} \ln \left\{ -\frac{1}{2}(1-e)L - \frac{1}{2}(1+e)G \right\} A_0 - \beta A_0 - \frac{1}{2\ell\omega} \ln \left(\frac{1}{1-e^2} \right) A_0 \cdot \\
&\quad + (1+r_f)C_I + (1+r_f)(A_0 - W_{0,E})
\end{aligned} \tag{48}$$

After the contract has been signed, that is, with given λ (and C_I), γ , and β , the entrepreneur decides for a level of effort that maximizes her certainty equivalent from (48):

$$\frac{dCE_E^{PLS}}{de} = 0 \Leftrightarrow e = \frac{\sqrt{(L+G)^2 + 4\ell(\ell-1)(L-G)^2} + (L+G)}{2(\ell-1)(-L+G)}. \tag{49}$$

For simplicity, only the case when $a = 0 \Leftrightarrow \ell = 1$ is analytically considered in this section. In the numerical analyses, cases with $a = 1$ are also illustrated. When $a = 0 \Leftrightarrow \ell = 1$, the optimal effort of the entrepreneur is

$$e^* = \frac{L-G}{L+G} \tag{50}$$

(see Appendix B). This can be considered the general form of optimal effort with all three financing alternatives. When $\lambda = \gamma = 0$, the financing under consideration is debt financing. In

that case, $e^* = \frac{L-G}{L+G} = \operatorname{erf} \left(\frac{\omega\sigma}{\sqrt{2}} \right)$ (see (19)). When $\lambda = \gamma = \alpha$ and $\beta = 0$, the result is

financing with equity capital. Then, $e^* = \frac{L-G}{L+G} = \operatorname{erf} \left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}} \right)$ (see (31)).

It cannot be determined how the optimal effort (50) in case of nonlinear PLS contracting compares to the first-best and second-best levels of effort. This effort depends on how the losses and profits are shared (λ and optimal γ), the modified risk aversion coefficient ω of the entrepreneur, and the standard deviation σ . These dependencies will be elaborated on in the next section with the help of graphical illustrations.

Considering $a = 0 \Leftrightarrow \ell = 1$, the effort from (50), and the fixed payment (see (42)), the certainty equivalent of the entrepreneur in case of nonlinear PLS contracting is

$$\begin{aligned}
 CE_E^{PLS} &= \left\{ \begin{aligned} &\frac{1}{2\omega} \ln \left[\frac{1}{LG} \right] \\ &+ \frac{G}{L+G} \lambda \left[1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right] \\ &+ \frac{L}{L+G} \gamma \left[1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right] \end{aligned} \right\} A_0 \\
 &\quad - (1+r_f)(A_0 - W_{0,E}) \\
 &= \left\{ \begin{aligned} &1 - \frac{1}{4} \omega \left[(1-\lambda)^2 + (1-\gamma)^2 \right] \sigma^2 \\ &+ \frac{1}{2\omega} \ln \left(\frac{1}{\left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right]} \right) \\ &+ \frac{1}{2\omega} \ln \left(\frac{1}{\left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right]} \right) \\ &+ \frac{1}{2} \lambda \left[\frac{G}{L+G} \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) - 1 \right] \\ &+ \frac{1}{2} \gamma \left[\frac{L}{L+G} \left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) - 1 \right] \end{aligned} \right\} A_0 \\
 &\quad - (1+r_f)(A_0 - W_{0,E})
 \end{aligned} \tag{51}$$

The entrepreneur's effort decision depends on λ and γ . The entrepreneur asks the investor for a capital contribution in the amount of $C_I = \lambda A_0$ and chooses her effort after determining the optimal share γ that maximizes her certainty equivalent from (51). Solving this optimization problem mathematically is not possible since γ is a term of the error function. Instead, numerical analyses were conducted, whose results will be presented in section 5.

After determining the optimal γ , the fixed payment will be calculated based on the investor's participation constraint, and the entrepreneur will decide whether to finance her

business via nonlinear PLS contracting by comparing her certainty equivalent with her outside option (i.e., risk-free investment):

$$\left. \begin{aligned}
 & 1 - \frac{1}{4} \omega \left[(1-\lambda)^2 + (1-\gamma)^2 \right] \sigma^2 \\
 & + \frac{1}{2\omega} \ln \left(\frac{1}{\left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right]} \right) \\
 & + \frac{1}{2\omega} \ln \left(\frac{1}{\left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right]} \right) \\
 & + \frac{1}{2} \lambda \left[\frac{G}{L+G} \left(1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) - 1 \right] \\
 & + \frac{1}{2} \gamma \left[\frac{L}{L+G} \left(1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right) - 1 \right]
 \end{aligned} \right\} A_0 - (1+r_f)(A_0 - W_{0,E}) > (1+r_f)W_{0,E} \quad (52)$$

$$\Leftrightarrow \frac{1}{2\omega} \left\{ \ln \left(\frac{1}{\left[1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right) \right]} \right) + \ln \left(\frac{1}{\left[1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right) \right]} \right) \right\} \\
 + \left\{ \begin{aligned}
 & \frac{G}{L+G} \lambda \left[1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right] \\
 & + \frac{L}{L+G} \gamma \left[1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right]
 \end{aligned} \right\} - \frac{1}{4} \omega \left[(1-\lambda)^2 + (1-\gamma)^2 \right] \sigma^2 > r_f \\
 - \frac{1}{2} (\lambda + \gamma) \quad (53)$$

This means that the difference between the net benefit from extending effort and her risk premium, expressed as a rate of return on the investment, should exceed the risk-free rate of return. This difference, i.e., the LHS of (53) can be considered the entrepreneur's risk-adjusted rate of return on the investment:

$$\begin{aligned}
r_{PLS} = & \frac{1}{2\omega} \left\{ \ln \left[\frac{1}{1 + \operatorname{erf} \left(\frac{\omega(1-\lambda)\sigma}{\sqrt{2}} \right)} \right] + \ln \left[\frac{1}{1 - \operatorname{erf} \left(\frac{\omega(1-\gamma)\sigma}{\sqrt{2}} \right)} \right] \right\} \\
& + \left\{ \frac{G}{L+G} \lambda \left[1 - \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right] + \frac{L}{L+G} \gamma \left[1 + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \right] - \frac{1}{2}(\lambda + \gamma) \right\} - \frac{1}{4} \omega [(1-\lambda)^2 + (1-\gamma)^2] \sigma^2
\end{aligned} \quad . (54)$$

Since with nonlinear PLS contracting, the entrepreneur shares both the outcome of the project and the risk with the investor, both the benefit from her effort (the sum of the first two terms on the RHS of (54)) and the risk premium (the third term in the RHS of (54)) depend additionally on the sharing rule, i.e., λ and the optimal γ .

Analyzing the risk-adjusted return from (54) in case of nonlinear PLS financing and comparing it with the risk-adjusted returns in case of debt and equity financing are not possible without determining the optimal equity and nonlinear PLS contracts. Therefore, instead of analytically comparing the optimal levels of effort and resulting risk-adjusted returns from three financing alternatives, the next section will explore the differences by presenting and interpreting the results of the conducted numerical analyses.

5. Numerical Analyses

The determinants of the risk-averse entrepreneur's expected utility are her risk premium and net benefit from effort, both of which depend on the structure of the financial contract entered into by the entrepreneur to finance her project. In case of debt financing, there is no profit and loss or risk sharing involved, therefore, the risk premium depends only on the standard deviation of the random variable \tilde{x} and the risk aversion of the entrepreneur. The higher the standard deviation or the risk aversion coefficient, the higher the risk premium. On the other hand, the net benefit from effort depends also on the standard deviation and risk aversion coefficient, and additionally on the ability of the entrepreneur to manage the project. The net benefit from effort increases in all of these three variables.

The risk-adjusted return (*RAR*) of the entrepreneur will then depend on

- the standard deviation of the random variable \tilde{x} , $\sigma \in (0;1]$,
- the ability of the entrepreneur, $a \in [0;1]$, and
- the modified risk aversion coefficient of the entrepreneur, $\omega \in (0;1]$.

When the entrepreneur finances her project with equity capital from a partner, her risk premium additionally depends on the optimal share α of the partner and his risk aversion. The net benefit from effort also depends on α . All other things being equal, the risk premium of the entrepreneur and the net benefit from effort decrease in α . For a given α , the entrepreneur's risk premium increases as the standard deviation or the risk aversion coefficient of the entrepreneur increases, and the net benefit from effort increases in standard deviation, risk aversion coefficient of the entrepreneur, and her ability to manage the project. The optimal share α also depends on the standard deviation, the entrepreneur's ability, and the risk aversion coefficients of the entrepreneur and the partner (see, for example, Lemma 1 and Lemma 2).

The risk-adjusted return (*RAR*) of the entrepreneur with equity financing depends on

- the standard deviation of the random variable \tilde{x} , $\sigma \in (0;1]$,
- the ability of the entrepreneur, $a \in [0;1]$,
- the modified risk aversion coefficient of the entrepreneur, $\omega \in (0;1]$, and
- the modified risk aversion coefficient of the partner, $\omega_p \in (0;1]$ (see Appendix C).

In case of financing via nonlinear PLS contract with an investor, the entrepreneur's risk premium and net benefit from effort depend on λ (the share of the capital provided by the investor and the share of the losses he will be liable for) and the optimal share γ (the share of the positive outcome the investor is entitled to), which also depends on λ , the standard deviation of the random variable \tilde{x} , the entrepreneur's ability to manage the project, and the risk aversion coefficients of the entrepreneur and the investor. All other things equal, the risk premium of the entrepreneur and the net benefit from effort decrease in γ . For a given λ and γ , the entrepreneur's risk premium increases as the standard deviation or the risk aversion coefficient of the entrepreneur increases, and the net benefit from effort increases in standard deviation, risk aversion coefficient of the entrepreneur, and her ability to manage the project.

The risk-adjusted return (*RAR*) of the entrepreneur in case of nonlinear PLS depends ultimately on

- the standard deviation of the random variable \tilde{x} , $\sigma \in (0;1]$,
- the ability of the entrepreneur, $a \in [0;1]$,
- the share of capital provided by the investor, $\lambda \in (0;1]$,

- the modified risk aversion coefficient of the entrepreneur, $\omega \in (0;1]$, and
- the modified risk aversion coefficient of the investor, $\omega_i \in (0;1]$ (see Appendix C).

The standard deviation of the random variable \tilde{x} is varied within the interval $(0;1]$ for the numerical analyses. The results are presented for only two values of the standard deviation, $\sigma \in \{0.5, 1.0\}$, to represent a low-risk and a high-risk project. The entrepreneur's level of ability is allowed to take any value in the range $a \in [0;1]$ for the numerical analyses. Two values indicating an entrepreneur with the highest and lowest level of managerial ability are presented: $a \in \{0.0, 1.0\}$. The modified risk aversion coefficient of the entrepreneur is (artificially) assumed to be not larger than 1, i.e., $\omega \in (0;1]$, since the numerical analyses with a higher risk aversion coefficient lead to very low (and even negative) certainty equivalents of the entrepreneur with all three financing alternatives. Two values of the modified risk aversion coefficient, $\omega \in \{0.5, 1.0\}$, are selected to represent a moderately and a very risk-averse entrepreneur. In the analyses, risk sharing with either risk-neutral or very risk-averse outside financiers are considered: $\omega_p = \omega_i \in \{\emptyset, 1.0\}$.

The parameters listed above are considered as given. The risk-adjusted returns of the entrepreneur in case of equity financing and nonlinear PLS contracting are then maximized (using Microsoft Excel Solver) by the choice of optimal α and optimal γ , respectively. The optimal efforts are then calculated based on the given parameters and the optimal α and γ . The fixed payment from the nonlinear PLS contract is then determined based on the participation constraint of the investor.

The results of the numerical analyses are graphically illustrated and presented. On the x -axis of every graph is $\lambda \in (0;1]$, that is, the investor's share of capital (which is relevant only for nonlinear PLS contracting), for a clear juxtaposition between the financing alternatives. The risk-free rate of return is assumed to be $r_f = 0.05$. The effects of the standard deviation of the random variable \tilde{x} , the risk aversion coefficient(s) of the entrepreneur (and the outside financiers), and the ability of the entrepreneur to manage the project on the optimal structures of the financial contracts involving risk sharing, on the optimal level of effort of the entrepreneur, and on her risk-adjusted return are explored for the financing alternatives of the entrepreneur. Five different scenarios are considered:

Scenario 1. *Graphs 1-3*: low-risk project ($\sigma = 0.50$), moderately risk-averse ($\omega = 0.50$) entrepreneur with no managerial ability ($a = 0.00$), and risk-neutral financiers;

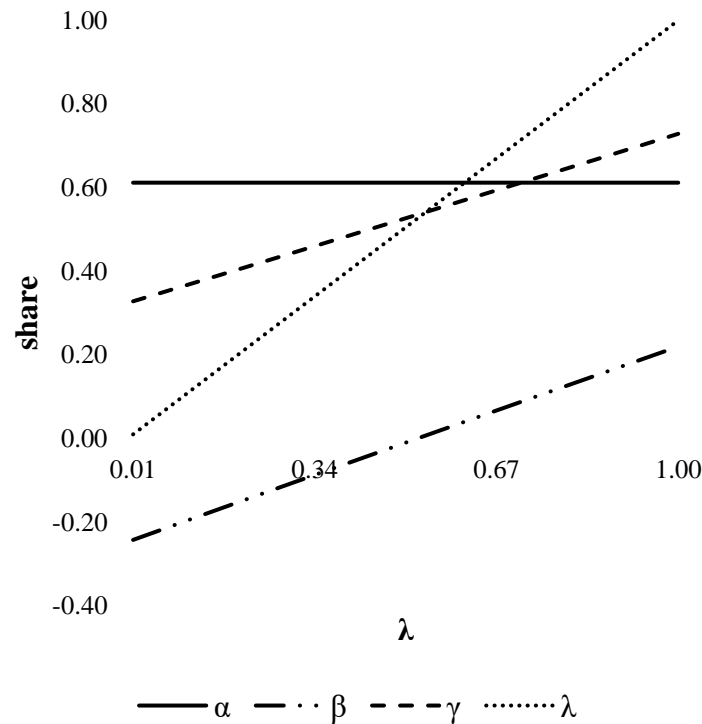
Scenario 2. *Graphs 4-6*: low-risk project ($\sigma = 0.50$), moderately risk-averse ($\omega = 0.50$) entrepreneur with the highest level of managerial ability ($a = 1.00$), and risk-neutral financiers;

Scenario 3. *Graphs 7-9*: low-risk project ($\sigma = 0.50$), moderately risk-averse ($\omega = 0.50$) entrepreneur with the highest level of managerial ability ($a = 1.00$), and risk-averse ($o_p = o_t = 1.00$) financiers;

Scenario 4. *Graphs 10-12*: low-risk project ($\sigma = 0.50$), very risk-averse ($\omega = 1.00$) entrepreneur with the highest level of managerial ability ($a = 1.00$), and risk-averse ($o_p = o_t = 1.00$) financiers;

Scenario 5. *Graphs 13-15*: high-risk project ($\sigma = 1.00$), very risk-averse ($\omega = 1.00$) entrepreneur with the highest level of managerial ability ($a = 1.00$), and risk-averse ($o_p = o_t = 1.00$) financiers.

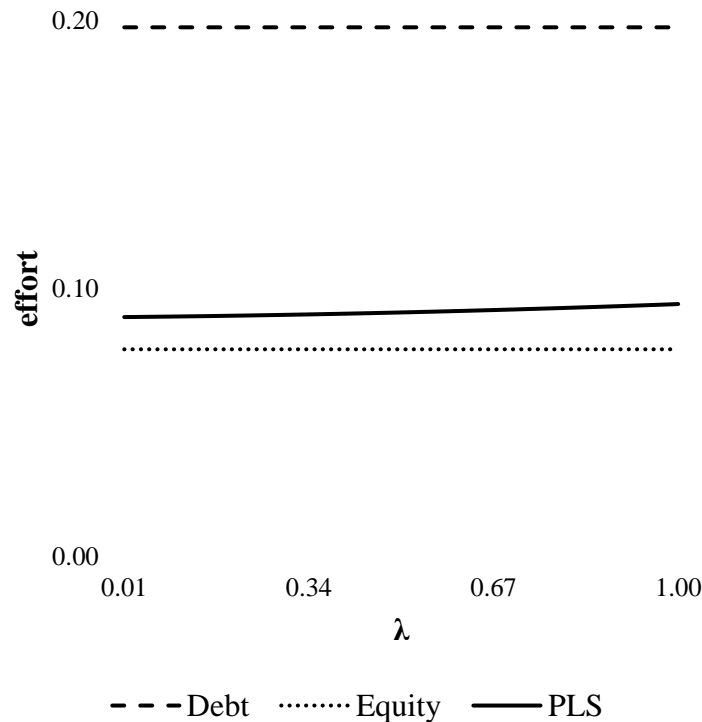
The first scenario as the baseline represents the underlying dependencies when the entrepreneur has no managerial ability. The second scenario illustrates the effects of the entrepreneur having the highest level of managerial ability. The ramifications of risk sharing with a risk-averse financier can be inferred from the third scenario. The fourth scenario shows the consequences of the entrepreneur being more risk-averse (as risk-averse as the outside financiers). The fifth scenario depicts the effects of risk sharing when the risk involved is very high.



Graph 1. Optimal contract ($\sigma = 0.50$; $a = 0.00$; $\omega = 0.50$; o_p & $o_l \in \emptyset$)

Graph 1 illustrates the optimal structure of the nonlinear PLS contract dependent on the share λ of the capital the investor provides and the optimal $\alpha \approx 0.6119$ for the case of equity financing. Both the optimal γ and the fixed payment component β increase as λ increases. That is, the higher the amount of capital the investor provides, the larger the share of the positive outcome he receives and the higher the fixed payment he will agree to.

The optimal γ has to trade off the benefits of risk sharing and the disadvantage of sharing the outcome with the investor. For small values of λ , $\gamma > \lambda$ and $\beta < 0$. To achieve the optimal level of risk sharing when the investor's capital contribution is small, the entrepreneur is willing to give up a higher γ share of the positive outcome (higher than the λ share of the negative outcome) and receives a fixed wage ($\beta < 0$) in return. The optimal γ equals λ when $\lambda \approx 0.54$. This value is smaller than the optimal α from equity financing due to the fixed payment component. For larger values of λ , $\gamma < \lambda$ and $\beta > 0$. When the investor's capital contribution is large, the optimal level of risk sharing is fulfilled with a lower γ share of the positive outcome (lower than the λ share of the negative outcome). The entrepreneur is then less willing to share the positive outcome with the investor, but agrees to pay him a fixed compensation in return.

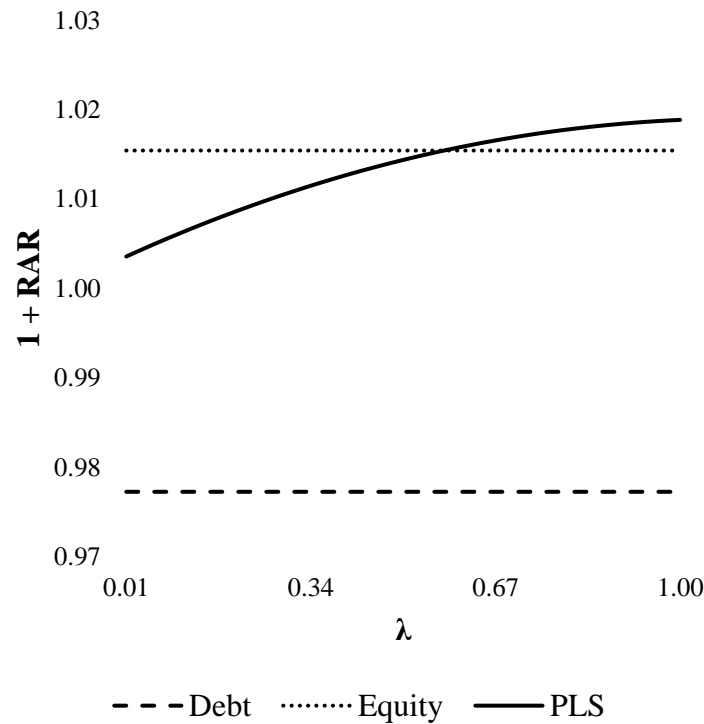


Graph 2. Optimal effort ($\sigma = 0.50; a = 0.00; \omega = 0.50; o_p \text{ \& } o_l \in \emptyset$)

Graph 2 juxtaposes the optimal efforts of the entrepreneur in case of debt, equity, and nonlinear PLS financing. The entrepreneur’s optimal effort $e_{debt} \approx 0.1974$ when she finances her project with debt capital is first-best since she does not have to share the outcome with an outside financier. Her effort in case of equity financing, $e_{equity} \approx 0.0773$, is much lower than the first-best level of effort; the incentive of the entrepreneur to extend effort is diluted since she has to share the outcome with the equity partner.

Financing with nonlinear PLS contracting results in a higher level of effort than with equity financing, but still lower than the first-best effort since it is a form of hybrid financing. With nonlinear PLS contracting, the disadvantage of sharing the outcome of the project for the entrepreneur is not as high as it is with equity financing. Moreover, the optimal level of effort of the entrepreneur with PLS financing increases as λ increases. There are two effects to consider here. On one hand, as λ increases, the optimal γ increases, which dilutes the entrepreneur’s effort since it is not desirable for the entrepreneur to give up a share of a positive outcome to the investor, and therefore, she is less motivated to extend effort to increase the probability of the positive outcome. On the other hand, as λ increases, the entrepreneur becomes more motivated to extend effort since a higher λ indicates that she will be entitled to a lower share of the outcome (if the value of the project decreases). The entrepreneur will then

be motivated to extend a higher level of effort to decrease the probability of the negative outcome. In this case, the positive effect of the increase in λ overpowers the negative effect of the increase in γ since γ increases at a lower rate than λ .

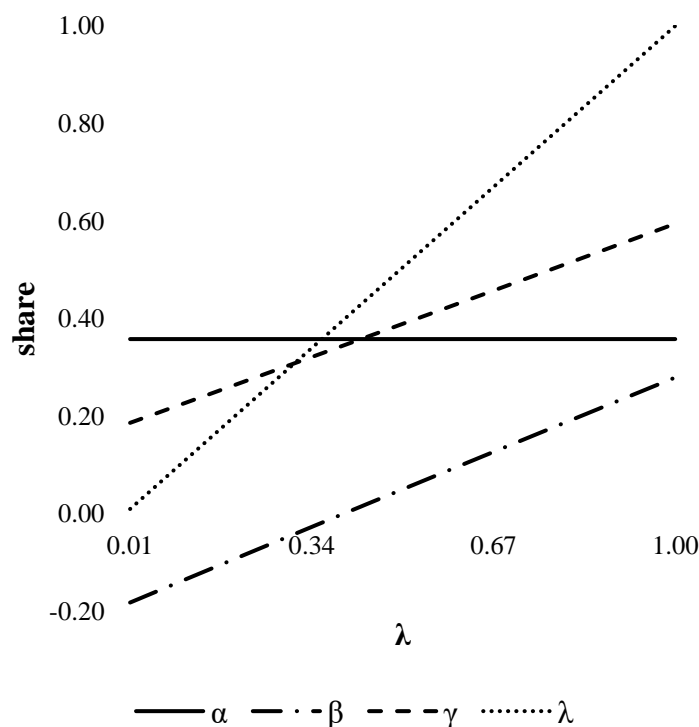


Graph 3. Risk-adjusted return ($\sigma = 0.50$; $a = 0.00$; $\omega = 0.50$; o_p & $o_l \in \emptyset$)

Graph 3 compares the entrepreneur’s risk-adjusted returns on the project when she finances it with debt, equity, or via nonlinear PLS contracting. First of all, it is to be mentioned that none of the alternatives result in a higher risk-adjusted return than the risk-free rate of return $r_f = 0.05$. This is due to the fact that the entrepreneur is risk-averse and has no managerial ability. Her cost of extending effort is too high. However, the risk-adjusted returns can still be compared among themselves. It is apparent that financing with debt capital provides the lowest risk-adjusted return on the project even though the entrepreneur’s effort in this case is first-best. The entrepreneur is risk-averse and does not share the risk with anyone. This disadvantage combined with the high cost of effort results in a negative risk-adjusted return, $r_{debt} \approx -0.0227$. Even though the entrepreneur’s effort is lower when she finances the project with equity capital from a partner, the risk-adjusted return in this case is higher than with debt financing: $r_{equity} \approx 0.0154$.

Financing via nonlinear PLS transaction also results in a higher risk-adjusted return due to the benefits of risk sharing. In fact, nonlinear PLS contracting is more advantageous for the

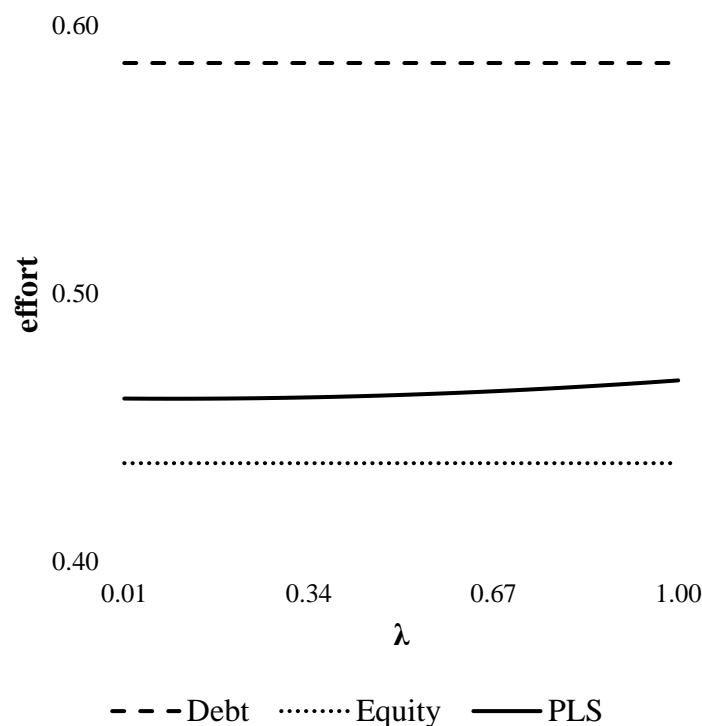
entrepreneur than equity financing for high enough values of λ . When $\lambda \approx 0.58$, both equity financing and nonlinear PLS contracting result in the same risk-adjusted return on the project. Since the investor is risk-neutral, the optimal λ for the entrepreneur is 1, that is, the risk-adjusted return is highest when the investor provides all the necessary capital to finance the project. In this case, the optimal share of the positive outcome of the investor is $\gamma \approx 0.7288$. Then, anticipating the effort decision of the entrepreneur based on $\lambda = 1$ and $\gamma \approx 0.7288$, which is $e_{PLS} \approx 0.0942$, the investor agrees to a fixed payment based on $\beta \approx 0.2200$, i.e., he will receive a compensation $\beta A_0 \approx 0.2200 A_0$, where A_0 is the initial investment amount. If the entrepreneur were to finance the project via nonlinear PLS arrangement where the investor provides all the capital necessary for the project, her risk-adjusted return would be $r_{PLS} \approx 0.0189$. Since none of the financing alternatives provide a risk-adjusted return higher than the risk-free rate of return due the entrepreneur having no managerial ability and being cost inefficient in extending effort, the entrepreneur will opt out of the project and invest her available capital risk-free in the market.



Graph 4. Optimal contract ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; o_p & $o_t \in \emptyset$)

Graph 4 illustrates the optimal structure of the nonlinear PLS contract dependent on the share λ of the capital the investor provides and the optimal $\alpha \approx 0.3582$ for the case of equity financing. In comparison to the previous case, it can be observed that both the optimal α from

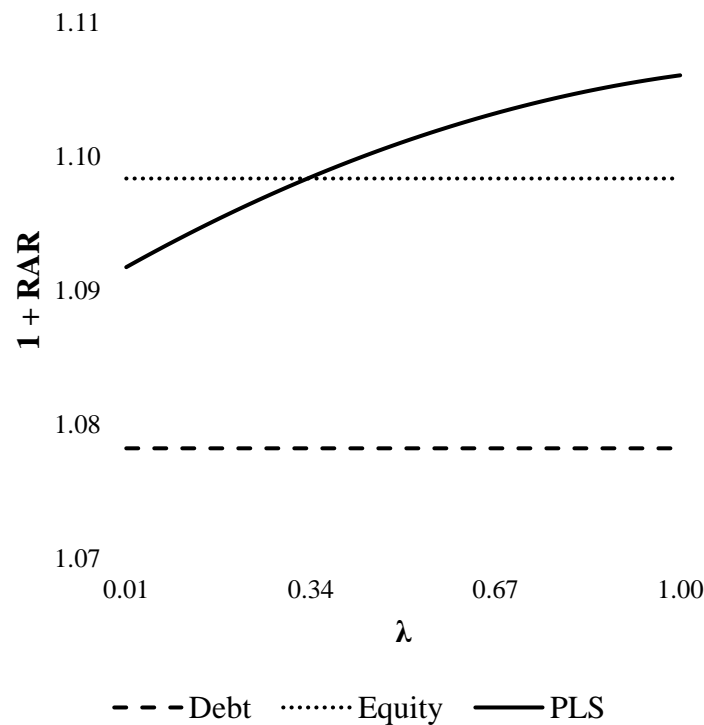
equity financing and the optimal γ from nonlinear PLS contracting are lower when the entrepreneur’s ability is higher. If the entrepreneur is more cost efficient when exerting effort, she is willing to give up a lower share of the outcome to the outside financier, and she can still achieve the sufficient level of risk sharing. That is, the optimal α and γ that trade off the benefit of risk sharing and the disadvantage from sharing the outcome with the outside financier are lower if the entrepreneur’s cost of effort is lower. The optimal γ is equal to λ when $\lambda \approx 0.31$, which is smaller than the optimal α with equity financing due to the fixed payment component. Since the optimal γ is lower when the entrepreneur has the highest level of managerial ability, the fixed payment component β is higher for any given λ .



Graph 5. Optimal effort ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; o_p & $o_l \in \emptyset$)

As depicted in Graph 5, the entrepreneur’s first-best level of effort, $e_{debt} \approx 0.5862$, with debt financing is higher compared to the previous case in which the entrepreneur has no managerial ability. Since she has to share the outcome of the project with the partner, her level of effort with equity financing is lower than the first-best effort: $e_{equity} \approx 0.4369$. The optimal effort in case of nonlinear PLS contracting is again higher than the effort the entrepreneur extends when she finances the project with equity capital. The entrepreneur’s optimal effort

when she chooses nonlinear PLS increases as λ increases. As in the previous case, the positive effect of the increase in λ outweighs the negative effect of the increase in γ .

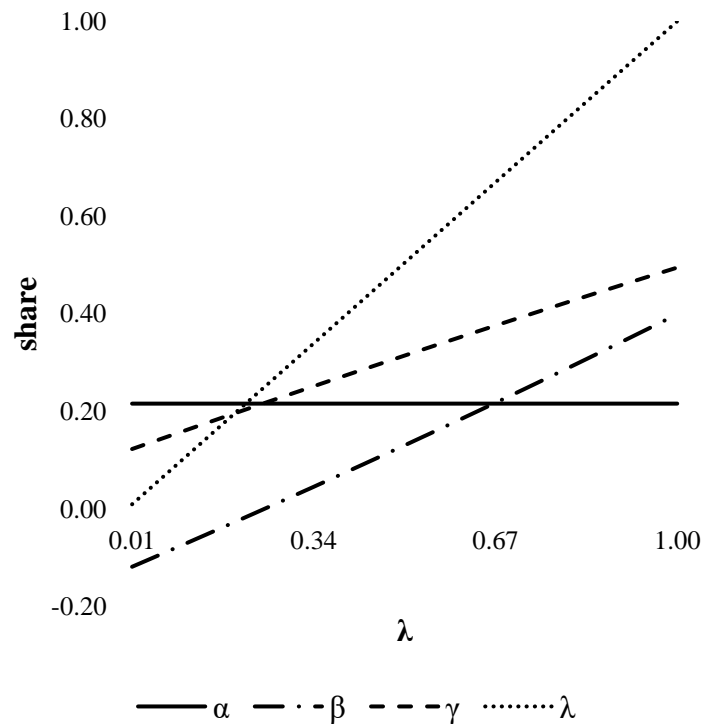


Graph 6. Risk-adjusted return ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; o_p & $o_l \in \emptyset$)

Graph 6 depicts the risk-adjusted returns on the project in case of debt, equity, and nonlinear PLS financing when the entrepreneur is highly capable of managing the project. All three alternatives of financing result in a risk-adjusted return higher than the risk-free rate of return $r_f = 0.05$. With equity financing, the risk-adjusted return $r_{equity} \approx 0.0983$ is higher than with debt financing, $r_{debt} \approx 0.0782$, due to the merits of risk sharing despite the fact that the entrepreneur’s incentives to extend effort are diminished when she finances her project with equity capital from an outside partner.

The same observation can be made for the case of nonlinear PLS contracting. When financed via nonlinear PLS arrangement, the risk-adjusted return on the project is even higher than with equity financing for large enough values of λ . This is the result of λ being sufficiently larger than γ . The value of λ ($\lambda \approx 0.33$) that results in a risk-adjusted return from nonlinear PLS contracting as high as with equity financing is smaller than when the entrepreneur has no managerial ability. This is because a higher ability of the entrepreneur leads to a lower optimal γ for any given λ , which means $\lambda > \gamma$ for a greater range of values

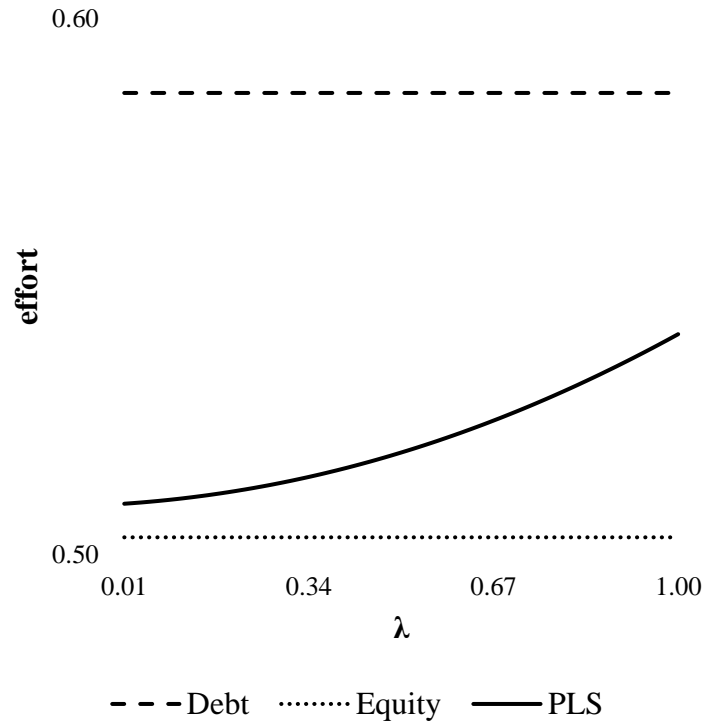
of λ . If the capital the investor provides is more than approximately 33% of the required amount, financing via nonlinear PLS transaction is more beneficial for the entrepreneur compared to optimal equity financing with the partner. Since the investor is risk-neutral, the optimal λ for the entrepreneur is 1, that is, the risk-adjusted return is highest when the investor provides all the necessary capital to finance the project. In this case, the optimal share of the positive outcome of the investor is $\gamma \approx 0.5947$. Then, anticipating the effort decision of the entrepreneur based on $\lambda = 1$ and $\gamma \approx 0.5947$, which is $e_{PLS} \approx 0.4678$, the investor will ask for a fixed payment based on $\beta \approx 0.2794$, i.e., he will receive a compensation $\beta A_0 \approx 0.2794 A_0$, where A_0 is the initial investment amount. If the entrepreneur were to finance the project via nonlinear PLS arrangement where the investor provides all the capital necessary for the project, the risk-adjusted return would be $r_{PLS} \approx 0.1060$.



Graph 7. Optimal contract ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; $o_p = o_t = 1.00$)

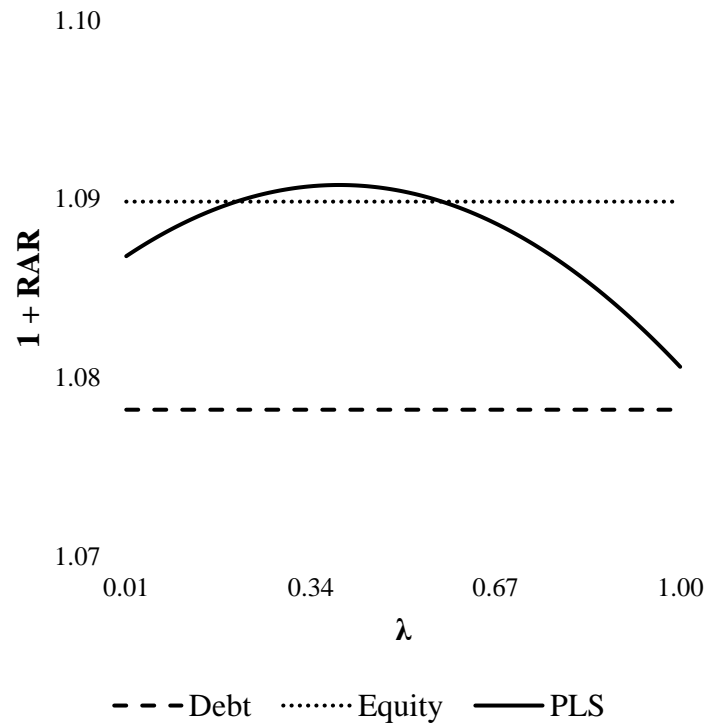
Graph 7 depicts the optimal structure of the nonlinear PLS contract dependent on the share λ of the capital the investor provides and the optimal $\alpha \approx 0.2163$ for the case of equity financing. The benefit of risk sharing diminishes as the financiers become risk-averse. Therefore, the optimal α is lower than in the previous case where the equity partner was risk-neutral. The same reasoning applies also to the optimal γ in case of nonlinear PLS contracting.

The optimal γ equals λ when $\lambda \approx 0.20$, which is smaller than the optimal α with equity financing due to the fixed payment component. Since the optimal γ is lower when the investor is more risk-averse, the fixed payment component β is higher for any given λ .



Graph 8. Optimal effort ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; $o_p = o_l = 1.00$)

As it can be seen in Graph 8, the optimal first-best effort of the entrepreneur in case of debt financing is same as in the previous case since debt financing does not involve any risk or profit and loss sharing with the outside financiers: $e_{debt} \approx 0.5862$. Because the optimal α is lower when the equity partner is risk-averse, the entrepreneur will have to give up a lower share of the outcome of her effort to the partner. Therefore, her optimal effort in this case is higher: $e_{equity} \approx 0.5033$. The entrepreneur’s optimal effort when she chooses nonlinear PLS financing increases as λ increases. The increase in the entrepreneur’s effort as λ increases is larger compared to the previous case because the difference between λ and the optimal γ increases at a more rapid rate.

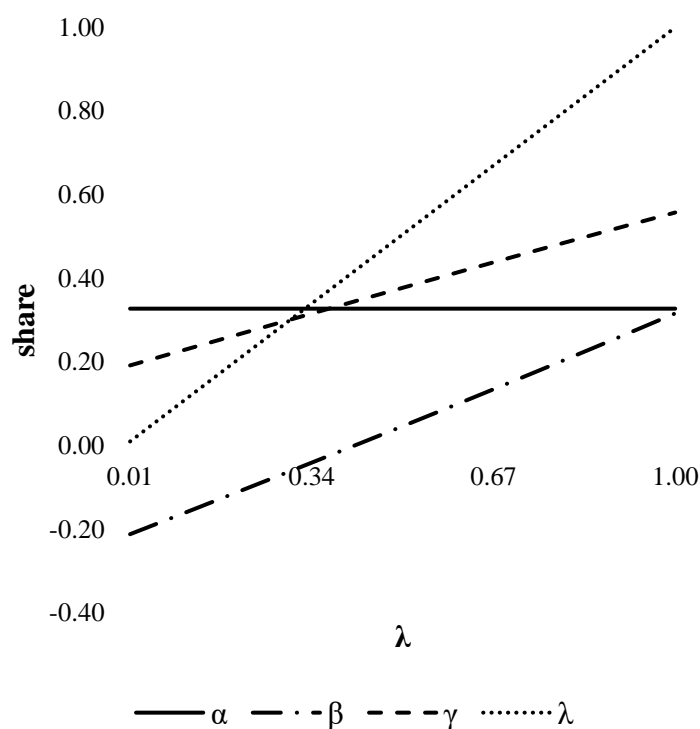


Graph 9. Risk-adjusted return ($\sigma = 0.50$; $a = 1.00$; $\omega = 0.50$; $\sigma_p = \sigma_l = 1.00$)

Graph 9 illustrates the risk-adjusted returns on the project in case of debt, equity, and nonlinear PLS financing. All three financing alternatives result in a higher risk-adjusted return than the risk-free rate of return $r_f = 0.05$ even when the outside financiers are risk-averse. Debt financing does not involve any risk sharing. Therefore, the risk-adjusted return in case the entrepreneur finances the project with debt does not change when the outside financiers become more risk-averse: $r_{debt} \approx 0.0782$. Financing the project with equity capital from a risk-averse partner (and sharing the risk with him), however, is less beneficial for the entrepreneur than risk sharing with a risk-neutral equity partner. Hence, the risk-adjusted return on the project is lower compared to the previous case: $r_{equity} \approx 0.0899$, but still higher than the risk-adjusted return from debt financing; risk sharing, even with a risk-averse partner, is beneficial for the entrepreneur.

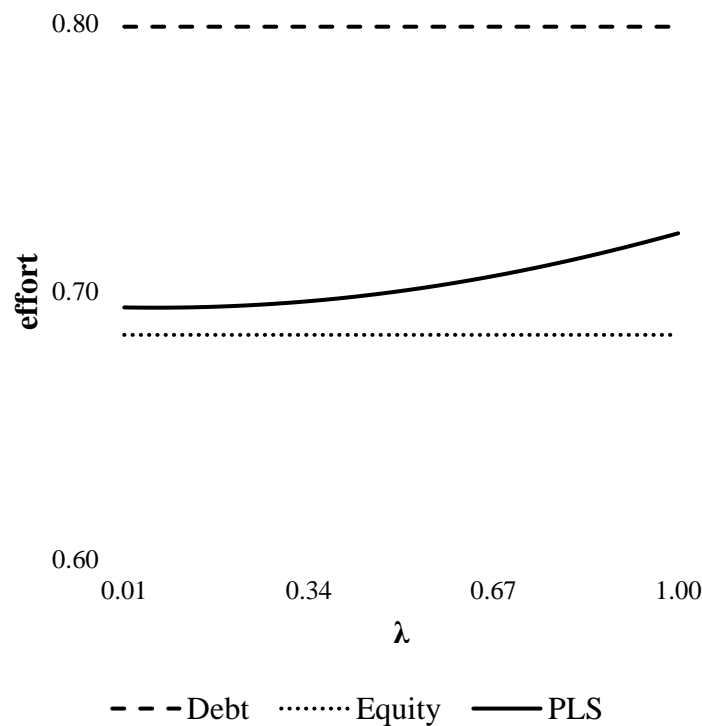
Financing the project via a nonlinear PLS contract with a risk-averse investor is also more advantageous for the entrepreneur than debt financing due to the merits of risk sharing. However, nonlinear PLS contracting results in a higher risk-adjusted return than equity financing only for a certain range of values of λ . Nonlinear PLS contracting is more beneficial for the entrepreneur than equity financing only if the share of the capital provided by the investor is approximately $0.21 \leq \lambda \leq 0.57$. This advantage is a result of the trade-off between

the benefits of risk sharing and the disadvantage of sharing the outcome with the investor. The share $\lambda \approx 0.3906$ of the capital to be provided by the investor is, in this case, optimal for the entrepreneur. For $\lambda < 0.3906$, the difference between the net benefit from entrepreneur's effort and the cumulative risk premium of the entrepreneur and the investor for the project increases as λ increases and becomes larger than with equity financing when $\lambda \geq 0.21$, that is, when λ becomes greater than optimal γ . This difference starts to decrease from $\lambda > 0.3906$, since the risk premium required by the investor becomes larger, and eventually, that is, when $\lambda > 0.57$, risk sharing with the equity partner becomes more beneficial than with the nonlinear PLS investor. Therefore, if the share of capital the investor provides is $0.21 \leq \lambda \leq 0.57$, financing the project via the nonlinear PLS contract with the investor is more advantageous for the entrepreneur than equity financing. When $\lambda \approx 0.3906$, the optimal share of the positive outcome of the investor is $\gamma \approx 0.2714$. Anticipating the effort decision $e_{PLS} \approx 0.5158$ of the entrepreneur based on $\lambda \approx 0.3906$ and $\gamma \approx 0.2714$, the investor will ask for a fixed payment based on $\beta \approx 0.0718$, i.e., he will receive a compensation $\beta A_0 \approx 0.0718A_0$, where A_0 is the initial investment amount. If the entrepreneur were to finance the project via nonlinear PLS arrangement where the investor provides $\lambda \approx 0.3906$ share of the capital necessary for the project, the risk-adjusted return would be $r_{PLS} \approx 0.0908$.



Graph 10. Optimal contract ($\sigma = 0.50$; $a = 1.00$; $\omega = 1.00$; $\omega_p = \omega_l = 1.00$)

Graph 10 depicts the optimal structure of the nonlinear PLS contract dependent on the share λ of the capital the investor provides and the optimal $\alpha \approx 0.3276$ for the case of equity financing. The benefit of risk sharing increases as the entrepreneur becomes more risk-averse. Therefore, the optimal α is higher than in the previous case where the entrepreneur is less risk-averse, which also applies to the optimal γ in case of nonlinear PLS contracting. The optimal γ equals λ when $\lambda \approx 0.30$, which is smaller than the optimal α with equity financing due to the fixed payment. Since the optimal γ is higher when the entrepreneur is more risk-averse, the fixed payment component β is lower for any given λ .

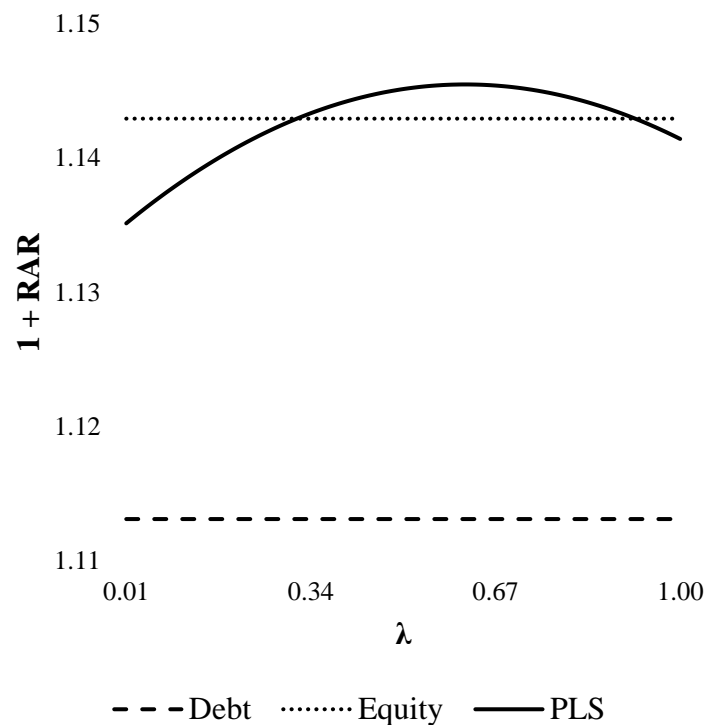


Graph 11. Optimal effort ($\sigma = 0.50$; $a = 1.00$; $\omega = 1.00$; $o_p = o_l = 1.00$)

The optimal effort of the entrepreneur increases as she becomes more risk-averse. As it is depicted in Graph 11, this results in a higher level of first-best effort of the entrepreneur in case of debt financing, $e_{debt} \approx 0.7988$, compared to the previous case. In case of equity financing, the optimal effort of entrepreneur decreases as optimal α increases. On one hand, an increase in the risk aversion level of the entrepreneur results in an increase in her optimal level of effort. On the other hand, a higher risk aversion coefficient indicates a higher optimal α , which leads to a decrease in the entrepreneur’s effort. In this case, the optimal effort of the entrepreneur when she finances the project with equity capital from the partner is

$e_{equity} \approx 0.6838$, that is, higher compared to the previous case, where the entrepreneur is less risk-averse. Therefore, the direct effect of a higher risk aversion of the entrepreneur overpowers the indirect effect (in the opposite direction) of a higher optimal α .

The same can be observed when it comes to financing with nonlinear PLS contract. The optimal γ for any given λ is higher when the entrepreneur is risk-averse, which results in a lower level of effort. However, since the increase in the risk aversion level of the entrepreneur also has a greater direct positive effect on her effort, an increase in the entrepreneur’s risk aversion coefficient results in a higher level of optimal effort for any given λ . Furthermore, the optimal effort in case of nonlinear PLS financing increases as λ increases. The increase in the entrepreneur’s effort as λ increases is smaller compared to the previous case because the difference between λ and the optimal γ increases at a less rapid rate.

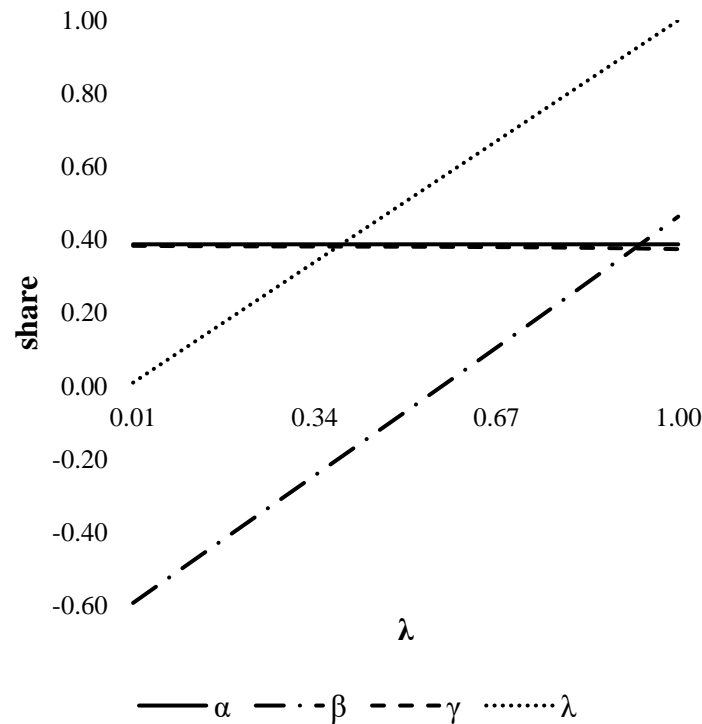


Graph 12. Risk-adjusted return ($\sigma = 0.50 ; a = 1.00 ; \omega = 1.00 ; o_p = o_l = 1.00$).

Graph 12 illustrates the risk-adjusted returns on the project. Thanks to the higher level of effort, it is higher with all three financing alternatives compared to the previous case despite the fact the entrepreneur is more risk-averse. Financing with debt capital results in a lower risk-adjusted return than equity or PLS financing since the entrepreneur does not benefit from risk sharing: $r_{debt} \approx 0.1131$. Even though the entrepreneur extends a lower effort when she finances

the project with equity capital, the risk-adjusted return is higher than with debt financing thanks to risk sharing: $r_{equity} \approx 0.1429$.

Financing the project via a nonlinear PLS contract with an investor is also more advantageous for the entrepreneur than debt financing due to the benefits of risk sharing. However, nonlinear PLS contracting results in a higher risk-adjusted return than equity financing only for a certain range of values of λ . Nonlinear PLS contracting is more beneficial for the entrepreneur than equity financing only if the share of the capital provided by the investor is approximately $0.31 \leq \lambda \leq 0.91$. When λ becomes larger than γ , the nonlinear PLS contracting becomes more advantageous for the entrepreneur than equity financing. The share $\lambda \approx 0.6158$ of the capital to be provided by the investor is, in this case, optimal for the entrepreneur. For $\lambda < 0.6158$, the difference between the net benefit from entrepreneur's effort and the cumulative risk premium of the entrepreneur and the investor for the project increases as λ increases and becomes larger than with equity financing when $\lambda \geq 0.31$, that is, when λ becomes greater than optimal γ . This difference starts to decrease from $\lambda > 0.6158$, since the risk premium required by the investor becomes larger, and eventually, that is, when $\lambda > 0.91$, risk sharing with the equity partner becomes more beneficial than with the nonlinear PLS investor. Therefore, if the share of capital the investor provides is $0.31 \leq \lambda \leq 0.91$, financing the project via the nonlinear PLS contract with the investor is more advantageous for the entrepreneur than equity financing. Assuming that the share of capital the investor provides is optimal for the entrepreneur, that is, $\lambda \approx 0.6158$, the optimal share of the positive outcome of the investor is $\gamma \approx 0.4185$. Then, anticipating the effort decision of the entrepreneur based on $\lambda \approx 0.6158$ and $\gamma \approx 0.4185$, which is $e_{PLS} \approx 0.7037$, the investor will agree to a fixed payment based on $\beta \approx 0.1057$, i.e., he will receive a compensation $\beta A_0 \approx 0.1057 A_0$, where A_0 is the initial investment amount. If the entrepreneur were to finance the project via nonlinear PLS arrangement where the investor provides $\lambda \approx 0.6158$ share of the capital necessary for the project, the risk-adjusted return would be $r_{PLS} \approx 0.1455$.

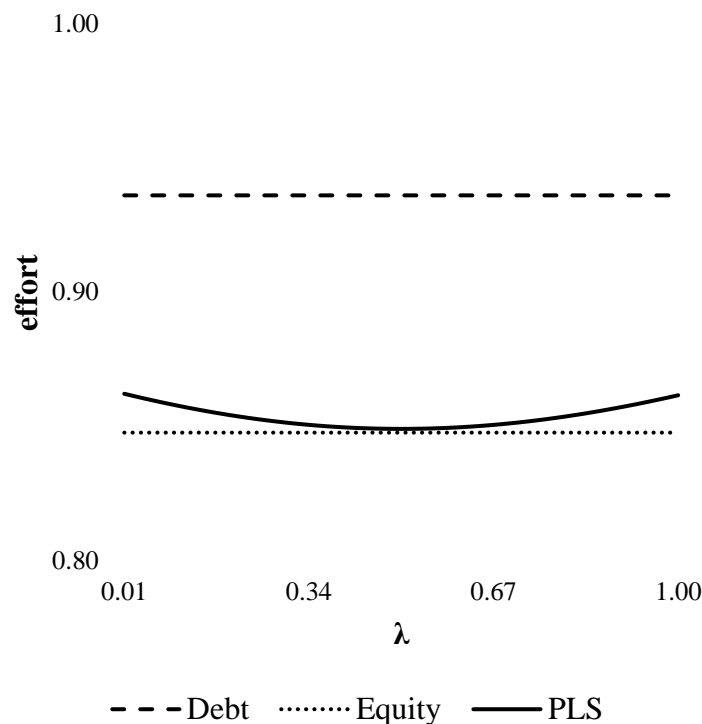


Graph 13. Optimal contract ($\sigma = 1.00$; $a = 1.00$; $\omega = 1.00$; $\sigma_p = \sigma_l = 1.00$)

Graph 13 depicts the optimal structure of the nonlinear PLS contract dependent on the share λ of the capital the investor provides and the optimal $\alpha \approx 0.3880$ for the case of equity financing. With a higher standard deviation, both the risk and the expected return are higher. Therefore, an increase in the standard deviation has two opposite effects on the optimal α . On one hand, when the risk is higher, the entrepreneur will want to share more of the risk with the equity partner resulting in a higher α . On the other hand, the entrepreneur will be more reluctant to share the outcome with the partner since the expected return is also higher resulting in a decrease in the optimal α . In this case, the optimal α is higher compared to the previous case, where $\sigma = 0.50$. Therefore, the benefit of sharing the higher risk with the equity partner outweighs the disadvantage from sharing the higher expected return with him.

The effect of an increase in the standard deviation on the optimal γ in case of nonlinear PLS contracting is less straightforward. As standard deviation increases, the slope of the curve depicting the dependence between optimal γ and λ decreases while the slope of the curve depicting the dependence between β and λ increases. $\frac{d\gamma^*}{d\lambda} < 0$, i.e., the optimal γ decreases as λ increases when $\sigma = 1.00$. This can be explained by considering the trade-off between the benefits of risk sharing and the disadvantage from sharing the positive outcome with the

investor. A higher standard deviation indicates not only a higher risk, but also a higher expected positive return. For small values of λ , the risk-averse entrepreneur would be willing to give up a higher share of the positive return to the investor in exchange for more protection against the higher risk. As λ increases, this protection is provided thanks to the higher λ , and the optimal γ decreases since the expected positive return is high, and the entrepreneur is less willing to give up a share of the positive outcome. The optimal γ equals λ when $\lambda \approx 0.38$. Compared to the previous case where $\sigma = 0.50$, the optimal γ is higher when $\lambda \leq 0.51$ (because the risk is higher, and λ is too small to provide sufficient level of risk sharing, which needs to be compensated for by the higher optimal γ) and lower when $\lambda > 0.51$ (because the expected positive return is higher, and λ is large enough that the sufficient level of risk sharing is possible with smaller optimal γ).

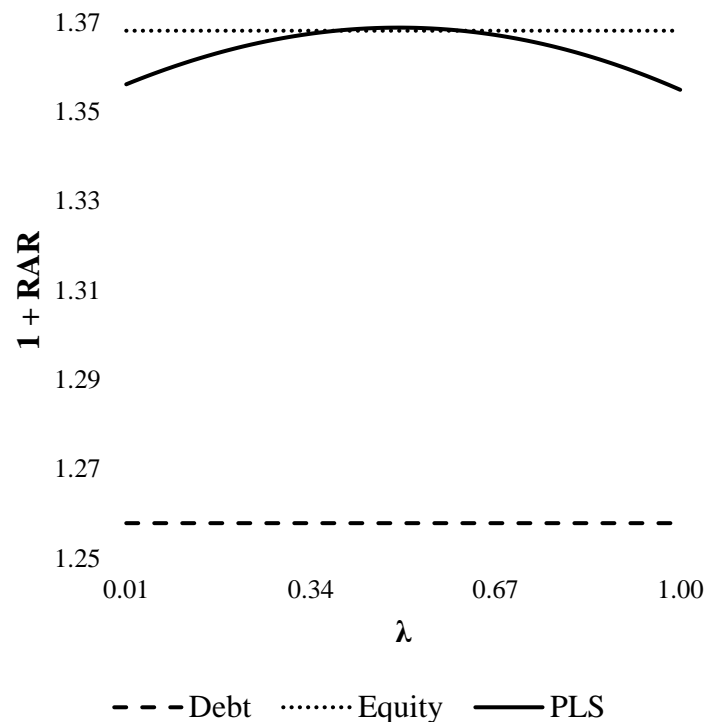


Graph 14. Optimal effort ($\sigma = 1.00$; $a = 1.00$; $\omega = 1.00$; $o_p = o_t = 1.00$)

Graph 14 shows the optimal efforts of the entrepreneur with debt, equity, and nonlinear PLS financing. The entrepreneur extends a higher level of effort as the standard deviation increases, which can be deduced by comparing Graph 11 and Graph 14. This results in a higher first-best effort in case of debt financing, $e_{debt} \approx 0.9361$, compared to the previous case. In case of equity financing, the optimal effort of entrepreneur decreases as optimal α increases. On one hand, an increase in the standard deviation results in an increase in her optimal level of

effort. On the other hand, a higher standard deviation indicates a higher optimal α , which leads to a decrease in the entrepreneur’s effort. In this case, the optimal effort of the entrepreneur when she finances the project with equity capital from the partner is $e_{equity} \approx 0.8476$, i.e., higher compared to the previous case where $\sigma = 0.50$. Thus, the direct effect of a higher standard deviation overpowers the indirect effect (in the opposite direction) of a higher optimal α .

In case of nonlinear PLS financing, the optimal effort of the entrepreneur is also higher than in the previous case. Here, the effort first decreases, then increases as λ increases. For small values of λ , as λ increases, the benefit of risk sharing increases at a higher rate than the disadvantage of sharing the negative outcome with the investor since the risk is very high, and the expected negative return is very low. Therefore, the entrepreneur becomes less motivated to extend effort to decrease the probability of the negative outcome. The negative effect of the increase in λ overpowers the positive effect of the decrease in γ on the optimal effort for small values of λ . For larger values of λ , that is, after the optimal level of risk sharing is reached, the benefit of risk sharing increases at a lower rate than the disadvantage of sharing the negative outcome with the investor. Therefore, the entrepreneur becomes more motivated to extend effort to decrease the probability of the negative outcome. Then, both the increase in λ and the decrease in optimal γ have a positive effect on the entrepreneur’s optimal effort.



Graph 15. Risk-adjusted return ($\sigma = 1.00$; $a = 1.00$; $\omega = 1.00$; $\sigma_p = \sigma_l = 1.00$)

Graph 15 illustrates the risk-adjusted returns on the project. Thanks to the higher standard deviation of the rate of return on the project, it is higher with all three financing alternatives compared to the previous case despite the higher risk premium. Financing with debt financing again results in a lower risk-adjusted return than equity or PLS financing since the entrepreneur does not benefit from risk sharing: $r_{debt} \approx 0.2578$. Even though the entrepreneur extends a lower effort when she finances the project with equity capital, the risk-adjusted return is higher than with debt due to risk sharing: $r_{equity} \approx 0.3681$.

Financing the project via a nonlinear PLS contract with an investor is also more advantageous for the entrepreneur than debt financing due to the merits of risk sharing. Nonlinear PLS contracting results in a higher risk-adjusted return than equity financing only for a certain range of values of λ . Nonlinear PLS contracting is more beneficial for the entrepreneur than equity financing only if the share of the capital provided by the investor is approximately $0.39 \leq \lambda \leq 0.60$. When λ becomes larger than γ , the nonlinear PLS contracting becomes more advantageous for the entrepreneur than equity financing. The share $\lambda \approx 0.4971$ of the capital to be provided by the investor is, in this case, optimal for the entrepreneur. For $\lambda < 0.4971$, the difference between the net benefit from entrepreneur's effort and the cumulative risk premium of the entrepreneur and the investor for the project increases as λ increases and becomes larger than with equity financing when $\lambda \geq 0.39$, that is, when λ becomes greater than optimal γ . This difference starts to decrease from $\lambda > 0.4971$, since the risk premium required by the investor becomes larger, and eventually, that is, when $\lambda > 0.60$, risk sharing with the equity partner becomes more beneficial than with the nonlinear PLS investor. Therefore, if the share of capital the investor provides is $0.39 \leq \lambda \leq 0.60$, financing the project via the nonlinear PLS contract with the investor is more advantageous for the entrepreneur than equity financing. Assuming that the share of capital the investor provides is optimal for the entrepreneur, that is, $\lambda \approx 0.4971$, the optimal share of the positive outcome of the investor is $\gamma \approx 0.3815$. Then, anticipating the effort decision of the entrepreneur based on $\lambda \approx 0.4971$ and $\gamma \approx 0.3815$, which is $e_{PLS} \approx 0.8490$, the investor will agree to pay the entrepreneur a wage based on $\beta \approx -0.0778$, i.e., he will pay the entrepreneur $-\beta A_0 \approx 0.0778 A_0$, where A_0 is the initial investment amount. If the entrepreneur were to finance the project via nonlinear PLS arrangement where the investor provides $\lambda \approx 0.4971$ share of the capital necessary for the project, the risk-adjusted return would be $r_{PLS} \approx 0.3688$.

Table 1 summarizes the results referred to above for debt, equity, and nonlinear PLS financing. The results with different levels of ability of the entrepreneur, risk aversion of the outside financiers, risk aversion of the entrepreneur, and riskiness of the project on the optimal equity and PLS contract structures, the optimal levels of effort of the entrepreneur, and the risk-adjusted returns resulting from the three financing alternatives are presented. It can be seen easily that, in all five cases, the optimal (i.e., when λ is optimal) nonlinear PLS results in a higher risk-adjusted return than debt or equity financing. The lowest risk-adjusted return is achieved when the entrepreneur forgoes the benefits of risk sharing as she finances her project with debt capital despite the first-best level of effort. Moreover, compared to equity financing, the managerial incentives of the entrepreneur to extend effort are diluted to a lesser extent with nonlinear profit-and-loss-sharing contracting arrangement.

Table 1. The results of the numerical analyses

		<i>Project</i>				
		$\sigma = 0.50$	$\sigma = 0.50$	$\sigma = 0.50$	$\sigma = 0.50$	$\sigma = 1.00$
		<i>Entrepreneur</i>				
		$\omega = 0.50$	$\omega = 0.50$	$\omega = 0.50$	$\omega = 1.00$	$\omega = 1.00$
		$a = 0.00$	$a = 1.00$	$a = 1.00$	$a = 1.00$	$a = 1.00$
		<i>Financiers</i>				
		risk-neutral	risk-neutral	$\sigma_p = 1.00$ $\sigma_l = 1.00$	$\sigma_p = 1.00$ $\sigma_l = 1.00$	$\sigma_p = 1.00$ $\sigma_l = 1.00$
<i>Debt financing</i>	Optimal effort	0.1974	0.5862	0.5862	0.7988	0.9361
	RAR	-0.0227	0.0782	0.0782	0.1131	0.2578
<i>Equity financing</i>	Optimal α	0.6119	0.3582	0.2163	0.3276	0.3880
	Optimal effort	0.0773	0.4369	0.5033	0.6838	0.8476
	RAR	0.0154	0.0983	0.0899	0.1429	0.3681
<i>Nonlinear PLS contracting</i>	Optimal λ	1.0000	1.0000	0.3906	0.6158	0.4971
	Optimal γ	0.7288	0.5947	0.2714	0.4185	0.3815
	Optimal β	0.2200	0.2794	0.0718	0.1057	-0.0778
	Optimal effort	0.0942	0.4678	0.5158	0.7037	0.8490
	RAR	0.0189	0.1060	0.0908	0.1455	0.3688

6. Conclusion

The principal-agent relationship between an entrepreneur (the agent) and a financier (the principal) is explored for three different financial contracts. The type of financial contract used has an impact on the extent of risk sharing, and the benefits thereof, and on managerial incentives. Three alternatives for an entrepreneur seeking capital to finance her project have been explored: i) risk-free debt financing; ii) equity financing with capital provided by a partner; iii) nonlinear profit-and-loss-sharing contracting with an investor. The effects of risk sharing with equity financing and nonlinear PLS contracting on the optimal effort and expected utility of the entrepreneur have been analyzed. It is assumed that the entrepreneur can influence the outcome of the project with her effort, which also results in private costs for the entrepreneur. She can, with her effort, increase the probability of a profit, subsequently decreasing the probability of a loss. The nonlinear PLS contracting considered in this paper involves sharing of profits (an increase in the value of the project) and losses (a decrease in the value of the project) based on different ratios between the entrepreneur and the investor and a fixed payment either the entrepreneur or the investor is to receive. In that case, the negative outcome is shared between the parties based on their initial capital contributions to the project. The entrepreneur and investor also agree on their shares of the positive outcome and the fixed payment.

The optimal effort of the entrepreneur as well as her risk-adjusted return (consequently, her expected utility) ultimately (either directly or indirectly by affecting the optimal contract structures) depend on four variables: 1) the ability of the entrepreneur to manage the project; 2) the risk aversion level of the outside financiers; 3) the risk aversion level of the entrepreneur; 4) the riskiness the project. Holding the other three variables constant,

- the higher the entrepreneur's level of managerial ability, the higher her optimal effort and risk-adjusted return.
- the more risk-averse the outside financier, the higher the optimal effort of the entrepreneur and the lower the risk-adjusted return.
- the more risk-averse the entrepreneur, the higher her optimal effort and risk-adjusted return.
- the riskier the project, the higher the optimal effort and risk-adjusted return.

The numerical analyses in this paper indicate that the optimal nonlinear profit-and-loss-sharing contracting as considered here is a better financing alternative than financing with debt

or equity capital. Financing with debt capital does not involve any profit and loss or risk sharing. In that case, the optimal effort of the entrepreneur is first-best since she does not have to share the outcome with an outside financier. With optimal equity financing, even though the incentives of the entrepreneur to extend effort are diminished, resulting in a lower level of effort, she will achieve a higher risk-adjusted return on the project thanks to the benefits of risk sharing. The incentives of the entrepreneur, although to a lesser extent compared to equity financing, are also diluted with nonlinear PLS contracting, which also results in a higher risk-adjusted return compared to debt financing. The entrepreneur can achieve a higher expected utility with nonlinear PLS contracting than with equity financing in certain cases depending on the share of capital provided by the investor. The optimal nonlinear profit-and-loss-sharing contracting, that is, when λ , the share of the capital the investor provides, is optimal, will be more advantageous for the entrepreneur than equity financing. The risk-adjusted return on the project is highest when it is financed via an optimal nonlinear PLS contract with an investor and lowest with debt financing.

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Appendix A. Entrepreneur's expected utility of income

The entrepreneur's income consists of her share S of the uncertain future value of the project and a fixed payment P :

$$i_E = (1 + \tilde{r})S + P. \quad (55)$$

The expected utility of this income is then

$$\begin{aligned} E\{u(i_E)\} &= \int_{-\infty}^{+\infty} -\exp\{-\theta i_E\} f(\tilde{r}) d\tilde{r} = \int_{-\infty}^{+\infty} -\exp\{-\theta[(1 + \tilde{r})S + P]\} f(\tilde{r}) d\tilde{r} \\ &= \int_{-\infty}^0 -\exp\{-\theta[(1 + \tilde{r})S_{loss} + P_{loss}]\} f(\tilde{r}) d\tilde{r} \\ &\quad + \int_0^{+\infty} -\exp\{-\theta[(1 + \tilde{r})S_{profit} + P_{profit}]\} f(\tilde{r}) d\tilde{r} \end{aligned} \quad (56)$$

Considering the probability density function $f(\tilde{r})$ from (4),

$$\begin{aligned} &\int_{-\infty}^0 -\exp\{-\theta[(1 + \tilde{r})S_{loss} + P_{loss}]\} f(\tilde{r}) d\tilde{r} \\ &= \int_{-\infty}^0 -\exp\{-\theta[(1 + \tilde{r})S_{loss} + P_{loss}]\} \frac{1-e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] d\tilde{r}, \\ &= -\frac{1-e}{\sigma\sqrt{2\pi}} \int_{-\infty}^0 \exp\left\{-\theta[(1 + \tilde{r})S_{loss} + P_{loss}] - \frac{\tilde{r}^2}{2\sigma^2}\right\} d\tilde{r} \end{aligned} \quad (57)$$

and

$$\begin{aligned} &\int_0^{+\infty} -\exp\{-\theta[(1 + \tilde{r})S_{profit} + P_{profit}]\} f(\tilde{r}) d\tilde{r} \\ &= \int_0^{+\infty} -\exp\{-\theta[(1 + \tilde{r})S_{profit} + P_{profit}]\} \frac{1+e}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] d\tilde{r}. \\ &= -\frac{1+e}{\sigma\sqrt{2\pi}} \int_{-\infty}^0 \exp\left\{-\theta[(1 + \tilde{r})S_{profit} + P_{profit}] - \frac{\tilde{r}^2}{2\sigma^2}\right\} d\tilde{r} \end{aligned} \quad (58)$$

$$\begin{aligned} \Leftrightarrow E\{u(i_E)\} &= -\frac{1-e}{\sigma\sqrt{2\pi}} \int_{-\infty}^0 \exp\left\{-\theta[(1 + \tilde{r})S_{loss} + P_{loss}] - \frac{\tilde{r}^2}{2\sigma^2}\right\} d\tilde{r} \\ &\quad - \frac{1+e}{\sigma\sqrt{2\pi}} \int_{-\infty}^0 \exp\left\{-\theta[(1 + \tilde{r})S_{profit} + P_{profit}] - \frac{\tilde{r}^2}{2\sigma^2}\right\} d\tilde{r}. \end{aligned} \quad (59)$$

The integrals were solved using the computer algebra system Maxima:

```
wxMaxima document

assume(σ>0);
[σ>0]
integrate(%e^(-θ*((1+r)·S_loss+P_loss)-r^2/(2·σ^2)),r,0,inf);


$$\frac{\sqrt{\pi} \sigma e^{-\frac{S_{\text{loss}}^2 \theta^2 \sigma^2}{2} - S_{\text{loss}} \theta - P_{\text{loss}} \theta} \operatorname{erf}\left(\frac{S_{\text{loss}} \theta \sigma}{\sqrt{2}}\right)}{\sqrt{2}} + \frac{\sqrt{\pi} \sigma e^{-\frac{S_{\text{loss}}^2 \theta^2 \sigma^2}{2} - S_{\text{loss}} \theta - P_{\text{loss}} \theta}}{\sqrt{2}}$$


integrate(%e^(-θ*((1+r)·S_profit+P_profit)-r^2/(2·σ^2)),r,0,inf);


$$\frac{\sqrt{\pi} \sigma e^{-\frac{S_{\text{profit}}^2 \theta^2 \sigma^2}{2} - S_{\text{profit}} \theta - P_{\text{profit}} \theta} \operatorname{erf}\left(\frac{S_{\text{profit}} \theta \sigma}{\sqrt{2}}\right)}{\sqrt{2}} - \frac{\sqrt{\pi} \sigma e^{-\frac{S_{\text{profit}}^2 \theta^2 \sigma^2}{2} - S_{\text{profit}} \theta - P_{\text{profit}} \theta}}{\sqrt{2}}$$

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Figure 3. wxMaxima 23.05.1 (Windows 11 (build 22621), 64-bit edition)

By simplifying the Maxima outputs and inserting them in (59), the entrepreneur's expected utility of income can be represented as

$$E\{u(i_E)\} = -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta S_{\text{loss}} \sigma}{\sqrt{2}}\right) \right] \exp\left(-\theta S_{\text{loss}} + \frac{1}{2} \theta^2 S_{\text{loss}}^2 \sigma^2 - \theta P_{\text{loss}}\right) - \frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta S_{\text{profit}} \sigma}{\sqrt{2}}\right) \right] \exp\left(-\theta S_{\text{profit}} + \frac{1}{2} \theta^2 S_{\text{profit}}^2 \sigma^2 - \theta P_{\text{profit}}\right). \quad (60)$$

In case of debt financing,

$$S_{\text{loss}}^{\text{debt}} = S_{\text{profit}}^{\text{debt}} = A_0$$

$$P_{\text{loss}}^{\text{debt}} = P_{\text{profit}}^{\text{debt}} = -(1+r_f)(A_0 - W_{0,E}). \quad (61)$$

Therefore,

$$E\{u(i_E^{\text{debt}})\} = -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta A_0 \sigma}{\sqrt{2}}\right) \right] \exp\left(-\theta A_0 + \frac{1}{2} \theta^2 A_0^2 \sigma^2 + \theta(1+r_f)(A_0 - W_{0,E})\right) - \frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta A_0 \sigma}{\sqrt{2}}\right) \right] \exp\left(-\theta A_0 + \frac{1}{2} \theta^2 A_0^2 \sigma^2 + \theta(1+r_f)(A_0 - W_{0,E})\right) - \left[1 - e \cdot \operatorname{erf}\left(\frac{\theta A_0 \sigma}{\sqrt{2}}\right) \right] \exp\left(-\theta A_0 + \frac{1}{2} \theta^2 A_0^2 \sigma^2 + \theta(1+r_f)(A_0 - W_{0,E})\right). \quad (62)$$

Based on the definition of certainty equivalent (see (13)), the certainty equivalent of entrepreneur's uncertain future income when she chooses to finance her project with risk-free debt can be derived as follows:

$$\begin{aligned}
CE(i_E^{debt}) &= -\frac{1}{\theta} \ln \left[-E \left\{ u(i_E^{debt}) \right\} \right] \\
&= -\frac{1}{\theta} \ln \left\{ - \left[1 - e \cdot \operatorname{erf} \left(\frac{\theta A_0 \sigma}{\sqrt{2}} \right) \right] \exp \left(-\theta A_0 + \frac{1}{2} \theta^2 A_0^2 \sigma^2 + \theta(1+r_f)(A_0 - W_{0,E}) \right) \right\} \\
&= -\frac{1}{\theta} \ln \left\{ - \left[1 - e \cdot \operatorname{erf} \left(\frac{\theta A_0 \sigma}{\sqrt{2}} \right) \right] \right\} \\
&\quad - \frac{1}{\theta} \ln \left\{ \exp \left(-\theta A_0 + \frac{1}{2} \theta^2 A_0^2 \sigma^2 + \theta(1+r_f)(A_0 - W_{0,E}) \right) \right\} \\
&= \frac{1}{\theta} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta A_0 \sigma}{\sqrt{2}} \right)} \right] + A_0 - \frac{1}{2} \theta A_0^2 \sigma^2 - (1+r_f)(A_0 - W_{0,E}) \\
&= \left\{ 1 - \frac{1}{2} \theta A_0 \sigma^2 + \frac{1}{\theta A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta A_0 \sigma}{\sqrt{2}} \right)} \right] \right\} A_0 - (1+r_f)(A_0 - W_{0,E}) \tag{63}
\end{aligned}$$

In case of equity financing,

$$\begin{aligned}
S_{loss}^{equity} &= S_{profit}^{equity} = (1-\alpha) A_0 \\
P_{loss}^{equity} &= P_{profit}^{equity} = (1+r_f) C_P - (1+r_f)(A_0 - W_{0,E}) \tag{64}
\end{aligned}$$

Therefore,

$$\begin{aligned}
E \left\{ u(i_E^{equity}) \right\} &= -\frac{1}{2} (1-e) \left[1 + \operatorname{erf} \left(\frac{\theta(1-\alpha) A_0 \sigma}{\sqrt{2}} \right) \right] \exp \begin{pmatrix} -\theta(1-\alpha) A_0 \\ +\frac{1}{2} \theta^2 (1-\alpha)^2 A_0^2 \sigma^2 \\ -\theta(1+r_f) C_P \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \\
&\quad - \frac{1}{2} (1+e) \left[1 - \operatorname{erf} \left(\frac{\theta(1-\alpha) A_0 \sigma}{\sqrt{2}} \right) \right] \exp \begin{pmatrix} -\theta(1-\alpha) A_0 \\ +\frac{1}{2} \theta^2 (1-\alpha)^2 A_0^2 \sigma^2 \\ -\theta(1+r_f) C_P \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \tag{65} \\
&= - \left[1 - e \cdot \operatorname{erf} \left(\frac{\theta(1-\alpha) A_0 \sigma}{\sqrt{2}} \right) \right] \exp \begin{pmatrix} -\theta(1-\alpha) A_0 + \frac{1}{2} \theta^2 (1-\alpha)^2 A_0^2 \sigma^2 \\ -\theta(1+r_f) C_P + \theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix}
\end{aligned}$$

Based on the definition of certainty equivalent (see (13)), the certainty equivalent of entrepreneur's uncertain future income when she chooses to finance her project with equity capital from a partner can be derived as follows:

$$\begin{aligned}
CE(i_E^{equity}) &= -\frac{1}{\theta} \ln \left[-E \left\{ u(i_E^{equity}) \right\} \right] \\
&= -\frac{1}{\theta} \ln \left\{ - \left[1 - e \cdot \operatorname{erf} \left(\frac{\theta(1-\alpha)A_0\sigma}{\sqrt{2}} \right) \right] \exp \left\{ \begin{array}{l} -\theta(1-\alpha)A_0 \\ +\frac{1}{2}\theta^2(1-\alpha)^2 A_0^2\sigma^2 \\ -\theta(1+r_f)C_P \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{array} \right\} \right\} \\
&= -\frac{1}{\theta} \ln \left\{ \left[1 - e \cdot \operatorname{erf} \left(\frac{\theta(1-\alpha)A_0\sigma}{\sqrt{2}} \right) \right] \right\} \\
&\quad -\frac{1}{\theta} \ln \left\{ - \exp \left\{ \begin{array}{l} -\theta(1-\alpha)A_0 + \frac{1}{2}\theta^2(1-\alpha)^2 A_0^2\sigma^2 \\ -\theta(1+r_f)C_P + \theta(1+r_f)(A_0 - W_{0,E}) \end{array} \right\} \right\} \\
&= \frac{1}{\theta} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta(1-\alpha)A_0\sigma}{\sqrt{2}} \right)} \right] \\
&\quad + (1-\alpha)A_0 - \frac{1}{2}\theta(1-\alpha)^2 A_0^2\sigma^2 + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}) \\
&= \left\{ (1-\alpha) - \frac{1}{2}\theta(1-\alpha)^2 \sigma^2 + \frac{1}{\theta A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\theta(1-\alpha)A_0\sigma}{\sqrt{2}} \right)} \right] \right\} A_0 \\
&\quad + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}) \tag{66}
\end{aligned}$$

In case of nonlinear PLS contracting,

$$\begin{aligned}
S_{loss}^{PLS} &= (1-\lambda)A_0 \\
S_{profit}^{PLS} &= (1-\gamma)A_0 \\
P_{loss}^{PLS} = P_{profit}^{PLS} &= -\beta A_0 + (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E}) \tag{67}
\end{aligned}$$

Therefore,

$$E\{u(i_E^{PLS})\} = -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta(1-\lambda)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\lambda)A_0 + \frac{1}{2}\theta^2(1-\lambda)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 - \theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \\ -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta(1-\gamma)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\gamma)A_0 + \frac{1}{2}\theta^2(1-\gamma)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 - \theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix}. \quad (68)$$

Based on the definition of certainty equivalent (see (13)), the certainty equivalent of entrepreneur's uncertain future income when she chooses to finance her project via nonlinear PLS arrangement with an investor can be derived as follows:

$$CE(i_E^{PLS}) = -\frac{1}{\theta} \ln \left[-E\{u(i_E^{PLS})\} \right] \\ = -\frac{1}{\theta} \ln \left\{ \begin{array}{l} -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta(1-\lambda)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\lambda)A_0 \\ +\frac{1}{2}\theta^2(1-\lambda)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 - \theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \\ -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta(1-\gamma)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\gamma)A_0 \\ +\frac{1}{2}\theta^2(1-\gamma)^2 A_0^2\sigma^2 \\ +\theta\beta A_0 - \theta(1+r_f)C_I \\ +\theta(1+r_f)(A_0 - W_{0,E}) \end{pmatrix} \end{array} \right\} \\ = -\frac{1}{\theta} \ln \left\{ \begin{array}{l} -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\theta(1-\lambda)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\lambda)A_0 \\ +\frac{1}{2}\theta^2(1-\lambda)^2 A_0^2\sigma^2 \end{pmatrix} \\ -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\theta(1-\gamma)A_0\sigma}{\sqrt{2}}\right) \right] \exp \begin{pmatrix} -\theta(1-\gamma)A_0 \\ +\frac{1}{2}\theta^2(1-\gamma)^2 A_0^2\sigma^2 \end{pmatrix} \end{array} \right\} \\ -B + (1+r_f)C_I - (1+r_f)(A_0 - W_{0,E}) \quad (69)$$

Appendix B. Entrepreneur's optimal effort

The entrepreneur's certainty equivalent of her uncertain future wealth in case of debt financing is

$$\begin{aligned}
 CE_E^{debt} &= CE(i_E^{debt}) - c(e) \\
 &= \left\{ 1 - \frac{1}{2} \omega \sigma^2 + \frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell \omega} \ln \left(\frac{1}{1 - e^2} \right) \right\} A_0, \quad \text{see (17)} \\
 &\quad - (1 + r_f)(A_0 - W_{0,E})
 \end{aligned}$$

which is maximized by the entrepreneur's choice of optimal effort:

$$\begin{aligned}
 \frac{dCE_E^{debt}}{de} &= \left\{ \frac{\operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}{\omega \left[1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right) \right]} - \frac{e}{\ell \omega (1 - e^2)} \right\} A_0 = 0 \\
 \Leftrightarrow \frac{\operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} &= \frac{e}{\ell (1 - e^2)} \quad (70) \\
 \Leftrightarrow e &= \frac{-1 \pm \sqrt{1 + 4\ell(\ell - 1) \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)}}{2(\ell - 1) \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}
 \end{aligned}$$

Since $\ell > 1$, $\frac{-1 - \sqrt{1 + 4\ell(\ell - 1) \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)}}{2(\ell - 1) \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} < 0$. Therefore, the entrepreneur's optimal effort

when she finances her project with debt capital is $e = \frac{\sqrt{1 + 4\ell(\ell - 1) \operatorname{erf}^2 \left(\frac{\omega \sigma}{\sqrt{2}} \right)} - 1}{2(\ell - 1) \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}$ (see (18)).

When $a = 0 \Leftrightarrow \ell = 1$, the certainty equivalent from (17) becomes

$$CE_E^{debt} = \left\{ 1 - \frac{1}{2} \omega \sigma^2 + \frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\omega} \ln \left(\frac{1}{1 - e^2} \right) \right\} A_0 - (1 + r_f)(A_0 - W_{0,E}). \quad (71)$$

The entrepreneur with no managerial ability then chooses her optimal effort to maximize her certainty equivalent from (71):

$$\begin{aligned} \frac{dCE_E^{debt}}{de} &= \left\{ \frac{\operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}{\omega \left[1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right) \right]} - \frac{e}{\omega(1 - e^2)} \right\} A_0 = 0 \\ &\Leftrightarrow \frac{\operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)}{1 - e \cdot \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right)} = \frac{e}{1 - e^2} \\ &\Leftrightarrow e - \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right) = 0 \\ &\Leftrightarrow e = \operatorname{erf} \left(\frac{\omega \sigma}{\sqrt{2}} \right) \end{aligned} \quad (72)$$

With equity financing, the entrepreneur's certainty equivalent of her uncertain future wealth is

$$\begin{aligned} CE_E^{equity} &= CE(i_E^{equity}) - c(e) \\ &= \left\{ (1 - \alpha) - \frac{1}{2} \omega (1 - \alpha)^2 \sigma^2 \right. \\ &\quad \left. + \frac{1}{\omega} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\omega(1 - \alpha)\sigma}{\sqrt{2}} \right)} \right] - \frac{1}{2\ell\omega} \ln \left(\frac{1}{1 - e^2} \right) \right\} A_0, \quad \text{see (29)} \\ &\quad + (1 + r_f)C_P - (1 + r_f)(A_0 - W_{0,E}) \end{aligned}$$

which is maximized by the entrepreneur's choice of optimal effort:

$$\begin{aligned}
 \frac{dCE_E^{equity}}{de} &= \left\{ \frac{\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}{\omega\left[1-e \cdot \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)\right]} - \frac{e}{\ell\omega(1-e^2)} \right\} A_0 = 0 \\
 &\Leftrightarrow \frac{\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}{1-e \cdot \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} = \frac{e}{\ell(1-e^2)} \quad (73) \\
 &\Leftrightarrow e = \frac{-1 \pm \sqrt{1+4\ell(\ell-1)\operatorname{erf}^2\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}}{2(\ell-1)\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}
 \end{aligned}$$

Since $\ell > 1$, $\frac{-1 - \sqrt{1+4\ell(\ell-1)\operatorname{erf}^2\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}}{2(\ell-1)\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} < 0$. Therefore, the entrepreneur's optimal

effort when she finances her project with equity capital is

$$e = \frac{\sqrt{1+4\ell(\ell-1)\operatorname{erf}^2\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} - 1}{2(\ell-1)\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} \quad (\text{see (30)}).$$

When $a = 0 \Leftrightarrow \ell = 1$, the certainty equivalent from (29) becomes

$$\begin{aligned}
 CE_E^{equity} &= \left\{ \begin{aligned} &(1-\alpha) - \frac{1}{2}\omega(1-\alpha)^2\sigma^2 \\ &+ \frac{1}{\omega} \ln \left[\frac{1}{1-e \cdot \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} \right] - \frac{1}{2\omega} \ln \left(\frac{1}{1-e^2} \right) \end{aligned} \right\} A_0. \quad (74) \\
 &+ (1+r_f)C_P - (1+r_f)(A_0 - W_{0,E})
 \end{aligned}$$

The entrepreneur with no managerial ability then chooses her optimal effort to maximize her certainty equivalent from (74):

$$\begin{aligned}
\frac{dCE_E^{equity}}{de} &= \left\{ \frac{\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}{\omega\left[1-e \cdot \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)\right]} - \frac{e}{\omega(1-e^2)} \right\} A_0 = 0 \\
&\Leftrightarrow \frac{\operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)}{1-e \cdot \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)} = \frac{e}{1-e^2} \quad (75) \\
&\Leftrightarrow e - \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right) = 0 \\
&\Leftrightarrow e = \operatorname{erf}\left(\frac{\omega(1-\alpha)\sigma}{\sqrt{2}}\right)
\end{aligned}$$

The entrepreneur's certainty equivalent of her uncertain future wealth in case of nonlinear PLS contracting is

$$\begin{aligned}
CE_E^{PLS} &= CE(i_E^{PLS}) - c(e) \\
&= \left[-\frac{1}{\omega} \ln \left\{ -\frac{1}{2}(1-e)L - \frac{1}{2}(1+e)G \right\} - \beta - \frac{1}{2\ell\omega} \ln \left(\frac{1}{1-e^2} \right) \right] A_0, \quad \text{see (48)} \\
&\quad + (1+r_f)C_I + (1+r_f)(A_0 - W_{0,E})
\end{aligned}$$

which is maximized by the entrepreneur's choice of optimal effort:

$$\begin{aligned}
\frac{dCE_E^{PLS}}{de} &= \left\{ \frac{\frac{L}{2} - \frac{G}{2}}{\omega\left[\frac{1}{2}(1-e)L + \frac{1}{2}(1+e)G\right]} - \frac{e}{\ell\omega(1-e^2)} \right\} A_0 = 0 \\
&\Leftrightarrow \frac{L-G}{(1-e)L + (1+e)G} = \frac{e}{\ell(1-e^2)} \quad (76) \\
&\Leftrightarrow e = \frac{-(L+G) \pm \sqrt{(L+G)^2 + 4\ell(\ell-1)(L-G)^2}}{2(\ell-1)(L-G)}
\end{aligned}$$

Since $\ell > 1$ and $L < G < 0$, $\frac{-(L+G) + \sqrt{(L+G)^2 + 4\ell(\ell-1)(L-G)^2}}{2(\ell-1)(L-G)} < 0$. Therefore, the

entrepreneur's optimal effort when she finances her project via nonlinear PLS contracting is

$$e = \frac{\sqrt{(L+G)^2 + 4\ell(\ell-1)(L-G)^2} + (L+G)}{2(\ell-1)(-L+G)} \quad (\text{see (49)}).$$

When $a = 0 \Leftrightarrow \ell = 1$, the certainty equivalent from (48) becomes

$$CE_E^{PLS} = \left[-\frac{1}{\omega} \ln \left\{ -\frac{1}{2}(1-e)L - \frac{1}{2}(1+e)G \right\} - \beta - \frac{1}{2\omega} \ln \left(\frac{1}{1-e^2} \right) \right] A_0 + (1+r_f)C_I + (1+r_f)(A_0 - W_{0,E}) \quad (77)$$

The entrepreneur with no managerial ability then chooses her optimal effort to maximize her certainty equivalent from (77):

$$\begin{aligned} \frac{dCE_E^{PLS}}{de} &= \left\{ \frac{\frac{L}{2} - \frac{G}{2}}{\omega \left[\frac{1}{2}(1-e)L + \frac{1}{2}(1+e)G \right]} - \frac{e}{\omega(1-e^2)} \right\} A_0 = 0 \\ &\Leftrightarrow \frac{L-G}{(1-e)L + (1+e)G} = \frac{e}{1-e^2} \\ &\Leftrightarrow e(L+G) - (L-G) = 0 \\ &\Leftrightarrow e = \frac{L-G}{L+G} \end{aligned} \quad (78)$$

Appendix C. Participation constraints of risk-averse financiers

A potential outside financier's utility function is given by

$$V(\tilde{W}_{1,F}) = -\exp(-\tau_F \tilde{W}_{1,F}), \quad (79)$$

where $\tilde{W}_{1,F}$ is the financier's uncertain future wealth, and $\tau_F = -\frac{V''(\tilde{W}_{1,F})}{V'(\tilde{W}_{1,F})}$ is his constant

absolute risk aversion coefficient (Pratt, 1964). In the formulae below, $o_p = \tau_p A_0$ and $o_I = \tau_I A_0$ are, respectively, the equity partner's and the PLS investor's modified risk aversion coefficients.

The risk-averse financier expects the same level of expected utility from his participation in the entrepreneur's project as his outside option (i.e., the risk-free investment). The financier's certainty equivalent of the risk-free investment is $(1+r_f)W_{0,F}$. The future wealth of the financier consists of his share of the outcome from the project and proceeds from leftover capital invested at the risk-free rate of return. The certainty equivalent of the financier's uncertain future wealth can be derived both with and without truncating the normal distribution function of the rate of return \tilde{r} . For the theoretical and numerical analyses in this paper, no truncation was applied.

The rate of return on the investment has the probability density function

$$f(\tilde{r}) = \frac{1 + e^{\frac{\tilde{r}}{|\tilde{r}|}}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right]. \quad \text{see (3)}$$

The rate of return on the investment with truncated normal distribution ($-1 \leq \tilde{r} \leq 1$) has the probability density function

$$f(\tilde{r}) = \begin{cases} \frac{1 + e^{\frac{\tilde{r}}{|\tilde{r}|}}}{\text{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)\sigma\sqrt{2\pi}} \exp\left[-\frac{\tilde{r}^2}{2\sigma^2}\right] & \text{if } -1 \leq \tilde{r} \leq 1 \\ 0 & \text{otherwise} \end{cases}. \quad (80)$$

In the following formulae, $\mathcal{X}_{\{\tilde{r}<0\}} = \begin{cases} 1 & \text{if } \tilde{r} < 0 \\ 0 & \text{if } \tilde{r} > 0 \end{cases}$ and $\mathcal{X}_{\{\tilde{r}>0\}} = \begin{cases} 1 & \text{if } \tilde{r} > 0 \\ 0 & \text{if } \tilde{r} < 0 \end{cases}$ are

indicator functions.

Equity financing (normal distribution of rate of return)

With the probability distribution function (3), the expected utility of the equity partner with initial wealth $W_{0,P}$, who provides capital C_P entitling him to share α of the outcome, can be determined as

$$E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} < 0\}} = \int_{-\infty}^0 -\exp\left\{-\tau_P \left[\alpha(1+\tilde{r})A_0 + (1+r_f)(W_{0,P} - C_P)\right]\right\} f(\tilde{r}) d\tilde{r} \quad , \quad (81)$$

$$= -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_P \alpha A_0 \\ +\frac{1}{2} \tau_P^2 \alpha^2 A_0^2 \sigma^2 \\ -\tau_P (1+r_f)(W_{0,P} - C_P) \end{array}\right\}$$

$$E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} > 0\}} = \int_0^{+\infty} -\exp\left\{-\tau_P \left[\alpha(1+\tilde{r})A_0 + (1+r_f)(W_{0,P} - C_P)\right]\right\} f(\tilde{r}) d\tilde{r} \quad , \quad (82)$$

$$= -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_P \alpha A_0 \\ +\frac{1}{2} \tau_P^2 \alpha^2 A_0^2 \sigma^2 \\ -\tau_P (1+r_f)(W_{0,P} - C_P) \end{array}\right\}$$

and

$$E[V(\tilde{W}_{1,P})] = E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} < 0\}} + E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} > 0\}} = \quad (83)$$

$$= -\left[1 - e \cdot \operatorname{erf}\left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_P \alpha A_0 \\ +\frac{1}{2} \tau_P^2 \alpha^2 A_0^2 \sigma^2 \\ -\tau_P (1+r_f)(W_{0,P} - C_P) \end{array}\right\}$$

in case of loss, profit, and in total, respectively.

The certainty equivalent of the equity partner in case he decides to invest in the entrepreneur's project should equal his future wealth in case he would invest his funds risk-free in the market:

$$CE_P = \left\{ \begin{array}{l} \alpha - \frac{1}{2} \tau_P \alpha^2 A_0 \sigma^2 \\ + \frac{1}{\tau_P A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}} \right)} \right] \end{array} \right\} A_0 + (1 + r_f)(W_{0,P} - C_P) \quad (84)$$

$$= (1 + r_f) W_{0,P}$$

The participation constraint of the equity partner determines how much capital he is willing to provide:

$$C_P = \frac{1}{1 + r_f} \left\{ \alpha - \frac{1}{2} \tau_P \alpha^2 \sigma^2 + \frac{1}{\tau_P A_0} \ln \left[\frac{1}{1 - e \cdot \operatorname{erf} \left(\frac{\tau_P \alpha \sigma}{\sqrt{2}} \right)} \right] \right\} A_0 \quad (85)$$

Equity financing (truncated normal distribution of rate of return)

With the probability distribution function (80), the expected utility of the equity partner with initial wealth $W_{0,P}$, who provides capital C_P entitling him to share α of the outcome, can be determined as

$$E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} < 0\}} = \int_{-1}^0 -\exp\left\{-\tau_P \left[\alpha(1 + \tilde{r})A_0 + (1 + r_f)(W_{0,P} - C_P) \right]\right\} f(\tilde{r}) d\tilde{r} \quad (86)$$

$$= -\frac{1 - e}{2 \operatorname{erf} \left(\frac{1}{\sigma \sqrt{2}} \right)} \left[\frac{\operatorname{erf} \left(\frac{1 - \tau_P \alpha A_0 \sigma^2}{\sigma \sqrt{2}} \right)}{+ \operatorname{erf} \left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}} \right)} \right] \exp \left\{ \begin{array}{l} -\tau_P \alpha A_0 \\ + \frac{1}{2} \tau_P^2 \alpha^2 A_0^2 \sigma^2 \\ -\tau_P (1 + r_f)(W_{0,P} - C_P) \end{array} \right\}$$

$$E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} > 0\}} = \int_0^1 -\exp\left\{-\tau_P \left[\alpha(1 + \tilde{r})A_0 + (1 + r_f)(W_{0,P} - C_P) \right]\right\} f(\tilde{r}) d\tilde{r} \quad (87)$$

$$= -\frac{1 + e}{2 \operatorname{erf} \left(\frac{1}{\sigma \sqrt{2}} \right)} \left[\frac{\operatorname{erf} \left(\frac{1 + \tau_P \alpha A_0 \sigma^2}{\sigma \sqrt{2}} \right)}{-\operatorname{erf} \left(\frac{\tau_P \alpha A_0 \sigma}{\sqrt{2}} \right)} \right] \exp \left\{ \begin{array}{l} -\tau_P \alpha A_0 \\ + \frac{1}{2} \tau_P^2 \alpha^2 A_0^2 \sigma^2 \\ -\tau_P (1 + r_f)(W_{0,P} - C_P) \end{array} \right\}$$

and

$$\begin{aligned}
 E[V(\tilde{W}_{1,P})] &= E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} < 0\}} + E[V(\tilde{W}_{1,P})] \cdot \chi_{\{\tilde{r} > 0\}} = \\
 &= -\frac{1}{\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\begin{array}{l} \frac{1}{2}(1+e)\operatorname{erf}\left(\frac{1+\tau_p\alpha A_0\sigma^2}{\sigma\sqrt{2}}\right) \\ +\frac{1}{2}(1-e)\operatorname{erf}\left(\frac{1-\tau_p\alpha A_0\sigma^2}{\sigma\sqrt{2}}\right) \\ -e \cdot \operatorname{erf}\left(\frac{\tau_p\alpha A_0\sigma}{\sqrt{2}}\right) \end{array} \right] \exp \left\{ \begin{array}{l} -\tau_p\alpha A_0 \\ +\frac{1}{2}\tau_p^2\alpha^2 A_0^2\sigma^2 \\ -\tau_p(1+r_f)(W_{0,P}-C_P) \end{array} \right\}
 \end{aligned} \tag{88}$$

in case of loss, profit, and in total, respectively.

The certainty equivalent of the equity partner in case he decides to invest in the entrepreneur's project should equal his future wealth in case he would invest his funds risk-free in the market:

$$\begin{aligned}
 CE_p &= \left\{ \begin{array}{l} \alpha - \frac{1}{2}\tau_p\alpha^2 A_0\sigma^2 + \frac{1}{\tau_p A_0} \ln \left[\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right) \right] \\ -\frac{1}{\tau_p A_0} \ln \left[\begin{array}{l} \frac{1}{2}(1+e)\operatorname{erf}\left(\frac{1+\tau_p\alpha A_0\sigma^2}{\sigma\sqrt{2}}\right) \\ +\frac{1}{2}(1-e)\operatorname{erf}\left(\frac{1-\tau_p\alpha A_0\sigma^2}{\sigma\sqrt{2}}\right) \\ -e \cdot \operatorname{erf}\left(\frac{\tau_p\alpha A_0\sigma}{\sqrt{2}}\right) \end{array} \right] \end{array} \right\} A_0 + (1+r_f)(W_{0,P}-C_P), \\
 &= (1+r_f)W_{0,P}
 \end{aligned} \tag{89}$$

The participation constraint of the equity partner determines how much capital he is willing to provide:

$$\begin{aligned}
 C_P &= \frac{1}{1+r_f} \left\{ \begin{array}{l} \alpha - \frac{1}{2}o_p\alpha^2\sigma^2 \\ +\frac{1}{o_p} \ln \left[\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right) \right] - \frac{1}{o_p} \ln \left[\begin{array}{l} \frac{1}{2}(1+e)\operatorname{erf}\left(\frac{1+o_p\alpha\sigma^2}{\sigma\sqrt{2}}\right) \\ +\frac{1}{2}(1-e)\operatorname{erf}\left(\frac{1-o_p\alpha\sigma^2}{\sigma\sqrt{2}}\right) \\ -e \cdot \operatorname{erf}\left(\frac{o_p\alpha\sigma}{\sqrt{2}}\right) \end{array} \right] \end{array} \right\} A_0.
 \end{aligned} \tag{90}$$

Nonlinear PLS (normal distribution of rate of return)

With the probability distribution function (3), the expected utility of the PLS investor with initial wealth $W_{0,I}$, who provides capital $C_I = \lambda A_0$ making him liable for share λ of the losses (and possibly, the fixed payment; $B = \beta A_0 < 0$) and entitling him to share γ of profits (and possibly, the fixed payment; $B = \beta A_0 > 0$) can be determined as

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} < 0\}} &= \int_{-\infty}^0 -\exp\left\{-\tau_I \left[\lambda(1+\tilde{r})A_0 + B + (1+r_f)(W_{0,I} - C_I)\right]\right\} f(\tilde{r}) d\tilde{r} \\
 &= -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\tau_I \lambda A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_I \lambda A_0 \\ +\frac{1}{2} \tau_I^2 \lambda^2 A_0^2 \sigma^2 \\ -\tau_I B - \tau_I (1+r_f)(W_{0,I} - C_I) \end{array}\right\}
 \end{aligned} \tag{91}$$

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} > 0\}} &= \int_0^{+\infty} -\exp\left\{-\tau_I \left[\gamma(1+\tilde{r})A_0 + B + (1+r_f)(W_{0,I} - C_I)\right]\right\} f(\tilde{r}) d\tilde{r} \\
 &= -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\tau_I \gamma A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_I \gamma A_0 \\ +\frac{1}{2} \tau_I^2 \gamma^2 A_0^2 \sigma^2 \\ -\tau_I B - \tau_I (1+r_f)(W_{0,I} - C_I) \end{array}\right\}
 \end{aligned} \tag{92}$$

and

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] &= E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} < 0\}} + E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} > 0\}} = \\
 &= -\frac{1}{2}(1-e) \left[1 + \operatorname{erf}\left(\frac{\tau_I \lambda A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_I \lambda A_0 \\ +\frac{1}{2} \tau_I^2 \lambda^2 A_0^2 \sigma^2 \\ -\tau_I B - \tau_I (1+r_f)(W_{0,I} - C_I) \end{array}\right\} \\
 &\quad -\frac{1}{2}(1+e) \left[1 - \operatorname{erf}\left(\frac{\tau_I \gamma A_0 \sigma}{\sqrt{2}}\right)\right] \exp\left\{\begin{array}{l} -\tau_I \gamma A_0 \\ +\frac{1}{2} \tau_I^2 \gamma^2 A_0^2 \sigma^2 \\ -\tau_I B - \tau_I (1+r_f)(W_{0,I} - C_I) \end{array}\right\}
 \end{aligned} \tag{93}$$

in case of loss, profit, and in total, respectively.

The certainty equivalent of the PLS investor in case he decides to invest in the entrepreneur's project should equal his future wealth in case he would invest his funds risk-free in the market:

$$\begin{aligned}
 CE_I &= -\frac{1}{\tau_I A_0} \ln \left\{ \begin{aligned} &\frac{1}{2}(1-e) \left[1 + \operatorname{erf} \left(\frac{\tau_I \lambda A_0 \sigma}{\sqrt{2}} \right) \right] \exp \left(-\tau_I \lambda A_0 + \frac{1}{2} \tau_I^2 \lambda^2 A_0^2 \sigma^2 \right) \\ &+ \frac{1}{2}(1+e) \left[1 - \operatorname{erf} \left(\frac{\tau_I \gamma A_0 \sigma}{\sqrt{2}} \right) \right] \exp \left(-\tau_I \gamma A_0 + \frac{1}{2} \tau_I^2 \gamma^2 A_0^2 \sigma^2 \right) \end{aligned} \right\} A_0 \\
 &+ B + (1+r_f)(W_{0,I} - C_I) \\
 &= (1+r_f)W_{0,I}
 \end{aligned} \quad . \quad (94)$$

The participation constraint of the PLS investor determines the fixed payment:

$$B = \left\{ (1+r_f)\lambda + \frac{1}{o_I} \ln \left[\begin{aligned} &\frac{1}{2}(1-e) \left[1 + \operatorname{erf} \left(\frac{o_I \lambda \sigma}{\sqrt{2}} \right) \right] \exp \left(-o_I \lambda + \frac{1}{2} o_I^2 \lambda^2 \sigma^2 \right) \\ &+ \frac{1}{2}(1+e) \left[1 - \operatorname{erf} \left(\frac{o_I \gamma \sigma}{\sqrt{2}} \right) \right] \exp \left(-o_I \gamma + \frac{1}{2} o_I^2 \gamma^2 \sigma^2 \right) \end{aligned} \right] \right\} A_0. \quad (95)$$

Nonlinear PLS (truncated normal distribution of rate of return)

With the probability distribution function (80), the expected utility of the PLS investor with initial wealth $W_{0,I}$, who provides capital $C_I = \lambda A_0$ making him liable for share λ of the losses (and possibly, the fixed payment $B = \beta A_0 < 0$) and entitling him to share γ of profits (and possibly, the fixed payment $B = \beta A_0 > 0$) can be determined as

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} \leq 0\}} &= \int_{-1}^0 -\exp \left\{ -\tau_I \left[\lambda(1+\tilde{r})A_0 + B + (1+r_f)(W_{0,I} - C_I) \right] \right\} f(\tilde{r}) d\tilde{r} \\
 &= -\frac{1-e}{2 \operatorname{erf} \left(\frac{1}{\sigma \sqrt{2}} \right)} \left[\begin{aligned} &\operatorname{erf} \left(\frac{1-\tau_I \lambda A_0 \sigma^2}{\sigma \sqrt{2}} \right) \\ &+ \operatorname{erf} \left(\frac{\tau_I \lambda A_0 \sigma}{\sqrt{2}} \right) \end{aligned} \right] \exp \left\{ \begin{aligned} &-\tau_I \alpha A_0 \\ &+ \frac{1}{2} \tau_I^2 \lambda^2 A_0^2 \sigma^2 \\ &-\tau_I B \\ &-\tau_I (1+r_f)(W_{0,I} - C_I) \end{aligned} \right\}
 \end{aligned} \quad , \quad (96)$$

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} \leq 0\}} &= \int_{-1}^0 -\exp\left\{-\tau_I \left[\lambda(1+\tilde{r})A_0 + B + (1+r_f)(W_{0,I} - C_I)\right]\right\} f(\tilde{r}) d\tilde{r} \\
 &= -\frac{1-e}{2\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\operatorname{erf}\left(\frac{1-\tau_I\lambda A_0\sigma^2}{\sigma\sqrt{2}}\right) \right. \\
 &\quad \left. + \operatorname{erf}\left(\frac{\tau_I\lambda A_0\sigma}{\sqrt{2}}\right) \right] \exp\left\{ \begin{array}{l} -\tau_I\alpha A_0 \\ +\frac{1}{2}\tau_I^2\lambda^2 A_0^2\sigma^2 \\ -\tau_I B \\ -\tau_I(1+r_f)(W_{0,I} - C_I) \end{array} \right\}, \quad (97)
 \end{aligned}$$

and

$$\begin{aligned}
 E[V(\tilde{W}_{1,I})] &= E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} \leq 0\}} + E[V(\tilde{W}_{1,I})] \cdot \chi_{\{\tilde{r} > 0\}} = \\
 &= -\frac{1-e}{2\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\operatorname{erf}\left(\frac{1-\tau_I\lambda A_0\sigma^2}{\sigma\sqrt{2}}\right) \right. \\
 &\quad \left. + \operatorname{erf}\left(\frac{\tau_I\lambda A_0\sigma}{\sqrt{2}}\right) \right] \exp\left\{ \begin{array}{l} -\tau_I\lambda A_0 \\ +\frac{1}{2}\tau_I^2\lambda^2 A_0^2\sigma^2 \\ -\tau_I B \\ -\tau_I(1+r_f)(W_{0,I} - C_I) \end{array} \right\} \\
 &\quad - \frac{1+e}{2\operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\operatorname{erf}\left(\frac{1+\tau_I\gamma A_0\sigma^2}{\sigma\sqrt{2}}\right) \right. \\
 &\quad \left. - \operatorname{erf}\left(\frac{\tau_I\gamma A_0\sigma}{\sqrt{2}}\right) \right] \exp\left\{ \begin{array}{l} -\tau_I\gamma A_0 \\ +\frac{1}{2}\tau_I^2\gamma^2 A_0^2\sigma^2 \\ -\tau_I B \\ -\tau_I(1+r_f)(W_{0,I} - C_I) \end{array} \right\} \quad (98)
 \end{aligned}$$

in case of loss, profit, and in total, respectively.

The certainty equivalent of the PLS investor in case he decides to invest in the entrepreneur's project should equal his future wealth in case he would invest his funds risk-free in the market:

$$\begin{aligned}
 CE_I = & -\frac{1}{\tau_I A_0} \ln \left(\frac{1-e}{2 \operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\frac{\operatorname{erf}\left(\frac{1-\tau_I \lambda A_0 \sigma^2}{\sigma\sqrt{2}}\right)}{+\operatorname{erf}\left(\frac{\tau_I \lambda A_0 \sigma}{\sqrt{2}}\right)} \right] \exp \left\{ \begin{array}{l} -\tau_I \lambda A_0 + \frac{1}{2} \tau_I^2 \lambda^2 A_0^2 \sigma^2 \\ -\tau_I B \\ -\tau_I (1+r_f)(W_{0,I} - C_I) \end{array} \right\} \right) \\
 & + \frac{1+e}{2 \operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\frac{\operatorname{erf}\left(\frac{1+\tau_I \gamma A_0 \sigma^2}{\sigma\sqrt{2}}\right)}{-\operatorname{erf}\left(\frac{\tau_I \gamma A_0 \sigma}{\sqrt{2}}\right)} \right] \exp \left\{ \begin{array}{l} -\tau_I \gamma A_0 + \frac{1}{2} \tau_I^2 \gamma^2 A_0^2 \sigma^2 \\ -\tau_I B \\ -\tau_I (1+r_f)(W_{0,I} - C_I) \end{array} \right\} \\
 & + B + (1+r_f)(W_{0,I} - C_I) \\
 = & (1+r_f)W_{0,I}
 \end{aligned} \tag{99}$$

The participation constraint of the PLS investor determines the fixed payment:

$$B = \left\{ (1+r_f)\lambda + \frac{1}{o_I} \ln \left(\frac{(1-e)}{2 \operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\frac{\operatorname{erf}\left(\frac{1-o_I \lambda \sigma^2}{\sigma\sqrt{2}}\right)}{+\operatorname{erf}\left(\frac{o_I \lambda \sigma}{\sqrt{2}}\right)} \right] \exp \left\{ -o_I \lambda + \frac{1}{2} o_I^2 \lambda^2 \sigma^2 \right\} \right) \right. \\
 \left. + \frac{(1+e)}{2 \operatorname{erf}\left(\frac{1}{\sigma\sqrt{2}}\right)} \left[\frac{\operatorname{erf}\left(\frac{1+o_I \gamma \sigma^2}{\sigma\sqrt{2}}\right)}{-\operatorname{erf}\left(\frac{o_I \gamma \sigma}{\sqrt{2}}\right)} \right] \exp \left\{ -o_I \gamma + \frac{1}{2} o_I^2 \gamma^2 \sigma^2 \right\} \right\} A_0. \tag{100}$$

Bank Loan Credit Risk Pooling: Risk Diversification versus Moral Hazard

Bank Loan Credit Risk Pooling

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Posted on SSRN: <https://ssrn.com/abstract=5815622>

Abstract

Banks aim to manage their exposure to credit risk, which consists of default risk and recovery risk, with credit risk transfer instruments such as loan sales and credit derivatives. Credit risk pooling, with which banks pool their loans together and share the credit returns, is an alternative to credit risk transfer. Both credit risk transfer and credit risk pooling result in a trade-off between the benefits of risk transfer/sharing and the adverse effects of moral hazard problem, that is, diminished incentives to monitor loans whose credit risks have been completely or partially transferred outside. This paper presents a credit risk pooling arrangement that not only eliminates the moral hazard problem associated with credit risk sharing, but also improves the risk-return profiles of all the participating banks via diversification of recovery risk. The diversification effect of pooling lowers the probability of bank failure for each participating bank.

Keywords: bank loans, credit risk, credit risk transfer, credit risk pooling, default risk, recovery risk, risk diversification, moral hazard problem, costly monitoring, nonlinear risk sharing

1. Introduction

The credit risk of a bank loan can be characterized by default risk and recovery risk. Default risk refers to the likelihood that a borrower fails to fully repay principal and interest, resulting in financial loss for the bank. The recovery risk determines the magnitude of this loss following the default event. Even though the likelihood of a default is small, the losses a bank would incur if a borrower defaults can be significantly large. Credit risk contributes to a bank's insolvency risk, which is limited by the amount of equity capital available to absorb losses. Therefore, the capital structure of the bank should account for the credit risk of its loan portfolio. The equity that backs obligations to liability holders is a firm's risk capital (Erel et al., 2015, 2021). Regulatory capital refers to the amount of capital a bank is required to hold by a financial regulator. Bank regulation aims to increase financial stability by reducing the probability of bank failure, which depends on the riskiness of the bank's portfolio (Koehn & Santomero, 1980). Economic capital, on the other hand, is a measure of risk, and it is defined as the amount of risk capital a bank requires to remain solvent given the riskiness of its credit portfolio. A relatively high economic capital (given the bank's equity capital) may exacerbate the insolvency risk and result in a lower firm value. This underscores the importance of managing credit risk for a bank.

Banks can limit their exposure to credit risk with credit risk transfer instruments. The most straightforward way of transferring credit risk is loan sales. Credit derivatives such as credit default swaps, total return swaps, collateralized debt obligations, and credit-linked notes, which represent a more flexible alternative to loan sales, are used in portfolio management by banks in the European Union for risk shedding (protection buying) or risk taking (protection selling) depending on their risk management needs and strategies (ECB, 2004). Another method of credit risk management used by German regional banks, namely, credit risk pooling, is considered in this paper. In a risk pooling arrangement, bank loans are pooled together, and the credit returns are shared among the participating banks. Such an arrangement represents a trade-off between the benefits of risk diversification, which lowers the individual credit risks of the banks, and the adverse effects of the moral hazard problem, that is, the diminished incentives of the participating banks to monitor the loans they have contributed to the pool. The extent of the advantages and disadvantages of pooling would then mainly depend on how the risks are pooled and subsequently shared, meaning, what share of their credit risks banks contribute to the pool and what share of the credit returns from the pool banks receive.

A nonlinear risk-sharing arrangement between the banks participating in a credit risk pool, where the share of the credit returns each bank receives depends on its contribution to the pool and whether a default event occurs, is analyzed theoretically in this paper. An optimal sharing rule that eliminates the moral hazard problem is derived. In such an arrangement, the share of the return each bank in the pool receives from their own loan is different depending on whether default on the loan originated by the bank has occurred. The benefit of pooling is realized via diversification of recovery risk with the optimal sharing rule. Furthermore, it is shown that there is an optimal share of credit risk each bank contributes to the pool that maximizes the risk diversification benefits for the participating banks.

The rest of this paper is organized as follows. Section 2 provides a review of literature pertaining to credit risk transfer. In section 3, the baseline model of a single bank loan used for the analyses in this paper is presented, and the determinants of credit risk are defined. Credit risk pooling with first two, then three loans is analyzed in section 4. The analyses from section 4 are generalized in section 5 for multi-loan pooling arrangements, and the optimal pooling contract with risk diversification benefits for the participating banks and without the diminished incentives of the banks to monitor their loans is presented. Section 6 provides concluding remarks.

2. Literature Review: Credit Risk Transfer

Banks are interested in not only the return (i.e., the interest) they receive on a loan, but also its riskiness. The interest rate charged on a loan might, however, affect its riskiness. According to Stiglitz and Weiss (1981), interest rates act as both a screening device (by eliminating high-risk borrowers) and an incentive mechanism (by inducing the desired behavior from borrowers). They argue that it is usually the high-risk borrowers who are willing to borrow at high interest rates, and that high interest rates incentivize borrowers to invest in projects with higher risk and also higher return when successful. Banks can therefore alleviate asymmetric information problems such as adverse selection and moral hazard between borrowers and lenders with loan contract terms such as the interest rate charged or the collateral required (Berger & Udell, 1995). A bank can also improve the riskiness of its credit portfolio by actively screening the potential borrowers and by monitoring the existing ones.

The loan sales market offers a way for banks to reduce their exposure to credit risk by allowing a bank whose capital is not sufficient to support the risk associated with a loan it has originated to sell it to another bank with adequate capital (Carlstrom & Samolyk, 1995).

Furthermore, loan sales are a less expensive way of financing loans for banks than issuing deposits or equity and offer banks flexibility in liquidity creation (Parlour & Plantin, 2008). However, loan sales are also associated with a moral hazard problem since banks would have diminished incentives to monitor the loans they have sold.

The existence of loan sales market has led to the “originate-to-distribute model of bank credit” (Berndt & Gupta, 2009). According to Berndt and Gupta (2009), the underperformance of loans that are sold on secondary markets is due to the problems arising from informational asymmetries. They posit, these loans are either poor quality (adverse selection), or the banks do not monitor the sold loans efficiently (moral hazard). Parlour and Winton (2013) claim that credit risk transfer results in inefficient levels of monitoring; the riskier credits are monitored excessively while the level of monitoring of safer loans is insufficient. The loan sales market is therefore characterized by the trade-off between the (re)financing flexibility for banks and the negative consequences arising from informational asymmetries.

According to Pennacchi (1988), a piecewise-linear sharing rule would limit the extent of the moral hazard problem and the restriction on the amount of loans that can be sold. The optimal loan sale contract would give the buyer the entire return on the loan in case of default. The seller receives a share of the return only if no default occurs. Gorton and Pennacchi (1995) suggest that the moral hazard problem can be reduced if the bank were to sell only a share of a loan or to offer a guarantee on the value of the loan to the buyer. These results are similar to the ones derived in this paper, which offers a more efficient alternative to loan sales (namely, credit risk pooling) by deriving the optimal credit risk pooling arrangement that can eliminate the moral hazard problem completely in a risk-neutral setting.

Credit derivatives such as credit default swaps offer banks a more flexible way of transferring credit risk than loan sales. Duffee and Zhou (2001) argue that the use of credit derivatives could result in better risk sharing and reduce the expected deadweight costs of bank insolvency, but could also lead to the breakdown of the loan sales market. Various factors such as the type and severity of informational asymmetries (Duffee & Zhou, 2001), the bank’s cost of capital (Thompson, 2007), or the riskiness of the loans (Parlour & Winton, 2013) might determine which method of credit risk transfer is preferable for banks.

In this paper, credit risk pooling, a method of credit risk transfer used by German regional banks (Gintschel & Hackethal, 2004), is considered. In a risk pooling arrangement, bank loans are pooled together, and credit returns are shared between the participating banks.

Such an arrangement can have both benefits and drawbacks. On one hand, the exposure of the participating banks to credit risk could be reduced due to diversification. The incentives of the participating banks to monitor the loans they have originated might, however, be diminished. Gintschel and Hackethal (2004) analyze this trade-off inherent in multi-bank loan pool contracts assuming risk aversion for banks and suggest that small commercial banks benefit from diversification of credit risk, and the adverse effects of the moral hazard problem are limited (in comparison to other methods of credit risk transfer) in such an arrangement since informational asymmetries are less severe between loan pool participants than between a bank and an outside investor. They also show that the benefit of risk diversification in a loan pool outweighs the adverse effects of informational asymmetries. However, their analysis assumes, banks are risk-averse, which is conceptually problematic since institutions, unlike individuals, do not have intrinsic risk preferences.

In the pooling arrangement explored in this paper, on the other hand, the benefits of diversification and the negative effects of diminished incentives of the participating banks to extend monitoring effort are analyzed in a risk-neutral environment. The aim of this paper is to present a multi-loan pooling arrangement that benefits all the participating banks via diversification of risk and that eliminates the moral hazard problem of diminished incentives of the banks to monitor the loans they originate and contribute to the pool. The benefit of risk diversification (that is, lower credit risk) for a risk-neutral bank can be linked to a lower probability of bank failure, i.e., insolvency. Even in a risk-neutral setting, a bank would strive for a lower probability of insolvency for the sake of financial stability, due to regulatory pressure, or even to protect its reputation. The reduction in credit risk of the participating banks not only indicates a lower probability of bank failure, but also frees up economic capital for the banks to be able to take on additional risk by investing in further profitable loans.

The credit risk transfer instruments explored in literature always entail a trade-off between reducing the credit risk exposure of banks and the negative effects of informational asymmetries (i.e., moral hazard problem). The contribution of this paper is, therefore, a novel method of credit risk transfer in the form of credit risk pooling with asymmetrical risk-sharing rules that eliminates this trade-off, that is, it benefits the participating banks via risk diversification and eliminates the drawbacks resulting from the moral hazard problem, which is a common consequence of other methods of credit risk transfer such as loan sales and credit derivatives.

3. The Baseline Model without Pooling

A one-period model of a bank loan with a face value of F at $t = 0$ and an interest rate r is considered. The expected losses from the loan at maturity ($t = 1$) depend on the probability of default and the expected recovery rate in default.¹

Assumption 1.

The event of the loan's default is a Bernoulli-distributed random variable $D \sim Ber(p)$ with the probability mass function

$$f_D(d; p) = p^d (1-p)^{1-d} = \begin{cases} p & \text{if } d = 1 \\ 1-p & \text{if } d = 0 \end{cases} \quad (1)$$

where p is the probability of default.

Assumption 2.

The bank can influence (i.e., decrease) the probability of default by extending monitoring effort $m \in (0;1)$ that transforms the probability mass function from (1) into

$$f_D(d; p, m) = [(1-m)p]^d [1-(1-m)p]^{1-d} = \begin{cases} (1-m)p & \text{if } d = 1 \\ 1-(1-m)p & \text{if } d = 0 \end{cases} \quad (2)$$

¹ Default occurs when the borrower cannot fully repay the outstanding debt including interest. In case of partial default, any payment the borrower has made is part of the loan's recovery in default. The recovery rate is defined as a fraction of the outstanding debt including interest.

The treatment of default probability is the foremost challenge in credit risk modeling. Structural-form models (e.g., Merton (1974), Black and Cox (1976), Geske (1977), Vasicek (1984), Hull and White (1995), Longstaff and Schwartz (1995)) estimate the probability of default based on the values of the firm's assets and liabilities. Based on this approach, default occurs when the value of the firm's assets falls below the value of its liabilities. Reduced-form models (e.g., Madan and Unal (1998), Jarrow and Turnbull (1995), Lando (1998), Duffee (1999), Duffie and Singleton (1999)), on the other hand, treat default event as a Poisson process and assume an exogenous cause of default.

Alternatively, the time to default can be assumed to be exponentially distributed, in which case, the probability of no default in t years is $p(t) = e^{-\lambda t}$, where $\frac{1}{\lambda}$ is the expected time to default (Duffie & Singleton, 2003). In this paper, it is not specified how the probability of default is modeled and estimated. Instead, only certain assumptions regarding default events are made.

Assumption 3.

The monitoring effort results in monetary costs for the bank:

$$c(m) = \frac{1}{2} m^2 (1+r) F. \quad (3)$$

The monitoring costs are positively related to the interest rate and the outstanding amount. The larger the payment that is due, the higher the cost of monitoring since larger loans represent a higher risk exposure and involve more complex borrowers than small loans.

Assumption 4.

The recovery rate of the loan in default is a beta-distributed random variable $RR \sim \text{Beta}(\alpha, \beta)$ with the probability density function

$$f(RR; \alpha, \beta) = \frac{RR^{\alpha-1} (1-RR)^{\beta-1}}{B(\alpha, \beta)}. \quad (4)$$

Beta distribution of recovery rates is used in credit risk models such as CreditMetrics (JPMorgan Chase & Co.) (Gupton et al., 1997). For empirical modeling and forecasting of recovery rates, the use of a beta distribution is contested. When it comes to bank loans, the recovery rates depend on the size of the loan, industry-specific factors, and collateral and are found to be bimodally distributed (see Asarnow and Edwards (1995), Dermine and de Carvalho (2006)). For theoretical modeling of recovery rates, however, beta distribution of a variable defined on an interval is appropriate.

A beta distribution is continuous probability distribution suitable for variables with values between 0 and 1. The probability density function of a beta distribution of variable

$x \in (0;1)$ is given by $f(x; \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1}$, where the parameters α and β

determine the shape of the distribution (symmetry, skewness, whether the distribution is U-

shaped or bell-shaped), and the beta function, $B(\alpha, \beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!} = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt$, is

a normalization constant that ensures that the total probability equals 1 (Johnson et al., 1995).

The expected value and variance of the beta-distributed variable $x \in (0;1)$ are, respectively,

$$E(x) = \frac{\alpha}{\alpha + \beta} \quad \text{and} \quad \text{Var}(x) = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}.$$

The beta-distribution of the recovery rate represented by the probability density function from (4) coincides with its conditional distribution given default since recovery rate is a variable defined only when default occurs. Gaspar and Slinko (2008) incorporate the correlation between default intensities and recovery rates in a reduced-form model of credit risk since probability of default and loss given default are also correlated and are both likely to be affected by macroeconomic factors (Altman et al., 2005). This correlation can be taken into account in the estimation of the parameters that define the distribution of the recovery rate.

The expected loss from the loan can be defined as

$$EL = \underbrace{(1-m)p}_{PD} \cdot \underbrace{[1-E(RR)]}_{LGD} \cdot \underbrace{(1+r)F}_{EAD}, \quad (5)$$

where $E(RR)$ is the expected recovery rate.² The expected payment from the loan at $t = 1$ is

$$\begin{aligned} E(C) &= (1+r)F - \underbrace{(1-m)p \cdot [1-E(RR)] \cdot (1+r)F}_{EL} \\ &= [1-(1-m)p] \cdot (1+r)F + (1-m)p \cdot E(RR) \cdot (1+r)F. \\ &= [1-(1-m)p] \cdot (1+r)F + (1-m)p \cdot \frac{\alpha}{\alpha+\beta} \cdot (1+r)F \end{aligned} \quad (6)$$

The bank chooses a level of effort that maximizes the expected payment from (6) less the cost of monitoring from (3):

$$\begin{aligned} \frac{d\{E(C) - c(m)\}}{dm} &= p \cdot (1+r)F - p \cdot \frac{\alpha}{\alpha+\beta} \cdot (1+r)F - m \cdot (1+r)F = 0 \\ \Leftrightarrow m^* &= p \frac{\beta}{\alpha+\beta} \end{aligned}, \quad (7)$$

which represents the first-best level of monitoring effort of the bank since the bank does not share the return on the loan and recovers the full marginal benefit of effort. It is straightforward to see that the optimal monitoring effort from (7) is positively related to the probability of default and negatively related to the expected recovery rate, $E(RR) = \frac{\alpha}{\alpha+\beta}$.

² Expected loss is defined as $EL = PD \cdot LGD \cdot EAD$, where PD indicates the probability of default, LGD is the measure of loss given default, and EAD refers to the exposure at default. Since LGD is a fraction of the EAD , the outstanding debt including interest is considered to be the exposure at default in this paper even though the face value or the market value of debt is usually referred to as exposure at default.

With the optimal monitoring effort from (7), the expected payment from the loan less the cost of monitoring equals

$$E(C^*) - c(m^*) = (1+r)F - p \cdot \frac{\beta}{\alpha + \beta} \cdot (1+r)F + \frac{1}{2} \left[p \cdot \frac{\beta}{\alpha + \beta} \right]^2 \cdot (1+r)F. \quad (8)$$

The risk-return profile of the loan can be characterized by the expected rate of return (adjusted for the cost of monitoring) and the variance of the rate of return on the loan, the derivations of which are detailed in Appendix A:

$$E(R^*) = \left\{ 1 - p \frac{\beta}{\alpha + \beta} \left[1 - \frac{1}{2} p \frac{\beta}{\alpha + \beta} \right] \right\} \cdot (1+r) - 1. \quad (9)$$

$$\begin{aligned} Var(R^*) = & \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \left[1 - \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot \frac{\beta^2}{(\alpha + \beta)^2} \cdot (1+r)^2 \\ & + \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \cdot (1+r)^2. \end{aligned} \quad (10)$$

Taking into account the law of total variance (Blitzstein & Hwang, 2015), the credit risk of the loan measured as the variance of the rate of return from (10) can be expressed as the sum of the default risk and the recovery risk:

$$\nu^{\text{default}} = Var[E(R^* | D)] = \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \left[1 - \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot \frac{\beta^2}{(\alpha + \beta)^2} \cdot (1+r)^2. \quad (11)$$

$$\nu^{\text{recovery}} = E[Var(R^* | D)] = \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \cdot (1+r)^2. \quad (12)$$

The default risk (see (11)) represents the variation in the expected rate of return that is attributed to and can be explained by the occurrence of the default event D . The recovery risk (see (12)), on the other hand, is attributed to the variation in the recovery rate given the occurrence of the default event D .

In a risk-neutral setting, the bank is not particularly interested in the riskiness of its loan portfolio, but rather in the risk of insolvency, that is, the probability of bank failure. Insolvency occurs when bank capital is not sufficient to absorb losses: $ROA + CAR < 0$, where ROA and CAR are return on assets and capital asset ratio, respectively. The probability of $ROA < -CAR$ for a bank can then be perceived as the likelihood of insolvency (Hannan & Hanweck, 1988).

Using Chebyshev's Inequality (Feller, 1968), the upper bound of this probability can be determined:

$$\Pr(ROA < -CAR) \leq \frac{Var(ROA)}{[E(ROA) + CAR]^2}. \quad (13)$$

This upper bound can be lowered by decreasing the variance of ROA . The variance of ROA is tied to the variance of rate of return on any loan in the bank's portfolio. Therefore, despite being risk-neutral, a bank would benefit from a lower variance of the rate of return on its loan portfolio, which can be achieved through diversification with the credit risk pooling arrangement presented in this paper. The credit risk pooling with two and three loans is explored in the next section, and it is shown that risk pooling and risk sharing can be arranged to improve the risk-return profiles of the participating banks by decreasing the variance of the rate of return for each bank. The pooling and sharing arrangement as constructed in this paper eliminates the moral hazard problem associated with risk sharing, that is, does not diminish the incentives of the banks to monitor their loans after pooling. This results in the first-best level of expected rate of return for all participating banks. Furthermore, all participating banks benefit from pooling due to the diversification of recovery risk. The recovery risk (see (12)) (consequently, the credit risk measured as the variance of the rate of return) for every bank is less with pooling thanks to the risk diversification effect.

4. Credit Risk Pooling

4.1. Pooling of two loans

Two loans with face values F_1 , F_2 and interest rates r_1 , r_2 from two different banks are considered. The loans are generated at $t = 0$ and mature at $t = 1$. The banks decide to pool their loans together: $P = F_1 + F_2$, where $F_1 = \lambda_1 P$ and $F_2 = \lambda_2 P$, with $\lambda_1 + \lambda_2 = 1$. Assumptions 1-4 apply to both of these loans as well.

When measuring the credit risk attributed to a loan portfolio, not only the individual default probabilities of the loans in the portfolio, but also the correlations of the risks play a role. Hamerle and Rösch (2006) differentiate systematic and idiosyncratic risk when it comes to default probabilities. They consider idiosyncratic risk factors for different borrowers (i.e., loans) to be independent. In their model, the default correlation between two loans is due to a systematic risk factor that affects both borrowers. These correlations between default events

may arise due to macroeconomic or industry-specific factors (Nagpal & Bahar, 2001) and can be estimated using a historical data approach or an option-theoretic approach (Zhou, 2001). Since banks cannot influence macroeconomic factors (i.e., systematic risk) by monitoring, the correlation between default events does not play a significant role in the analyses presented here. Furthermore, the estimation of the default correlations is beyond the scope of this paper. Instead, assumptions are made regarding joint default probabilities and the effects of monitoring on them.

Assumption 5.

The monitoring efforts m_1 and m_2 extended by the banks do not have an effect on the correlation coefficient $\rho_{1,2}$ between the default events:³

$$\begin{aligned} \rho_{1,2} &= \frac{p_{1,2} - p_1 p_2}{\sqrt{p_1 \{1 - p_1\} p_2 \{1 - p_2\}}} \\ &= \frac{p_{1,2}^m - (1 - m_1)(1 - m_2) p_1 p_2}{\sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} (1 - m_2) p_2 \{1 - (1 - m_2) p_2\}}} \end{aligned} \quad (14)$$

where $p_{1,2}$ is the estimated joint probability of default of the two loans in the absence of monitoring and $p_{1,2}^m$ represents the joint probability of default when the banks extend monitoring efforts.

The probabilities of only one loan being in default at $t = 1$ are $(1 - m_1) p_1 - p_{1,2}^m$ and $(1 - m_2) p_2 - p_{1,2}^m$, respectively. The probability of neither loan being in default is then $1 - (1 - m_1) p_1 - (1 - m_2) p_2 + p_{1,2}^m$. The joint probability of default $p_{1,2}^m$ can be calculated as

$$\begin{aligned} p_{1,2}^m &= (1 - m_1)(1 - m_2) p_1 p_2 \\ &+ \frac{p_{1,2} - p_1 p_2}{\sqrt{p_1 \{1 - p_1\} p_2 \{1 - p_2\}}} \sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} (1 - m_2) p_2 \{1 - (1 - m_2) p_2\}} \end{aligned} \quad (15)$$

Assumption 6.

Recovery rates of the loans are independently distributed.⁴

³ Since the default events are assumed to be Bernoulli-distributed, the correlation coefficient between the default events is $\rho_{1,2} = \frac{E[D_1 \cdot D_2] - E[D_1] \cdot E[D_2]}{\sqrt{\text{Var}[D_1] \cdot \text{Var}[D_2]}}$ (see Lucas (1995)).

⁴ The effect of the diversification of recovery risk is demonstrated for loans with uncorrelated recovery rates. It is, however, possible that the recovery rates of the two loans are correlated. The theoretical modeling of

The returns from the pool at $t = 1$ are shared nonlinearly (piecewise-linearly) between the banks. The banks receive γ_1 and γ_2 shares of the payment from their own loans, respectively, if the corresponding loan is not in default. That is, if loan 1 is not in default, its originator (bank 1) receives γ_1 share of the payment from loan 1. Bank 2 would then receive $1 - \gamma_1$ share of that payment. Similarly, if loan 2 is not in default, bank 2 receives γ_2 share of the payment from loan 2, leaving $1 - \gamma_2$ share of the payment for bank 1. If a loan is in default, the originating bank's share of the recovered amount is based on the bank's contribution to the pool. That is, if loan 1 (or 2) is in default, the originating bank receives λ_1 (or λ_2) share of the recovered amount:

Table 1. Payments to the banks in case of two-loan credit risk pooling arrangement

Event ⁵	Payment to Bank 1	Payment to Bank 2
D_{00}	$\gamma_1 \cdot (1+r_1)F_1 + (1-\gamma_2) \cdot (1+r_2)F_2$	$(1-\gamma_1) \cdot (1+r_1)F_1 + \gamma_2 \cdot (1+r_2)F_2$
D_{10}	$\lambda_1 RR_1 \cdot (1+r_1)F_1 + (1-\gamma_2) \cdot (1+r_2)F_2$	$\lambda_2 RR_1 \cdot (1+r_1)F_1 + \gamma_2 \cdot (1+r_2)F_2$
D_{01}	$\gamma_1 \cdot (1+r_1)F_1 + \lambda_1 RR_2 \cdot (1+r_2)F_2$	$(1-\gamma_1) \cdot (1+r_1)F_1 + \lambda_2 RR_2 \cdot (1+r_2)F_2$
D_{11}	$\lambda_1 RR_1 \cdot (1+r_1)F_1 + \lambda_1 RR_2 \cdot (1+r_2)F_2$	$\lambda_2 RR_1 \cdot (1+r_1)F_1 + \lambda_2 RR_2 \cdot (1+r_2)F_2$

The expected payments for each participating bank can then be derived as the sum of products of the expected values of the payments from Table 1 and the corresponding probabilities of the events (see Appendix A):

$$\begin{aligned}
 E(C_1) = & \gamma_1 \cdot (1+r_1)F_1 + (1-\gamma_2) \cdot (1+r_2)F_2 \\
 & - (1-m_1)p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1+r_1)F_1, \\
 & - (1-m_2)p_2 \cdot \left[(1-\gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_2)F_2
 \end{aligned} \tag{16}$$

and

correlated recovery rates (i.e., a bivariate beta distribution of recovery rates) is not mathematically tractable and hence omitted in this paper. A negative (positive) correlation would increase (decrease) the beneficial effect of risk diversification.

⁵ The events are denoted by D_{00} (neither loan in default), D_{10} (only loan 1 in default), D_{01} (only loan 2 in default), and D_{11} (both loans in default).

$$\begin{aligned}
E(C_2) &= (1-\gamma_1) \cdot (1+r_1)F_1 + \gamma_2 \cdot (1+r_2)F_2 \\
&\quad - (1-m_1)p_1 \cdot \left[(1-\gamma_1) - \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1+r_1)F_1 \cdot \\
&\quad - (1-m_2)p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_2)F_2
\end{aligned} \tag{17}$$

Assumption 7a.

Each bank is aware of the characteristics (interest rate, probability of default as a function of monitoring effort, expected value and variance of the recovery rate) of the loan originated (i.e., contributed to the pool) by their counterpart. The banks cannot observe how much monitoring effort their counterpart extends. The cost function of monitoring effort of each bank is, however, common knowledge.

Each participating bank chooses a level of effort that maximizes its expected payment from (16) and (17), respectively, less its cost of monitoring effort, given $\lambda_1, \lambda_2, \gamma_1$, and γ_2 :

$$\begin{aligned}
\frac{d\{E(C_1) - c(m_1)\}}{dm_1} &= p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1+r_1)F_1 - m_1 \cdot (1+r_1)F_1 = 0 \\
\Leftrightarrow m_1^* &= p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right]
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{d\{E(C_2) - c(m_2)\}}{dm_2} &= p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_2)F_2 - m_2 \cdot (1+r_2)F_2 = 0 \\
\Leftrightarrow m_2^* &= p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right]
\end{aligned} \tag{19}$$

If the payments were to be shared linearly, that is, if $\gamma_1 = \lambda_1$ and $\gamma_2 = \lambda_2$, the optimal efforts from (18) and (19) would be second-best, which indicates diminished incentives of the banks to monitor their loans. However, the shares γ_1 and γ_2 can be chosen so that it is optimal for both banks to extend the first-best level of effort (see (7)):

$$m_1^* = p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] = p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \Leftrightarrow \gamma_1^* = 1 - (1-\lambda_1) \frac{\alpha_1}{\alpha_1 + \beta_1}, \tag{20}$$

and

$$m_2^* = p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] = p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \Leftrightarrow \gamma_2^* = 1 - (1-\lambda_2) \frac{\alpha_2}{\alpha_2 + \beta_2}. \tag{21}$$

The moral hazard problem is thus eliminated, and the expected payment from the pool in total is first-best (see (8)):

$$\begin{aligned}
E(C_p^*) &= E(C_1^*) - c(m_1^*) + E(C_2^*) - c(m_2^*) \\
&= (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \frac{1}{2} \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 \\
&\quad + (1+r_2)F_2 - p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 + \frac{1}{2} \left[p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \right]^2 \cdot (1+r_2)F_2
\end{aligned} \quad (22)$$

With the optimal efforts from (18) and (19) and with the optimal shares γ_1^* and γ_2^* from (20) and (21), the expected payment each bank would receive is

$$\begin{aligned}
E(C_1^*) &= (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 \\
&\quad - \left\{ \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right\}
\end{aligned} \quad (23)$$

and

$$\begin{aligned}
E(C_2^*) &= (1+r_2)F_2 - p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 + \left[p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \right]^2 \cdot (1+r_2)F_2 \\
&\quad - \left\{ \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 - \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 \right\}
\end{aligned} \quad (24)$$

respectively. Under the premise that the banks would not agree to such an arrangement unless they receive the same expected payment as they would without pooling, they additionally agree on a fixed payment. $\left\{ \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right\}$ from (23), for example, represents the difference between the expected payments to bank 1 with and without pooling. If this difference is positive, the bank should receive a fixed payment for joining the pool. If the difference is negative, the bank would have to pay to join the pool.

$$\begin{aligned}
c_{f_{i_{1-2}}} &= \left| \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) F_2 \right| \\
&= \left| \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) \lambda_1 P - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) \lambda_2 P \right| \\
&= \left| \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) - \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) \right| \lambda_1 \lambda_2 P
\end{aligned} \tag{25}$$

is the payment the bank 1 receives (from bank 2) if $\frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) > \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)$ or makes

(to bank 2) if $\frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) > \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)$. Therefore, the originating bank of the loan

with the higher (conditional) expected (gross) return given the loan is in default receives a fixed payment from the other bank.

Pooling is only then beneficial for the participating banks if it improves the risk-return profile for each of them, which is only then the case if the risk each bank is exposed to is less with pooling since the expected payments are already first-best (considering the agreed upon fixed payment). The risk-return profile of each bank can be characterized by the expected rate of return (adjusted for the cost of monitoring and the fixed payment) and the variance of the rate of return (see Appendix A):

$$E(R_1^*) = \left\{ 1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \left[1 - \frac{1}{2} p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right] \right\} \cdot (1+r_1) - 1. \tag{26}$$

$$E(R_2^*) = \left\{ 1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \left[1 - \frac{1}{2} p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right] \right\} \cdot (1+r_2) - 1. \tag{27}$$

$$\begin{aligned}
\text{Var}(R_1^*) &= \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1+r_1)^2 \\
&+ \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1+r_1)^2 \\
&+ \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1+r_2)^2 .
\end{aligned} \tag{28}$$

$$= v_1^{\text{default}} + \lambda_1^2 v_1^{\text{recovery}} + \lambda_2^2 v_2^{\text{recovery}}$$

$$\begin{aligned}
\text{Var}(R_2^*) &= \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2}\right) p_2 \left[1 - \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2}\right) p_2\right] \cdot \frac{\beta_2^2}{(\alpha_2 + \beta_2)^2} \cdot (1 + r_2)^2 \\
&+ \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2}\right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 \\
&+ \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1}\right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 \quad . \quad (29) \\
&= v_2^{\text{default}} + \lambda_2^2 v_2^{\text{recovery}} + \lambda_1^2 v_1^{\text{recovery}}
\end{aligned}$$

Comparing (28) and (29) with the variance of the rate of return without pooling (see (10)), it can be deduced that the default risks remain the same for both banks while the recovery risks are diversified with pooling. The diversification of the recovery risk is beneficial for both banks if

$$\begin{aligned}
v_1^{\text{recovery}} &> \lambda_1^2 v_1^{\text{recovery}} + \lambda_2^2 v_2^{\text{recovery}} \\
\Leftrightarrow (1 - \lambda_1)(1 + \lambda_1) v_1^{\text{recovery}} &> (1 - \lambda_1)^2 v_2^{\text{recovery}} \quad , \quad (30) \\
\Leftrightarrow \lambda_1 > \frac{v_2^{\text{recovery}} - v_1^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} &\Leftrightarrow \lambda_2 < \frac{2v_1^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}}
\end{aligned}$$

and

$$\begin{aligned}
v_2^{\text{recovery}} &> \lambda_1^2 v_1^{\text{recovery}} + \lambda_2^2 v_2^{\text{recovery}} \\
\Leftrightarrow (1 - \lambda_2)(1 + \lambda_2) v_2^{\text{recovery}} &> (1 - \lambda_2)^2 v_1^{\text{recovery}} \quad . \quad (31) \\
\Leftrightarrow \lambda_2 > \frac{v_1^{\text{recovery}} - v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} &\Leftrightarrow \lambda_1 < \frac{2v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}}
\end{aligned}$$

Therefore, both banks benefit from the risk diversification resulting from pooling if

$$\frac{v_2^{\text{recovery}} - v_1^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} < \lambda_1 < \frac{2v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} \quad , \quad (32)$$

or

$$\frac{v_1^{\text{recovery}} - v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} < \lambda_2 < \frac{2v_1^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} \quad . \quad (33)$$

The banks' contributions to the pool, that is, λ_1 and λ_2 can be selected so that the reduction in the variance of the rate of return resulting from the diversification of the recovery risk is maximized for both banks. This reduction in risk equals $v_1^{\text{recovery}} - \lambda_1^2 v_1^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}}$ and

$v_2^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}} - \lambda_1^2 v_1^{\text{recovery}}$, respectively. Therefore, maximizing $-\lambda_1^2 v_1^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}}$ also maximizes the risk diversification benefit of pooling for both banks. This optimization problem can be solved using the method of Lagrange multipliers:

$$\begin{aligned} \max_{\lambda_1, \lambda_2} \quad & f(\lambda_1, \lambda_2) = -\lambda_1^2 v_1^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}} \\ \text{s.t.} \quad & g(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 - 1 = 0 \end{aligned} \quad (34)$$

The Lagrangian function is then

$$\begin{aligned} \mathcal{L}(\lambda_1, \lambda_2, \xi) &= f(\lambda_1, \lambda_2) + \xi \cdot g(\lambda_1, \lambda_2) \\ &= -\lambda_1^2 v_1^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}} + \xi \cdot (\lambda_1 + \lambda_2 - 1) \end{aligned} \quad (35)$$

with the Lagrange multiplier ξ . The first-order partial derivatives of the Lagrangian function with respect to λ_1 , λ_2 , and the Lagrange multiplier ξ are set equal to 0:

$$\begin{cases} \frac{d\mathcal{L}(\lambda_1, \lambda_2, \xi)}{d\lambda_1} = -2\lambda_1 \cdot v_1^{\text{recovery}} + \xi = 0 \\ \frac{d\mathcal{L}(\lambda_1, \lambda_2, \xi)}{d\lambda_2} = -2\lambda_2 \cdot v_2^{\text{recovery}} + \xi = 0 \\ \frac{d\mathcal{L}(\lambda_1, \lambda_2, \xi)}{d\xi} = \lambda_1 + \lambda_2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = \frac{\xi}{2v_1^{\text{recovery}}} \\ \lambda_2 = \frac{\xi}{2v_2^{\text{recovery}}} \\ \lambda_1 + \lambda_2 = 1 \end{cases} \quad (36)$$

$$\Leftrightarrow \frac{\xi}{2v_1^{\text{recovery}}} + \frac{\xi}{2v_2^{\text{recovery}}} = 1 \Leftrightarrow \xi = \frac{2}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}}} \quad (37)$$

From (36) and (37), it follows:

$$\lambda_1^* = \frac{1}{v_1^{\text{recovery}} \cdot \left[\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}} \right]} = \frac{v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} \quad (38)$$

$$\lambda_2^* = \frac{1}{v_2^{\text{recovery}} \cdot \left[\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}} \right]} = \frac{v_1^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} \quad (39)$$

With the optimal contributions (λ_1^* from (38) and λ_2^* from (39)) of the banks to the pool, the variances of the rates of return from (28) and (29) can be calculated as follows:

$$\text{Var}(R_1^*) = v_1^{\text{default}} + \frac{1}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}}}. \quad (40)$$

$$\text{Var}(R_2^*) = v_2^{\text{default}} + \frac{1}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}}}. \quad (41)$$

Comparing (40) and (41) with (10) (given (11) and (12)), it can be shown that the credit risk measured as the variance of the rate of return is with pooling lower for both banks.⁶ This is due to the diversification of the recovery risk. The lower variance of the rate of return also indicates a lower upper bound for the probability of bank failure (see (13)).

4.2. Pooling of three loans

Three loans, generated at $t = 0$ and maturing at $t = 1$, with face values F_1, F_2, F_3 and interest rates r_1, r_2, r_3 from three different banks are pooled together: $P = F_1 + F_2 + F_3$, where $F_1 = \lambda_1 P$, $F_2 = \lambda_2 P$, and $F_3 = \lambda_3 P$, with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. Assumptions 1-6 analogously apply to all of these loans as well.⁷

The returns from the pool at $t = 1$ are shared nonlinearly (piecewise-linearly) between the banks. The banks receive γ_1, γ_2 , and γ_3 shares of the payment from their own loans, respectively, if the corresponding loan is not in default. If a loan is in default, the originating bank's share of the recovered amount is based on the bank's contribution to the pool. That is, if the corresponding loan is in default, the banks receive λ_1, λ_2 and λ_3 shares of the recovered amount from their own loans, respectively. The rest of the recovered payment from the loan is shared between the other participating banks:

$$\left[v_1^{\text{default}} + \frac{1}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}}} \right] - [v_1^{\text{default}} + v_1^{\text{recovery}}] = \frac{v_1^{\text{recovery}} v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} - v_1^{\text{recovery}} = -\frac{(v_1^{\text{recovery}})^2}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} < 0$$

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$$\left[v_2^{\text{default}} + \frac{1}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}}} \right] - [v_2^{\text{default}} + v_2^{\text{recovery}}] = \frac{v_1^{\text{recovery}} v_2^{\text{recovery}}}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} - v_2^{\text{recovery}} = -\frac{(v_2^{\text{recovery}})^2}{v_1^{\text{recovery}} + v_2^{\text{recovery}}} < 0$$

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The individual loans are pairwise correlated (see Appendix B). The joint probability of all three loans being in default is $p_{1,2,3}$ in the absence of monitoring by the banks and $p_{1,2,3}^m$ with the monitoring efforts.

Table 2. Payments to the banks in case of three-loan credit risk pooling arrangement

Event ⁸	Payment to Bank 1	Payment to Bank 2	Payment to Bank 3
D_{000}	$\gamma_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\frac{\lambda_2}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \gamma_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_2}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\frac{\lambda_3}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \frac{\lambda_3}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \gamma_3 \cdot (1+r_3) F_3$
D_{100}	$\lambda_1 RR_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\lambda_2 RR_1 \cdot (1+r_1) F_1$ $+ \gamma_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_2}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\lambda_3 RR_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_3}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \gamma_3 \cdot (1+r_3) F_3$
D_{010}	$\gamma_1 \cdot (1+r_1) F_1$ $+ \lambda_1 RR_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\frac{\lambda_2}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \lambda_2 RR_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_2}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\frac{\lambda_3}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \lambda_3 RR_2 \cdot (1+r_2) F_2$ $+ \gamma_3 \cdot (1+r_3) F_3$
D_{001}	$\gamma_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \lambda_1 RR_3 \cdot (1+r_3) F_3$	$\frac{\lambda_2}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \gamma_2 \cdot (1+r_2) F_2$ $+ \lambda_2 RR_3 \cdot (1+r_3) F_3$	$\frac{\lambda_3}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \frac{\lambda_3}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \lambda_3 RR_3 \cdot (1+r_3) F_3$
D_{110}	$\lambda_1 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_1 RR_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\lambda_2 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_2 RR_2 \cdot (1+r_2) F_2$ $+ \frac{\lambda_2}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3$	$\lambda_3 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_3 RR_2 \cdot (1+r_2) F_2$ $+ \gamma_3 \cdot (1+r_3) F_3$
D_{101}	$\lambda_1 RR_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \lambda_1 RR_3 \cdot (1+r_3) F_3$	$\lambda_2 RR_1 \cdot (1+r_1) F_1$ $+ \gamma_2 \cdot (1+r_2) F_2$ $+ \lambda_2 RR_3 \cdot (1+r_3) F_3$	$\lambda_3 RR_1 \cdot (1+r_1) F_1$ $+ \frac{\lambda_3}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2$ $+ \lambda_3 RR_3 \cdot (1+r_3) F_3$

⁸ The events are denoted by D_{000} (no loan in default), D_{100} (only loan 1 in default), D_{010} (only loan 2 in default), D_{001} (only loan 3 in default), D_{110} (only loan 1 and loan 2 in default), D_{101} (only loan 1 and loan 3 in default), D_{011} (only loan 2 and loan 3 in default), and D_{111} (all three loans in default).

D_{011}	$\gamma_1 \cdot (1+r_1) F_1$ $+ \lambda_1 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_1 RR_3 \cdot (1+r_3) F_3$	$\frac{\lambda_2}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \lambda_2 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_2 RR_3 \cdot (1+r_3) F_3$	$\frac{\lambda_3}{\lambda_2 + \lambda_3} (1-\gamma_1) \cdot (1+r_1) F_1$ $+ \lambda_3 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_3 RR_3 \cdot (1+r_3) F_3$
D_{111}	$\lambda_1 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_1 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_1 RR_3 \cdot (1+r_3) F_3$	$\lambda_2 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_2 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_2 RR_3 \cdot (1+r_3) F_3$	$\lambda_3 RR_1 \cdot (1+r_1) F_1$ $+ \lambda_3 RR_2 \cdot (1+r_2) F_2$ $+ \lambda_3 RR_3 \cdot (1+r_3) F_3$

In case only loan 3 is in default (D_{001}), for example, bank 1 receives γ_1 share of the payment from loan 1. The rest of the payment from loan 1, $(1-\gamma_1) \cdot (1+r_1) F_1$, is shared between bank 2 and bank 3 based on their shares in the pool excluding loan 1 ($1-\lambda_1$). These shares are $\frac{\lambda_2}{1-\lambda_1} = \frac{\lambda_2}{\lambda_2 + \lambda_3}$ for bank 2 and $\frac{\lambda_3}{1-\lambda_1} = \frac{\lambda_3}{\lambda_2 + \lambda_3}$ for bank 3. Since loan 2 is not in default either, bank 2 receives γ_2 share of the payment from loan 2. The rest of the payment from loan 2, $(1-\gamma_2) \cdot (1+r_2) F_2$, is shared between bank 1 and bank 3 based on their shares in the pool excluding loan 2 ($1-\lambda_2$). These shares are $\frac{\lambda_1}{1-\lambda_2} = \frac{\lambda_1}{\lambda_1 + \lambda_3}$ for bank 1 and $\frac{\lambda_3}{1-\lambda_2} = \frac{\lambda_3}{\lambda_1 + \lambda_3}$ for bank 3. The payment from loan 3 for bank 3 equals λ_3 share of the recovered amount, that is, $\lambda_3 RR_3 \cdot (1+r_3) F_3$. The remainder of the recovered amount is analogously shared between bank 1 and bank 2, whose shares equal $\frac{\lambda_1}{1-\lambda_3} (1-\lambda_3) = \lambda_1$ and $\frac{\lambda_2}{1-\lambda_3} (1-\lambda_3) = \lambda_2$, respectively.

The expected payments for each participating bank can be derived as the sum of the products of the expected values of the payments from Table 2 and the corresponding probabilities of the events (see Appendix A):

$$\begin{aligned}
E(C_1) = & \gamma_1 \cdot (1+r_1) F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) \cdot (1+r_2) F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) \cdot (1+r_3) F_3 \\
& - (1-m_1) p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1+r_1) F_1 \\
& - (1-m_2) p_2 \cdot \left[\frac{\lambda_1}{\lambda_1 + \lambda_3} (1-\gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_2) F_2 \\
& - (1-m_3) p_3 \cdot \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} (1-\gamma_3) - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1+r_3) F_3
\end{aligned} \tag{42}$$

$$\begin{aligned}
E(C_2) &= \frac{\lambda_2}{\lambda_2 + \lambda_3} (1 - \gamma_1) \cdot (1 + r_1) F_1 + \gamma_2 \cdot (1 + r_2) F_2 + \frac{\lambda_2}{\lambda_1 + \lambda_2} (1 - \gamma_3) \cdot (1 + r_3) F_3 \\
&\quad - (1 - m_1) p_1 \cdot \left[\frac{\lambda_2}{\lambda_2 + \lambda_3} (1 - \gamma_1) - \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1 + r_1) F_1 \\
&\quad - (1 - m_2) p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1 + r_2) F_2 \\
&\quad - (1 - m_3) p_3 \cdot \left[\frac{\lambda_2}{\lambda_1 + \lambda_2} (1 - \gamma_3) - \lambda_2 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1 + r_3) F_3
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
E(C_3) &= \frac{\lambda_3}{\lambda_2 + \lambda_3} (1 - \gamma_1) \cdot (1 + r_1) F_1 + \frac{\lambda_3}{\lambda_1 + \lambda_3} (1 - \gamma_2) \cdot (1 + r_2) F_2 + \gamma_3 \cdot (1 + r_3) F_3 \\
&\quad - (1 - m_1) p_1 \cdot \left[\frac{\lambda_3}{\lambda_2 + \lambda_3} (1 - \gamma_1) - \lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1 + r_1) F_1 \\
&\quad - (1 - m_2) p_2 \cdot \left[\frac{\lambda_3}{\lambda_1 + \lambda_3} (1 - \gamma_2) - \lambda_3 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1 + r_2) F_2 \\
&\quad - (1 - m_3) p_3 \cdot \left[\gamma_3 - \lambda_3 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1 + r_3) F_3
\end{aligned} \tag{44}$$

Assumption 7b.

Every bank is aware of the characteristics (interest rate, probability of default as a function of monitoring effort, expected value and variance of the recovery rate) of the loan originated (i.e., contributed to the pool) by their counterparts. The banks cannot observe how much monitoring effort their counterparts extend. The cost function of monitoring effort of every bank is, however, common knowledge.

Each participating bank chooses a level of effort that maximizes its expected payment from (42), (43), and (44), respectively, less its cost of monitoring, given $\lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2,$ and γ_3 :

$$\begin{aligned}
\frac{d\{E(C_1) - c(m_1)\}}{dm_1} &= p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1 + r_1) F_1 - m_1 \cdot (1 + r_1) F_1 = 0 \\
&\Leftrightarrow m_1^* = p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right]
\end{aligned} \tag{45}$$

$$\begin{aligned} \frac{d\{E(C_2) - c(m_2)\}}{dm_2} &= p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_2)F_2 - m_2 \cdot (1+r_2)F_2 = 0 \\ &\Leftrightarrow m_2^* = p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \end{aligned} \quad (46)$$

$$\begin{aligned} \frac{d\{E(C_3) - c(m_3)\}}{dm_3} &= p_3 \cdot \left[\gamma_3 - \lambda_3 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1+r_3)F_3 - m_3 \cdot (1+r_3)F_3 = 0 \\ &\Leftrightarrow m_3^* = p_3 \cdot \left[\gamma_3 - \lambda_3 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \end{aligned} \quad (47)$$

If the payments were to be shared linearly, that is, if $\gamma_1 = \lambda_1$, $\gamma_2 = \lambda_2$, and $\gamma_3 = \lambda_3$, the optimal efforts from (45), (46), and (47) would be second-best. The shares γ_1 , γ_2 , and γ_3 can be chosen so that it is optimal for all banks to extend the first-best level of effort (see (7)):

$$m_1^* = p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] = p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \Leftrightarrow \gamma_1^* = 1 - (1 - \lambda_1) \frac{\alpha_1}{\alpha_1 + \beta_1} \quad (48)$$

$$m_2^* = p_2 \cdot \left[\gamma_2 - \lambda_2 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] = p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \Leftrightarrow \gamma_2^* = 1 - (1 - \lambda_2) \frac{\alpha_2}{\alpha_2 + \beta_2} \quad (49)$$

$$m_3^* = p_3 \cdot \left[\gamma_3 - \lambda_3 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] = p_3 \cdot \frac{\beta_3}{\alpha_3 + \beta_3} \Leftrightarrow \gamma_3^* = 1 - (1 - \lambda_3) \frac{\alpha_3}{\alpha_3 + \beta_3} \quad (50)$$

The moral hazard problem is thus eliminated, and the expected payment from the pool in total is first-best (see (8)):

$$\begin{aligned} E(C_p^*) &= E(C_1^*) - c(m_1^*) + E(C_2^*) - c(m_2^*) + E(C_3^*) - c(m_3^*) \\ &= (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \frac{1}{2} \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 \\ &\quad + (1+r_2)F_2 - p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 + \frac{1}{2} \left[p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \right]^2 \cdot (1+r_2)F_2 \\ &\quad + (1+r_3)F_3 - p_3 \cdot \frac{\beta_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 + \frac{1}{2} \left[p_3 \cdot \frac{\beta_3}{\alpha_3 + \beta_3} \right]^2 \cdot (1+r_3)F_3 \end{aligned} \quad (51)$$

With the optimal efforts from (45), (46), (47) and with the optimal shares γ_1^* , γ_2^* , γ_3^* from (48), (49), and (50), the expected payment each bank would receive is

$$E(C_1^*) = (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 - \left\{ \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right\} - \left[\lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 \right] \quad , \quad (52)$$

$$E(C_2^*) = (1+r_2)F_2 - p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 + \left[p_2 \cdot \frac{\beta_2}{\alpha_2 + \beta_2} \right]^2 \cdot (1+r_2)F_2 - \left[\lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 - \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 \right] - \left[\lambda_3 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 - \lambda_2 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 \right] \quad , \quad (53)$$

and

$$E(C_3^*) = (1+r_3)F_3 - p_3 \cdot \frac{\beta_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 + \left[p_3 \cdot \frac{\beta_3}{\alpha_3 + \beta_3} \right]^2 \cdot (1+r_3)F_3 - \left[\lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 - \lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 \right] - \left[\lambda_2 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 - \lambda_3 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right] \quad , \quad (54)$$

respectively. Under the premise that the banks would not agree to such an arrangement unless they receive the same expected payment as they would without pooling, they additionally agree on fixed payments. The originating bank of the loan with the higher (conditional) expected (gross) return given the loan is in default receives a fixed payment from the banks with a lower (conditional) expected (gross) return in default.

$$\begin{aligned}
c_{fix_{1-2}} &= \left| \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) F_2 \right| \\
&= \left| \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) \lambda_1 P - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) \lambda_2 P \right| \quad (55) \\
&= \left| \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) - \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) \right| \cdot \lambda_1 \lambda_2 P
\end{aligned}$$

is the payment the bank 1 receives (from bank 2) if $\frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) > \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)$ or makes

(to bank 2) if $\frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) > \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)$.

$$\begin{aligned}
c_{fix_{2-3}} &= \left| \lambda_3 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) F_2 - \lambda_2 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) F_3 \right| \\
&= \left| \lambda_3 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) \lambda_2 P - \lambda_2 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) \lambda_3 P \right| \quad (56) \\
&= \left| \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) - \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) \right| \cdot \lambda_2 \lambda_3 P
\end{aligned}$$

is the payment bank 2 receives (from bank 3) if $\frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2) > \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)$ or makes

(to bank 3) if $\frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) > \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)$.

$$\begin{aligned}
c_{fix_{3-1}} &= \left| \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) F_3 - \lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) F_1 \right| \\
&= \left| \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) \lambda_3 P - \lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) \lambda_1 P \right| \quad (57) \\
&= \left| \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) - \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) \right| \cdot \lambda_1 \lambda_3 P
\end{aligned}$$

is the payment bank 3 receives (from bank 1) if $\frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3) > \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)$ or makes (to

bank 1) if $\frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1) > \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)$.

Pooling is only then beneficial for the participating banks if it improves the risk-return profile for each of them, which is only the case if the risk each bank is exposed to is less with

pooling since the expected payments are already first-best (considering the agreed upon fixed payment). The risk-return profile of each bank can be characterized by the expected rate of return (adjusted for the cost of monitoring and the fixed payments) and the variance of the rate of return (see Appendix A):

$$E(R_1^*) = \left\{ 1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \left[1 - \frac{1}{2} p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right] \right\} \cdot (1 + r_1) - 1. \quad (58)$$

$$E(R_2^*) = \left\{ 1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \left[1 - \frac{1}{2} p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right] \right\} \cdot (1 + r_2) - 1. \quad (59)$$

$$E(R_3^*) = \left\{ 1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \left[1 - \frac{1}{2} p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \right] \right\} \cdot (1 + r_3) - 1. \quad (60)$$

$$\begin{aligned} \text{Var}(R_1^*) &= \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1 + r_1)^2 \\ &+ \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 \\ &+ \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 \\ &+ \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \right) p_3 \frac{\lambda_3^2 \alpha_3 \beta_3}{(\alpha_3 + \beta_3)(\alpha_3 + \beta_3 + 1)} \cdot (1 + r_3)^2 \\ &= \nu_1^{\text{default}} + \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_2^2 \nu_2^{\text{recovery}} + \lambda_3^2 \nu_3^{\text{recovery}} \end{aligned} \quad (61)$$

$$\begin{aligned} \text{Var}(R_2^*) &= \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \left[1 - \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \right] \cdot \frac{\beta_2^2}{(\alpha_2 + \beta_2)^2} \cdot (1 + r_2)^2 \\ &+ \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 \\ &+ \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 \\ &+ \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \right) p_3 \frac{\lambda_3^2 \alpha_3 \beta_3}{(\alpha_3 + \beta_3)(\alpha_3 + \beta_3 + 1)} \cdot (1 + r_3)^2 \\ &= \nu_2^{\text{default}} + \lambda_2^2 \nu_2^{\text{recovery}} + \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_3^2 \nu_3^{\text{recovery}} \end{aligned} \quad (62)$$

$$\begin{aligned}
\text{Var}(R_3^*) &= \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3}\right) p_3 \left[1 - \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3}\right) p_3\right] \cdot \frac{\beta_3^2}{(\alpha_3 + \beta_3)^2} \cdot (1 + r_3)^2 \\
&+ \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3}\right) p_3 \frac{(1 - \lambda_3^2) \alpha_3 \beta_3}{(\alpha_3 + \beta_3)(\alpha_3 + \beta_3 + 1)} \cdot (1 + r_3)^2 \\
&+ \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1}\right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 \\
&+ \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2}\right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 \\
&= \nu_3^{\text{default}} + \lambda_3^2 \nu_3^{\text{recovery}} + \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_2^2 \nu_2^{\text{recovery}}
\end{aligned} \tag{63}$$

Comparing (61), (62), and (63) with the variance of the rate of return without pooling (see (10)), it can be observed that the default risks remain the same for all three banks while the recovery risks are diversified. The diversification of the recovery risk is beneficial for all banks if

$$\nu_1^{\text{recovery}} > \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_2^2 \nu_2^{\text{recovery}} + \lambda_3^2 \nu_3^{\text{recovery}}, \tag{64}$$

$$\nu_2^{\text{recovery}} > \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_2^2 \nu_2^{\text{recovery}} + \lambda_3^2 \nu_3^{\text{recovery}}, \tag{65}$$

and

$$\nu_3^{\text{recovery}} > \lambda_1^2 \nu_1^{\text{recovery}} + \lambda_2^2 \nu_2^{\text{recovery}} + \lambda_3^2 \nu_3^{\text{recovery}}. \tag{66}$$

The banks' contributions to the pool, that is, λ_1 , λ_2 , and λ_3 can be selected so that the reduction in the variance of the rate of return is maximized for both banks. This reduction in risk is $\nu_1^{\text{recovery}} - \lambda_1^2 \nu_1^{\text{recovery}} - \lambda_2^2 \nu_2^{\text{recovery}} - \lambda_3^2 \nu_3^{\text{recovery}}$, $\nu_2^{\text{recovery}} - \lambda_1^2 \nu_1^{\text{recovery}} - \lambda_2^2 \nu_2^{\text{recovery}} - \lambda_3^2 \nu_3^{\text{recovery}}$, and $\nu_3^{\text{recovery}} - \lambda_1^2 \nu_1^{\text{recovery}} - \lambda_2^2 \nu_2^{\text{recovery}} - \lambda_3^2 \nu_3^{\text{recovery}}$, respectively. Therefore, maximizing $-\lambda_1^2 \nu_1^{\text{recovery}} - \lambda_2^2 \nu_2^{\text{recovery}} - \lambda_3^2 \nu_3^{\text{recovery}}$ also maximizes the risk diversification benefit of pooling for all three banks. The optimization problem can be solved using the method of Lagrange multipliers:

$$\begin{aligned}
\max_{\lambda_1, \lambda_2, \lambda_3} \quad & f(\lambda_1, \lambda_2, \lambda_3) = -\lambda_1^2 \nu_1^{\text{recovery}} - \lambda_2^2 \nu_2^{\text{recovery}} - \lambda_3^2 \nu_3^{\text{recovery}} \\
\text{s.t.} \quad & g(\lambda_1, \lambda_2, \lambda_3) = \lambda_1 + \lambda_2 + \lambda_3 - 1 = 0
\end{aligned} \tag{67}$$

The Lagrangian function is then

$$\begin{aligned}\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \lambda) &= f(\lambda_1, \lambda_2, \lambda_3) + \lambda \cdot g(\lambda_1, \lambda_2, \lambda_3) \\ &= -\lambda_1^2 v_1^{\text{recovery}} - \lambda_2^2 v_2^{\text{recovery}} - \lambda_3^2 v_3^{\text{recovery}} + \xi \cdot (\lambda_1 + \lambda_2 + \lambda_3 - 1).\end{aligned}\quad (68)$$

with the Lagrange multiplier ξ . The first-order partial derivatives of the Lagrangian function with respect to λ_1 , λ_2 , λ_3 , and the Lagrange multiplier ξ are set equal to 0 :

$$\begin{cases} \frac{d\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \xi)}{d\lambda_1} = -2\lambda_1 \cdot v_1^{\text{recovery}} + \xi = 0 \\ \frac{d\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \xi)}{d\lambda_2} = -2\lambda_2 \cdot v_2^{\text{recovery}} + \xi = 0 \\ \frac{d\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \xi)}{d\lambda_3} = -2\lambda_3 \cdot v_3^{\text{recovery}} + \xi = 0 \\ \frac{d\mathcal{L}(\lambda_1, \lambda_2, \lambda_3, \xi)}{d\xi} = \lambda_1 + \lambda_2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = \frac{\xi}{2v_1^{\text{recovery}}} \\ \lambda_2 = \frac{\xi}{2v_2^{\text{recovery}}} \\ \lambda_3 = \frac{\xi}{2v_3^{\text{recovery}}} \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{cases} \quad (69)$$

$$\Leftrightarrow \frac{\xi}{2v_1^{\text{recovery}}} + \frac{\xi}{2v_2^{\text{recovery}}} + \frac{\xi}{2v_3^{\text{recovery}}} = 1 \Leftrightarrow \xi = \frac{2}{\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}} + \frac{1}{v_3^{\text{recovery}}}}. \quad (70)$$

From (69) and (70), it follows:⁹

$$\lambda_1^* = \frac{1}{v_1^{\text{recovery}} \cdot \left[\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}} + \frac{1}{v_3^{\text{recovery}}} \right]} = \frac{v_2^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}}. \quad (71)$$

$$\lambda_2^* = \frac{1}{v_2^{\text{recovery}} \cdot \left[\frac{1}{v_1^{\text{recovery}}} + \frac{1}{v_2^{\text{recovery}}} + \frac{1}{v_3^{\text{recovery}}} \right]} = \frac{v_1^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}}. \quad (72)$$

⁹ It can be shown that these optimal values fulfill the conditions from (64), (65), and (66):

$$\begin{aligned}\lambda_1^2 v_1^{\text{recovery}} + \lambda_2^2 v_2^{\text{recovery}} + \lambda_3^2 v_3^{\text{recovery}} &= \frac{v_1^{\text{recovery}} v_2^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}} \\ &= v_1^{\text{recovery}} \left\{ 1 - \frac{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}} \right\} < v_1^{\text{recovery}} \\ &= v_2^{\text{recovery}} \left\{ 1 - \frac{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}} \right\} < v_2^{\text{recovery}} \\ &= v_3^{\text{recovery}} \left\{ 1 - \frac{v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}}{v_1^{\text{recovery}} v_2^{\text{recovery}} + v_1^{\text{recovery}} v_3^{\text{recovery}} + v_2^{\text{recovery}} v_3^{\text{recovery}}} \right\} < v_3^{\text{recovery}}\end{aligned}$$

$$\lambda_3^* = \frac{1}{\nu_3^{\text{recovery}} \cdot \left[\frac{1}{\nu_1^{\text{recovery}}} + \frac{1}{\nu_2^{\text{recovery}}} + \frac{1}{\nu_3^{\text{recovery}}} \right]} = \frac{\nu_1^{\text{recovery}} \nu_2^{\text{recovery}}}{\nu_1^{\text{recovery}} \nu_2^{\text{recovery}} + \nu_1^{\text{recovery}} \nu_3^{\text{recovery}} + \nu_2^{\text{recovery}} \nu_3^{\text{recovery}}} \quad (73)$$

With the optimal contributions (λ_1^* from (71), λ_2^* from (72), and λ_3^* from (73)) of the banks to the pool, the variances of the rates of return can be calculated as follows:

$$\text{Var}(R_1^*) = \nu_1^{\text{default}} + \frac{1}{\frac{1}{\nu_1^{\text{recovery}}} + \frac{1}{\nu_2^{\text{recovery}}} + \frac{1}{\nu_3^{\text{recovery}}}} \quad (74)$$

$$\text{Var}(R_2^*) = \nu_2^{\text{default}} + \frac{1}{\frac{1}{\nu_1^{\text{recovery}}} + \frac{1}{\nu_2^{\text{recovery}}} + \frac{1}{\nu_3^{\text{recovery}}}} \quad (75)$$

$$\text{Var}(R_3^*) = \nu_3^{\text{default}} + \frac{1}{\frac{1}{\nu_1^{\text{recovery}}} + \frac{1}{\nu_2^{\text{recovery}}} + \frac{1}{\nu_3^{\text{recovery}}}} \quad (76)$$

Comparing (74), (75), and (76) with (10) (given (11) and (12)), it is straightforward to see that the credit risk measured as the variance of the rate of return is with pooling lower for all three banks. This is due to the diversification of the recovery risk. The upper bound of the probability of bank failure is then also lower for each participating bank (see (13)).

5. Generalization with n Loans: Optimal Multi-Loan Credit Risk Pooling

Both the analyses of credit risk pooling with two and three loans in the previous section and their results are analogous. The simpler case with two loans presents the concept and mechanism of the pooling arrangement suggested in this paper. The extended case with three loans is outlined to clearly lay out the intricacies of such an arrangement when more than two loans are pooled together. In this section, the credit risk pooling arrangement is generalized for the case with multiple loans, which is more straightforward to derive parallelly to the case with three loans.

There are n banks who at $t=0$ respectively generate n loans maturing at $t=1$, with the face values F_i and the interest rates r_i , where $i \in \mathbb{N}^+$. The loans are then pooled together:

$P = \sum_{i=1}^n F_i$, where $F_i = \lambda_i P$, with $\sum_{i=1}^n \lambda_i = 1$. The assumptions pertaining to multi-loan credit risk

pooling arrangement are made similarly to pooling with two or three loans analyzed above:

- The default event of loan i is a Bernoulli-distributed random variable $D_i \sim Ber(p_i)$ with the probability mass function

$$f_{D_i}(d_i; p_i) = p_i^{d_i} (1-p_i)^{1-d_i} = \begin{cases} p_i & \text{if } d_i = 1 \\ 1-p_i & \text{if } d_i = 0 \end{cases}, \quad (77)$$

where p_i is the probability of default.

- Bank i can influence (i.e., decrease) the probability of default on loan i with monitoring effort $m_i \in (0;1)$ that transforms the probability mass function from (77) into

$$f_{D_i}(d_i; p_i, m_i) = [(1-m_i)p_i]^{d_i} [1-(1-m_i)p_i]^{1-d_i} = \begin{cases} (1-m_i)p_i & \text{if } d_i = 1 \\ 1-(1-m_i)p_i & \text{if } d_i = 0 \end{cases}. \quad (78)$$

- The monitoring effort m_i results in monetary costs for bank i :

$$c(m_i) = \frac{1}{2} m_i^2 (1+r_i) F_i. \quad (79)$$

- The recovery rate of loan i in default is a beta-distributed random variable $RR_i \sim Beta(\alpha_i, \beta_i)$ with the probability density function

$$f(RR_i; \alpha_i, \beta_i) = \frac{RR_i^{\alpha_i-1} (1-RR_i)^{\beta_i-1}}{B(\alpha_i, \beta_i)}. \quad (80)$$

- Default events are correlated. The monitoring efforts do not influence the correlation coefficients between default events, but they decrease the joint default probabilities.
- Recovery rates are independently distributed.
- Every bank is aware of the characteristics (interest rate, probability of default as a function of monitoring effort, expected value and variance of the recovery rate) of the loan originated (i.e., contributed to the pool) by their counterparts. The banks cannot observe how much monitoring effort their counterparts extend. The cost function of monitoring effort of every bank is, however, common knowledge.

The returns from the pool at $t=1$ are shared nonlinearly (piecewise-linearly) among the banks. Bank j receives γ_j share of the payment from loan j if the loan is not in default. If loan j is in default, the originating bank's share of the recovered amount is based on the

bank's contribution to the pool. That is, if loan j is in default, bank j receives λ_j share of the recovered amount. The rest of payment from loan j is distributed among the other banks. In case loan j does not default, bank k receives $\frac{\lambda_k}{1-\lambda_j} \cdot (1-\gamma_j)$ share of the payment from loan j . The share of bank k of the recovered amount from loan j in default is then $\frac{\lambda_k}{1-\lambda_j} \cdot (1-\lambda_j) = \lambda_k$.

Every bank chooses a level of monitoring effort

$$m_j^* = p_j \left[\gamma_j - \lambda_j \frac{\alpha_j}{\alpha_j + \beta_j} \right], \quad (81)$$

which maximizes its expected payment less its cost of monitoring, given λ_j and γ_j .

Proposition 1.

If bank j 's share of the payment from loan j when the loan is not in default equals

$$\gamma_j^* = 1 - (1-\lambda_j) \frac{\alpha_j}{\alpha_j + \beta_j}, \text{ the optimal effort of the bank is first-best: } m_j^* = p_j \frac{\beta_j}{\alpha_j + \beta_j}.$$

The moral hazard problem is thus eliminated, and the expected payment from the pool in total is first-best:

$$E(C_P^*) = \sum_{i=1}^n E(C_i^*) = \sum_{i=1}^n \left[(1+r_i)F_i - p_i \cdot \frac{\beta_i}{\alpha_i + \beta_i} \cdot (1+r_i)F_i + \left[p_i \cdot \frac{\beta_i}{\alpha_i + \beta_i} \right]^2 \cdot (1+r_i)F_i \right]. \quad (82)$$

The expected payment for bank j is

$$E(C_j^*) = (1+r_j)F_j - \left\{ 1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \right\} p_j \frac{\beta_j}{\alpha_j + \beta_j} \cdot (1+r_j)F_j - \left[\sum_{i=1}^n \lambda_i \cdot \frac{\alpha_j}{\alpha_j + \beta_j} \cdot (1+r_j)F_j - \lambda_j \cdot \sum_{i=1}^n \frac{\alpha_i}{\alpha_i + \beta_i} \cdot (1+r_i)F_i \right]. \quad (83)$$

The banks would not agree to such an arrangement unless they receive the same expected payment as they would without pooling. Therefore, they additionally agree on a fixed payment.

Proposition 2.

The originating bank of the loan with the higher (conditional) expected (gross) return given the loan is in default receives a fixed payment from the banks with a lower (conditional) expected (gross) return in default.

$$c_{\text{fix}_{j-k}} = \left| \frac{\alpha_j}{\alpha_j + \beta_j} \cdot (1+r_j) - \frac{\alpha_k}{\alpha_k + \beta_k} \cdot (1+r_k) \right| \lambda_j \lambda_k P \quad (84)$$

is the payment bank j receives (from bank k) if $\frac{\alpha_j}{\alpha_j + \beta_j} \cdot (1+r_j) > \frac{\alpha_k}{\alpha_k + \beta_k} \cdot (1+r_k)$ or makes

(to bank k) if $\frac{\alpha_k}{\alpha_k + \beta_k} \cdot (1+r_k) > \frac{\alpha_j}{\alpha_j + \beta_j} \cdot (1+r_j)$.

Pooling is only then beneficial for the participating banks if it improves the risk-return profile for every one of them, which is only the case if the risk every bank is exposed to is less with pooling since the expected payments are already first-best (considering the agreed upon fixed payments). The risk-return profile of every bank can be characterized by the expected rate of return (adjusted for the cost of monitoring and the fixed payments) and the variance of the rate of return:

$$E(R_j^*) = \left\{ 1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \left[1 - \frac{1}{2} p_j \frac{\beta_j}{\alpha_j + \beta_j} \right] \right\} \cdot (1+r_j) - 1. \quad (85)$$

$$\begin{aligned} \text{Var}(R_j^*) &= \left(1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \right) p_j \left[1 - \left(1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \right) p_j \right] \cdot \frac{\beta_j^2}{(\alpha_j + \beta_j)^2} \cdot (1+r_j)^2 \\ &+ \sum_{i=1}^n \left\{ \left(1 - p_i \frac{\beta_i}{\alpha_i + \beta_i} \right) p_i \frac{\lambda_i^2 \alpha_i \beta_i}{(\alpha_i + \beta_i)(\alpha_i + \beta_i + 1)} \cdot (1+r_i)^2 \right\}. \quad (86) \end{aligned}$$

$$= \nu_j^{\text{default}} + \sum_{i=1}^n \lambda_i^2 \nu_i^{\text{recovery}}$$

The variance of the rate of return for bank j is the sum of the default risk

$$\nu_j^{\text{default}} = \left(1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \right) p_j \left[1 - \left(1 - p_j \frac{\beta_j}{\alpha_j + \beta_j} \right) p_j \right] \cdot \frac{\beta_j^2}{(\alpha_j + \beta_j)^2} \cdot (1+r_j)^2 \quad (87)$$

and the recovery risk $\sum_{i=1}^n \lambda_i^2 \nu_i^{\text{recovery}}$.

$$v_i^{\text{recovery}} = \left(1 - p_i \frac{\beta_i}{\alpha_i + \beta_i}\right) p_i \frac{\alpha_i \beta_i}{(\alpha_i + \beta_i)(\alpha_i + \beta_i + 1)} \cdot (1 + r_i)^2 \quad (88)$$

represents the expected conditional variance of the rate of return on a loan given default.

The variance of the rate of return for every bank should be less with pooling:

$$v_j^{\text{recovery}} > \sum_{i=1}^n \lambda_i^2 v_i^{\text{recovery}}. \quad (89)$$

All banks benefit from the risk diversification resulting from pooling if the condition from (89) is fulfilled. The banks' contributions to the pool, that is, λ_i , can be selected so that the reduction in the variance of the rate of return is maximized for all banks. This reduction in risk equals $v_j^{\text{recovery}} - \sum_{i=1}^n \lambda_i^2 v_i^{\text{recovery}}$ for bank j . Therefore, maximizing $-\sum_{i=1}^n \lambda_i^2 v_i^{\text{recovery}}$ also maximizes the risk diversification benefit of pooling for all banks.

Proposition 3.

The risk diversification benefit of pooling for bank j is maximized when

$$\lambda_j^* = \frac{1}{v_j^{\text{recovery}} \cdot \sum_{i=1}^n \frac{1}{v_i^{\text{recovery}}}}. \quad (90)$$

With the optimal contributions (λ_j^* from (90)) of the banks to the pool, the variances of the rates of return are

$$\text{Var}(R_j^*) = v_j^{\text{default}} + \frac{1}{\sum_{i=1}^n \frac{1}{v_i^{\text{recovery}}}}, \quad (91)$$

with v_j^{default} from (87) and v_i^{recovery} from (88). Due to the diversification of recovery risk, the variance of the rate of return for each bank participating in pooling (see (91)) is lower than without pooling (see (10)). This also indicates a lower upper bound for the probability of failure for every participating bank (see (13)). Therefore, the credit risk pooling arrangement presented in this paper not only reduces the exposure of every participating bank to credit risk, but also does not result in any negative consequences (that is, diminished incentives to monitor loans) of informational asymmetries (i.e., moral hazard problem).

6. Conclusion

Banks aim to maximize their expected returns from their loans and limit their exposure to credit risk. Loan sales market and credit derivatives allow banks to transfer credit risk. Transferring credit risk outside also results in diminished incentives of the bank to monitor, resulting in a trade-off. Credit risk pooling is an alternative to credit risk transfer, which involves pooling multiple loans generated by different banks together. The banks then agree on how to share the credit returns from the pool. In this paper, an optimal credit risk pooling arrangement is presented, which eliminates the negative effects of the moral hazard problem, that is, the diminished incentives of the banks participating in the pool to monitor the loans they have contributed to the pool, and improves the risk-return profiles of all the participating banks by reducing their credit risk, measured as the variance of the rate of return for each bank, via diversification of recovery risk. Even in a risk-neutral setting, banks benefit from a lower credit risk in their loan portfolio. A lower risk not only indicates a lower probability of bank failure, i.e., a lower risk of insolvency, contributing to the financial stability of the bank, reducing the regulatory pressure, and safeguarding the bank's reputation, but also allows a bank with a given level of risk capital to take on additional risk with further profitable investments.

In this pooling arrangement with loans from multiple banks, the share of the credit return every bank receives from its own loan depends on the share of its contribution to the pool (in terms of face value) and whether the loan ends up in default. Every bank's share of the payment from the loan it has originated and contributed to the pool in case the loan is not in default can be selected so that the monitoring effort of the bank is first-best. This results in the first-best level of expected payment from the pool in total. The expected payments to individual banks might be less or more than first-best level, which needs to be balanced out by the banks who agree upon fixed payments depending on the expected rate of return given default of the loans. The originating bank of a loan with a higher conditional expected (gross) return given the loan is in default receives a fixed payment from the banks with a lower conditional expected (gross) return in default. This arrangement eliminates drawbacks from the moral hazard problem.

The benefit of pooling for the participating banks is represented via the reduction of recovery risk. Akin to risk diversification in a portfolio of assets with uncorrelated returns, the recovery risk is also diversified for every bank. The contributions of the banks to the pools (their shares in the pool in terms of face value) can be arranged so that the diversification of recovery risk is beneficial for all the banks. Furthermore, optimal shares of the banks in the

pool, which maximizes the reduction in credit risk (measured as the variance of rate of return for every bank), are derived. These optimal shares might not always align with the face values of the loans that the banks wish to pool together. This indicates that banks either should generate loans accordingly, that is, select such loan amounts (i.e., face values) that the contribution of every bank to the pool is optimal, or have to contribute only certain shares of their loans to the pool.

The results illustrated in this paper depend on the underlying assumptions, which have been made as realistically as possible. The monitoring effort of a bank is assumed to reduce the probability of default of its loan. An alternative approach would be assuming a positive effect of monitoring on the expected recovery rate of the loan in default instead. Then a similar nonlinear sharing of credit returns from a pool would depend on if the actual recovery rate of the loan in default turns out to be below or above average. A less realistic (possibly, the most critical) assumption underlying the analyses in this paper is that the participating banks in a credit pool are well-informed about the credit risks (that is, the loans) their counterparts contribute to the pool. Informational symmetry in this context can, however, be achieved with transparency and due diligence by the banks. Then a credit pool can be arranged so that it benefits all the participating banks by reducing their exposure to credit risk.

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Appendix A. Expected value and variance of the rate of return

Without pooling, the payment C to the bank equals $(1+r)F$ with probability $1-(1-m)p$ and $RR \cdot (1+r)F$ with probability $(1-m)p$. The expected payment and the expected value of C^2 are then calculated as

$$\begin{aligned}
 E(C) &= [1-(1-m)p] \cdot (1+r)F \\
 &\quad + [(1-m)p] \cdot E(RR) \cdot (1+r)F \\
 &= [1-(1-m)p] \cdot (1+r)F \\
 &\quad + [(1-m)p] \cdot \frac{\alpha}{\alpha+\beta} \cdot (1+r)F
 \end{aligned} \tag{92}$$

and

$$\begin{aligned}
 E(C^2) &= [1-(1-m)p] \cdot (1+r)^2 F^2 \\
 &\quad + [(1-m)p] \cdot E(RR^2) \cdot (1+r)^2 F^2 \\
 &= [1-(1-m)p] \cdot (1+r)^2 F^2 \\
 &\quad + [(1-m)p] \cdot \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \cdot (1+r)^2 F^2
 \end{aligned} \tag{93}$$

The variance of the payment to the bank is

$$\begin{aligned}
 \text{Var}(C) &= E(C^2) - E(C)^2 \\
 &= [1-(1-m)p] \cdot (1+r)^2 F^2 \\
 &\quad + [(1-m)p] \cdot \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \cdot (1+r)^2 F^2 \\
 &\quad - \left\{ [1-(1-m)p] \cdot (1+r)F + [(1-m)p] \cdot \frac{\alpha}{\alpha+\beta} \cdot (1+r)F \right\}^2
 \end{aligned} \tag{94}$$

Simplifying (94) results in

$$\begin{aligned}
 \text{Var}(C) &= (1-m)p [1-(1-m)p] \cdot \frac{\beta^2}{(\alpha+\beta)^2} \cdot (1+r)^2 F^2 \\
 &\quad + (1-m)p \frac{\alpha\beta}{(\alpha+\beta)^2 (\alpha+\beta+1)} \cdot (1+r)^2 F^2
 \end{aligned} \tag{95}$$

With the optimal effort m^* from (7), the expected value and the variance of the payment to the bank can be calculated:

$$E(C) = \left[1 - \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot (1+r)F + \left[\left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot \frac{\alpha}{\alpha + \beta} \cdot (1+r)F \quad (96)$$

$$Var(C) = \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \left[1 - \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot \frac{\beta^2}{(\alpha + \beta)^2} \cdot (1+r)^2 F^2 + \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \cdot (1+r)^2 F^2 \quad (97)$$

With the cost of monitoring effort from (3), the expected value and variance of the (cost-adjusted) rate of return can then be calculated as

$$E(R) = \frac{E(C) - c(m)}{F} - 1 = \left\{ 1 - p \frac{\beta}{\alpha + \beta} \left[1 - \frac{1}{2} p \frac{\beta}{\alpha + \beta} \right] \right\} \cdot (1+r) - 1 \quad (98)$$

and

$$Var(R) = \frac{Var(C)}{F^2} = \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \left[1 - \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \right] \cdot \frac{\beta^2}{(\alpha + \beta)^2} \cdot (1+r)^2 + \left(1 - p \frac{\beta}{\alpha + \beta} \right) p \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)} \cdot (1+r)^2 \quad (99)$$

In case of pooling with two loans, the payments to bank 1 and their probabilities are presented in the table below.

Table 3. Payments to bank 1 and their probabilities in a two-loan credit risk pooling arrangement

Event	Probability	Payment
D_{00}	$1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + (1 - \gamma_2) \cdot (1 + r_2)F_2$
D_{10}	$(1 - m_1)p_1 - p_{1,2}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + (1 - \gamma_2) \cdot (1 + r_2)F_2$
D_{01}	$(1 - m_2)p_2 - p_{1,2}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2$
D_{11}	$p_{1,2}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2$

The expected payment, the expected value of C_1^2 , and the variance of the payment are calculated as follows:

$$\begin{aligned}
E(C_1) &= \left[1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m\right] \cdot \left\{\gamma_1 \cdot (1 + r_1)F_1 + (1 - \gamma_2) \cdot (1 + r_2)F_2\right\} \\
&\quad + \left[(1 - m_1)p_1 - p_{1,2}^m\right] \cdot \left\{\lambda_1 E(RR_1) \cdot (1 + r_1)F_1 + (1 - \gamma_2) \cdot (1 + r_2)F_2\right\} \\
&\quad + \left[(1 - m_2)p_2 - p_{1,2}^m\right] \cdot \left\{\gamma_1 \cdot (1 + r_1)F_1 + \lambda_1 E(RR_2) \cdot (1 + r_2)F_2\right\} \\
&\quad + p_{1,2}^m \cdot \left\{\lambda_1 E(RR_1) \cdot (1 + r_1)F_1 + \lambda_1 E(RR_2) \cdot (1 + r_2)F_2\right\} \quad . \quad (100) \\
&= \gamma_1 \cdot (1 + r_1)F_1 - (1 - m_1)p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1}\right] \cdot (1 + r_1)F_1 \\
&\quad + (1 - \gamma_2) \cdot (1 + r_2)F_2 - (1 - m_2)p_2 \cdot \left[(1 - \gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2}\right] \cdot (1 + r_2)F_2
\end{aligned}$$

$$\begin{aligned}
E(C_1^2) &= \left[1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m\right] \cdot \left\{ \gamma_1^2 \cdot (1 + r_1)^2 F_1^2 + (1 - \gamma_2)^2 \cdot (1 + r_2)^2 F_2^2 + 2\gamma_1(1 - \gamma_2) \cdot (1 + r_1)(1 + r_2) F_1 F_2 \right\} \\
&\quad + \left[(1 - m_1)p_1 - p_{1,2}^m\right] \cdot \left\{ \lambda_1^2 E(RR_1^2) \cdot (1 + r_1)^2 F_1^2 + (1 - \gamma_2)^2 \cdot (1 + r_2)^2 F_2^2 + 2\lambda_1(1 - \gamma_2) E(RR_1) \cdot (1 + r_1)(1 + r_2) F_1 F_2 \right\} \\
&\quad + \left[(1 - m_2)p_2 - p_{1,2}^m\right] \cdot \left\{ \gamma_1^2 \cdot (1 + r_1)^2 F_1^2 + \lambda_1^2 E(RR_2^2) \cdot (1 + r_2)^2 F_2^2 + 2\gamma_1\lambda_1 E(RR_2) \cdot (1 + r_1)(1 + r_2) F_1 F_2 \right\} \\
&\quad + p_{1,2}^m \cdot \left\{ \lambda_1^2 E(RR_1^2) \cdot (1 + r_1)^2 F_1^2 + \lambda_1^2 E(RR_2^2) \cdot (1 + r_2)^2 F_2^2 + 2\lambda_1^2 E(RR_1) E(RR_2) \cdot (1 + r_1)(1 + r_2) F_1 F_2 \right\} \\
&= \left[1 - (1 - m_1)p_1\right] \cdot \gamma_1^2 \cdot (1 + r_1)^2 F_1^2 \\
&\quad + \left[(1 - m_1)p_1\right] \cdot \lambda_1^2 \frac{\alpha_1(\alpha_1 + 1)}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 F_1^2 \\
&\quad + \left[1 - (1 - m_2)p_2\right] \cdot (1 - \gamma_2)^2 \cdot (1 + r_2)^2 F_2^2 \\
&\quad + \left[(1 - m_2)p_2\right] \cdot \lambda_1^2 \frac{\alpha_2(\alpha_2 + 1)}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 F_2^2 \\
&\quad + 2 \cdot \left\{ \begin{aligned} &\left[1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m\right] \gamma_1(1 - \gamma_2) \\ &+ \left[(1 - m_1)p_1 - p_{1,2}^m\right] \lambda_1(1 - \gamma_2) \frac{\alpha_1}{\alpha_1 + \beta_1} \\ &+ \left[(1 - m_2)p_2 - p_{1,2}^m\right] \gamma_1 \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \\ &+ p_{1,2}^m \lambda_1^2 \frac{\alpha_1 \alpha_2}{(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)} \end{aligned} \right\} \cdot (1 + r_1)(1 + r_2) F_1 F_2 \quad . \quad (101)
\end{aligned}$$

$$\begin{aligned}
\text{Var}(C_1) &= [1 - (1 - m_1)p_1] \cdot \gamma_1^2 \cdot (1 + r_1)^2 F_1^2 + [(1 - m_1)p_1] \cdot \lambda_1^2 \frac{\alpha_1(\alpha_1 + 1)}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 F_1^2 \\
&+ [1 - (1 - m_2)p_2] \cdot (1 - \gamma_2)^2 \cdot (1 + r_2)^2 F_2^2 + [(1 - m_2)p_2] \cdot \lambda_1^2 \frac{\alpha_2(\alpha_2 + 1)}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 F_2^2 \\
&+ 2 \cdot \left\{ \begin{aligned} &[1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m] \gamma_1 (1 - \gamma_2) + p_{1,2}^m \lambda_1^2 \frac{\alpha_1 \alpha_2}{(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)} \\ &+ [(1 - m_1)p_1 - p_{1,2}^m] \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} (1 - \gamma_2) + [(1 - m_2)p_2 - p_{1,2}^m] \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \gamma_1 \end{aligned} \right\} \cdot (1 + r_1)(1 + r_2) F_1 F_2 \\
&- \left\{ \begin{aligned} &\gamma_1 \cdot (1 + r_1) F_1 - (1 - m_1)p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1 + r_1) F_1 + (1 - \gamma_2) \cdot (1 + r_2) F_2 - (1 - m_2)p_2 \cdot \left[(1 - \gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1 + r_2) F_2 \end{aligned} \right\}^2 \\
&= (1 - m_1)p_1 \cdot \left\{ \begin{aligned} &\frac{(\gamma_1 - \lambda_1)^2 \alpha_1(\alpha_1 + 1) + \gamma_1^2 \beta_1(\beta_1 + 1) + 2\gamma_1(\gamma_1 - \lambda_1)\alpha_1\beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} - (1 - m_1)p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right]^2 \end{aligned} \right\} \cdot (1 + r_1)^2 F_1^2 \\
&+ (1 - m_2)p_2 \cdot \left\{ \begin{aligned} &\frac{(1 - \gamma_2 - \lambda_1)^2 \alpha_2(\alpha_2 + 1) + (1 - \gamma_2)^2 \beta_2(\beta_2 + 1) + 2(1 - \gamma_2)(1 - \gamma_2 - \lambda_1)\alpha_2\beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \\ &- (1 - m_2)p_2 \cdot \left[(1 - \gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right]^2 \end{aligned} \right\} \cdot (1 + r_2)^2 F_2^2 \quad . (102) \\
&+ 2 \{ p_{1,2}^m - (1 - m_1)(1 - m_2)p_1 p_2 \} \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \left[(1 - \gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1 + r_1)(1 + r_2) F_1 F_2
\end{aligned}$$

The effect of diversification of default risk is represented by the last line in (102). Due to the nonlinear sharing of credit returns as described in

this paper (see (21)), however, this effect is eliminated: $1 - \gamma_2 = \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \Leftrightarrow (1 - \gamma_2) = \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2}$.

With the optimal efforts m_1^* from (18) and m_2^* from (19) and with the optimal shares γ_1^* from (20) and γ_2^* from (21), the expected value and the variance of the payment to bank 1 are

$$E(C_1) = (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 - \left\{ \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right\} \quad (103)$$

and

$$\begin{aligned} \text{Var}(C_1) = & \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1+r_1)^2 F_1^2 \\ & + \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)^2 (\alpha_1 + \beta_1 + 1)} \cdot (1+r_1)^2 F_1^2 + \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)^2 (\alpha_2 + \beta_2 + 1)} \cdot (1+r_2)^2 F_2^2 \end{aligned} \quad (104)$$

Considering the agreed upon fixed payment between the banks (see (25)) and with the optimal contributions (λ_1^* from (38) and λ_2^* from (39)) of the banks to the pool, the expected value and variance of the rate of return can be calculated:

$$E(R_1) = \frac{E(C_1) \pm c_{fx_1-2} - c(m_1)}{F_1} - 1 = \left\{ 1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \left[1 - \frac{1}{2} p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right] \right\} \cdot (1+r_1) - 1. \quad (105)$$

$$\begin{aligned} \text{Var}(R_1) = \frac{\text{Var}(C_1)}{F_1^2} = & \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1+r_1)^2 \\ & + \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)^2 (\alpha_1 + \beta_1 + 1)} \cdot (1+r_1)^2 + \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)^2 (\alpha_2 + \beta_2 + 1)} \cdot (1+r_2)^2 \end{aligned} \quad (106)$$

In case of pooling with three loans, the payments to bank 1 and their probabilities are presented in the table below.

Table 4. Payments to bank 1 and their probabilities in a three-loan credit risk pooling arrangement

	Probability	Payment
D_{000}	$1 - (1 - m_1)p_1 - (1 - m_2)p_2 - (1 - m_3)p_3 + p_{1,2}^m + p_{1,3}^m + p_{2,3}^m - p_{1,2,3}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3}(1 - \gamma_2) \cdot (1 + r_2)F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - \gamma_3) \cdot (1 + r_3)F_3$
D_{100}	$(1 - m_1)p_1 + p_{1,2,3}^m - p_{1,2}^m - p_{1,3}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3}(1 - \gamma_2) \cdot (1 + r_2)F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - \gamma_3) \cdot (1 + r_3)F_3$
D_{010}	$(1 - m_2)p_2 + p_{1,2,3}^m - p_{1,2}^m - p_{2,3}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - \gamma_3) \cdot (1 + r_3)F_3$
D_{001}	$(1 - m_3)p_3 + p_{1,2,3}^m - p_{1,3}^m - p_{2,3}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3}(1 - \gamma_2) \cdot (1 + r_2)F_2 + \lambda_1 RR_3 \cdot (1 + r_3)F_3$
D_{110}	$p_{1,2}^m - p_{1,2,3}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2}(1 - \gamma_3) \cdot (1 + r_3)F_3$
D_{101}	$p_{1,3}^m - p_{1,2,3}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3}(1 - \gamma_2) \cdot (1 + r_2)F_2 + \lambda_1 RR_3 \cdot (1 + r_3)F_3$
D_{011}	$p_{2,3}^m - p_{1,2,3}^m$	$\gamma_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2 + \lambda_1 RR_3 \cdot (1 + r_3)F_3$
D_{111}	$p_{1,2,3}^m$	$\lambda_1 RR_1 \cdot (1 + r_1)F_1 + \lambda_1 RR_2 \cdot (1 + r_2)F_2 + \lambda_1 RR_3 \cdot (1 + r_3)F_3$

To calculate the expected value of C_1 (C_1^2), the probabilities from Table 4 are first multiplied with the corresponding (squared) payments.

The sum of the products is the expected payment (the expected value of C_1^2).

$$\begin{aligned}
E(C_1) &= \left[\begin{array}{l} 1 - (1 - m_1) p_1 - (1 - m_2) p_2 - (1 - m_3) p_3 \\ + p_{1,2}^m + p_{1,3}^m + p_{2,3}^m - p_{1,2,3}^m \end{array} \right] \cdot \left\{ \gamma_1 \cdot (1 + r_1) F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) \cdot (1 + r_2) F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) \cdot (1 + r_3) F_3 \right\} \\
&+ \left[(1 - m_1) p_1 - p_{1,2}^m - p_{1,3}^m + p_{1,2,3}^m \right] \cdot \left\{ \lambda_1 RR_1 \cdot (1 + r_1) F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) \cdot (1 + r_2) F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) \cdot (1 + r_3) F_3 \right\} \\
&+ \left[(1 - m_2) p_2 - p_{1,2}^m - p_{2,3}^m + p_{1,2,3}^m \right] \cdot \left\{ \gamma_1 \cdot (1 + r_1) F_1 + \lambda_1 RR_2 \cdot (1 + r_2) F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) \cdot (1 + r_3) F_3 \right\} \\
&+ \left[(1 - m_3) p_3 - p_{1,3}^m - p_{2,3}^m + p_{1,2,3}^m \right] \cdot \left\{ \gamma_1 \cdot (1 + r_1) F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) \cdot (1 + r_2) F_2 + \lambda_1 RR_3 \cdot (1 + r_3) F_3 \right\} \\
&+ \left[p_{1,2}^m - p_{1,2,3}^m \right] \cdot \left\{ \lambda_1 RR_1 \cdot (1 + r_1) F_1 + \lambda_1 RR_2 \cdot (1 + r_2) F_2 + \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) \cdot (1 + r_3) F_3 \right\} \\
&+ \left[p_{1,3}^m - p_{1,2,3}^m \right] \cdot \left\{ \lambda_1 RR_1 \cdot (1 + r_1) F_1 + \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) \cdot (1 + r_2) F_2 + \lambda_1 RR_3 \cdot (1 + r_3) F_3 \right\} \\
&+ \left[p_{2,3}^m - p_{1,2,3}^m \right] \cdot \left\{ \gamma_1 \cdot (1 + r_1) F_1 + \lambda_1 RR_2 \cdot (1 + r_2) F_2 + \lambda_1 RR_3 \cdot (1 + r_3) F_3 \right\} \\
&+ p_{1,2,3}^m \cdot \left\{ \lambda_1 RR_1 \cdot (1 + r_1) F_1 + \lambda_1 RR_2 \cdot (1 + r_2) F_2 + \lambda_1 RR_3 \cdot (1 + r_3) F_3 \right\} \\
&= \gamma_1 (1 + r_1) F_1 - (1 - m_1) p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \cdot (1 + r_1) F_1 \\
&+ \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) (1 + r_2) F_2 - (1 - m_2) p_2 \cdot \left[\frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1 + r_2) F_2 \\
&+ \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) (1 + r_3) F_3 - (1 - m_3) p_3 \cdot \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1 + r_3) F_3
\end{aligned} \tag{107}$$

$$\begin{aligned}
E(C_1^2) = & [1 - (1 - m_1)p_1] \gamma_1^2 \cdot (1 + r_1)^2 F_1^2 + [(1 - m_1)p_1] \cdot \frac{\lambda_1^2 \alpha_1 (\alpha_1 + 1)}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 F_1^2 \\
& + [1 - (1 - m_2)p_2] \cdot \frac{\lambda_1^2 (1 - \gamma_2)^2}{(\lambda_1 + \lambda_3)^2} \cdot (1 + r_2)^2 F_2^2 + [(1 - m_2)p_2] \cdot \frac{\lambda_1^2 \alpha_2 (\alpha_2 + 1)}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2 F_2^2 \\
& + [1 - (1 - m_3)p_3] \cdot \frac{\lambda_1^2 (1 - \gamma_3)^2}{(\lambda_1 + \lambda_2)^2} \cdot (1 + r_3)^2 F_3^2 + [(1 - m_3)p_3] \cdot \frac{\lambda_1^2 \alpha_3 (\alpha_3 + 1)}{(\alpha_3 + \beta_3)(\alpha_3 + \beta_3 + 1)} \cdot (1 + r_3)^2 F_3^2 \\
& + 2 \cdot \left\{ \begin{aligned} & [1 - (1 - m_1)p_1 - (1 - m_2)p_2 + p_{1,2}^m] \gamma_1 \frac{\lambda_1 (1 - \gamma_2)}{\lambda_1 + \lambda_3} + p_{1,2}^m \lambda_1^2 \frac{\alpha_1 \alpha_2}{(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)} \\ & + [(1 - m_1)p_1 - p_{1,2}^m] \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \frac{\lambda_1 (1 - \gamma_2)}{\lambda_1 + \lambda_3} + [(1 - m_2)p_2 - p_{1,2}^m] \gamma_1 \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \end{aligned} \right\} \cdot (1 + r_1)(1 + r_2) F_1 F_2 \\
& + 2 \cdot \left\{ \begin{aligned} & [1 - (1 - m_1)p_1 - (1 - m_3)p_3 + p_{1,3}^m] \gamma_1 \frac{\lambda_1 (1 - \gamma_3)}{\lambda_1 + \lambda_2} + p_{1,3}^m \lambda_1^2 \frac{\alpha_1 \alpha_3}{(\alpha_1 + \beta_1)(\alpha_3 + \beta_3)} \\ & + [(1 - m_1)p_1 - p_{1,3}^m] \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \frac{\lambda_1 (1 - \gamma_3)}{\lambda_1 + \lambda_2} + [(1 - m_3)p_3 - p_{1,3}^m] \gamma_1 \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \end{aligned} \right\} \cdot (1 + r_1)(1 + r_3) F_1 F_3 \\
& + 2 \cdot \left\{ \begin{aligned} & [1 - (1 - m_2)p_2 - (1 - m_3)p_3 + p_{2,3}^m] \frac{\lambda_1 (1 - \gamma_2)}{\lambda_1 + \lambda_3} \frac{\lambda_1 (1 - \gamma_3)}{\lambda_1 + \lambda_2} + p_{2,3}^m \lambda_1^2 \frac{\alpha_2 \alpha_3}{(\alpha_2 + \beta_2)(\alpha_3 + \beta_3)} \\ & + [(1 - m_2)p_2 - p_{2,3}^m] \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \frac{\lambda_1 (1 - \gamma_3)}{\lambda_1 + \lambda_2} + [(1 - m_3)p_3 - p_{2,3}^m] \frac{\lambda_1 (1 - \gamma_2)}{\lambda_1 + \lambda_3} \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \end{aligned} \right\} \cdot (1 + r_2)(1 + r_3) F_2 F_3. \tag{108}
\end{aligned}$$

The variance of the payment to bank 1 is then the difference between $E(C_1^2)$ from (108) and $E(C_1)^2$ (see (107)).

$$\begin{aligned}
\text{Var}(C_1) = & (1-m_1)p_1 \cdot \left\{ \frac{(\gamma_1 - \lambda_1)^2 \alpha_1 (\alpha_1 + 1) + \gamma_1^2 \beta_1 (\beta_1 + 1) + 2\gamma_1 (\gamma_1 - \lambda_1) \alpha_1 \beta_1}{(\alpha_1 + \beta_1)(\alpha_1 + \beta_1 + 1)} \right\} \cdot (1+r_1)^2 F_1^2 \\
& - (1-m_1)p_1 \cdot \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right]^2 \\
& + (1-m_2)p_2 \cdot \left\{ \frac{\left(\frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} - \lambda_1 \right)^2 \alpha_2 (\alpha_2 + 1) + \frac{\lambda_1^2(1-\gamma_2)^2}{(\lambda_1 + \lambda_3)^2} \beta_2 (\beta_2 + 1) + 2 \frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} \left(\frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} - \lambda_1 \right) \alpha_2 \beta_2}{(\alpha_2 + \beta_2)(\alpha_2 + \beta_2 + 1)} \right\} \cdot (1+r_2)^2 F_2^2 \\
& - (1-m_2)p_2 \cdot \left[\frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right]^2 \\
& + (1-m_3)p_3 \cdot \left\{ \frac{\left(\frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} - \lambda_1 \right)^2 \alpha_3 (\alpha_3 + 1) + \frac{\lambda_1^2(1-\gamma_3)^2}{(\lambda_1 + \lambda_2)^2} \beta_3 (\beta_3 + 1) + 2 \frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} \left(\frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} - \lambda_1 \right) \alpha_3 \beta_3}{(\alpha_3 + \beta_3)(\alpha_3 + \beta_3 + 1)} \right\} \cdot (1+r_3)^2 F_3^2 \\
& - (1-m_3)p_3 \cdot \left[\frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \right]^2 \\
& + 2 \left\{ p_{1,2}^m - (1-m_1)(1-m_2)p_1 p_2 \right\} \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \left[\frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \cdot (1+r_1)(1+r_2) F_1 F_2 \\
& + 2 \left\{ p_{1,3}^m - (1-m_1)(1-m_3)p_1 p_3 \right\} \left[\gamma_1 - \lambda_1 \frac{\alpha_1}{\alpha_1 + \beta_1} \right] \left[\frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1+r_1)(1+r_3) F_1 F_3 \\
& + 2 \left\{ p_{2,3}^m - (1-m_2)(1-m_3)p_2 p_3 \right\} \left[\frac{\lambda_1(1-\gamma_2)}{\lambda_1 + \lambda_3} - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \right] \left[\frac{\lambda_1(1-\gamma_3)}{\lambda_1 + \lambda_2} - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \right] \cdot (1+r_2)(1+r_3) F_2 F_3
\end{aligned} \tag{109}$$

The effect of diversification of default risk is represented by the last three lines in (109). Due to the nonlinear sharing of credit returns as described in this paper (see (49) and (50)), however, this effect is eliminated: $1 - \gamma_2 = (\lambda_1 + \lambda_3) \frac{\alpha_2}{\alpha_2 + \beta_2} \Leftrightarrow \frac{\lambda_1}{\lambda_1 + \lambda_3} (1 - \gamma_2) = \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2}$ and

$$1 - \gamma_3 = (\lambda_1 + \lambda_2) \frac{\alpha_3}{\alpha_3 + \beta_3} \Leftrightarrow \frac{\lambda_1}{\lambda_1 + \lambda_2} (1 - \gamma_3) = \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3}.$$

With the optimal efforts m_1^* from (45), m_2^* from (46), and m_3^* from (47) and with the optimal shares γ_1^* from (48), γ_2^* from (49), and γ_3^* from (50), the expected value and the variance of the payment to bank 1 are

$$\begin{aligned} E(C_1) = & (1+r_1)F_1 - p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 + \left[p_1 \cdot \frac{\beta_1}{\alpha_1 + \beta_1} \right]^2 \cdot (1+r_1)F_1 \\ & - \left\{ \lambda_2 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_2}{\alpha_2 + \beta_2} \cdot (1+r_2)F_2 \right\} - \left[\lambda_3 \frac{\alpha_1}{\alpha_1 + \beta_1} \cdot (1+r_1)F_1 - \lambda_1 \frac{\alpha_3}{\alpha_3 + \beta_3} \cdot (1+r_3)F_3 \right] \end{aligned} \quad (110)$$

and

$$\begin{aligned} \text{Var}(C_1) = & \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1+r_1)^2 F_1^2 \\ & + \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)^2 (\alpha_1 + \beta_1 + 1)} \cdot (1+r_1)^2 F_1^2 \\ & + \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_1^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)^2 (\alpha_2 + \beta_2 + 1)} \cdot (1+r_2)^2 F_2^2 \\ & + \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \right) p_3 \frac{\lambda_1^2 \alpha_3 \beta_3}{(\alpha_3 + \beta_3)^2 (\alpha_3 + \beta_3 + 1)} \cdot (1+r_3)^2 F_3^2 \end{aligned} \quad (111)$$

Considering the agreed upon fixed payments (see (55) and (57)) and with the optimal contributions (λ_1^* from (71), λ_2^* from (72), and λ_3^* from (73)) of the banks to the pool, the expected value and variance of the rate of return (adjusted for the cost of monitoring and the fixed payments) can then be calculated as

$$E(R_1) = \frac{E(C_1) \pm c_{\text{fix}_{t-2}} \mp c_{\text{fix}_{t-1}} - c(m_1)}{F_1} - 1 = \left\{ 1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \left[1 - \frac{1}{2} p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right] \right\} \cdot (1 + r_1) - 1 \quad (112)$$

and

$$\begin{aligned} \text{Var}(R_1) = \frac{\text{Var}(C_1)}{F_1^2} = & \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \left[1 - \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \right] \cdot \frac{\beta_1^2}{(\alpha_1 + \beta_1)^2} \cdot (1 + r_1)^2 \\ & + \left(1 - p_1 \frac{\beta_1}{\alpha_1 + \beta_1} \right) p_1 \frac{\lambda_1^2 \alpha_1 \beta_1}{(\alpha_1 + \beta_1)^2 (\alpha_1 + \beta_1 + 1)} \cdot (1 + r_1)^2 \\ & + \left(1 - p_2 \frac{\beta_2}{\alpha_2 + \beta_2} \right) p_2 \frac{\lambda_2^2 \alpha_2 \beta_2}{(\alpha_2 + \beta_2)^2 (\alpha_2 + \beta_2 + 1)} \cdot (1 + r_2)^2, \\ & + \left(1 - p_3 \frac{\beta_3}{\alpha_3 + \beta_3} \right) p_3 \frac{\lambda_3^2 \alpha_3 \beta_3}{(\alpha_3 + \beta_3)^2 (\alpha_3 + \beta_3 + 1)} \cdot (1 + r_3)^2 \end{aligned} \quad (113)$$

respectively.

Appendix B. Correlated default events

In case of pooling with three bank loans, individual default events are pairwise correlated with the correlation coefficients

$$\rho_{1,2} = \frac{p_{1,2} - p_1 p_2}{\sqrt{p_1 \{1 - p_1\} p_2 \{1 - p_2\}}} = \frac{p_{1,2}^m - (1 - m_1)(1 - m_2) p_1 p_2}{\sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} (1 - m_2) p_2 \{1 - (1 - m_2) p_2\}}}, \quad (114)$$

$$\rho_{1,3} = \frac{p_{1,3} - p_1 p_3}{\sqrt{p_1 \{1 - p_1\} p_3 \{1 - p_3\}}} = \frac{p_{1,3}^m - (1 - m_1)(1 - m_3) p_1 p_3}{\sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} (1 - m_3) p_3 \{1 - (1 - m_3) p_3\}}}, \quad (115)$$

and

$$\rho_{2,3} = \frac{p_{2,3} - p_2 p_3}{\sqrt{p_2 \{1 - p_2\} p_3 \{1 - p_3\}}} = \frac{p_{2,3}^m - (1 - m_2)(1 - m_3) p_2 p_3}{\sqrt{(1 - m_2) p_2 \{1 - (1 - m_2) p_2\} (1 - m_3) p_3 \{1 - (1 - m_3) p_3\}}}. \quad (116)$$

The probability of all three loans being in default can be calculated as

$$\begin{aligned} p_{1,2,3}^m &= (1 - m_1) p_1 p_{2,3}^m + \rho_{1,2\&3} \cdot \sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} p_{2,3}^m [1 - p_{2,3}^m]} \\ &= (1 - m_2) p_2 p_{1,3}^m + \rho_{2,1\&3} \cdot \sqrt{(1 - m_2) p_2 \{1 - (1 - m_2) p_2\} p_{1,3}^m [1 - p_{1,3}^m]}, \quad (117) \\ &= (1 - m_3) p_3 p_{1,2}^m + \rho_{3,1\&2} \cdot \sqrt{(1 - m_3) p_3 \{1 - (1 - m_3) p_3\} p_{1,2}^m [1 - p_{1,2}^m]} \end{aligned}$$

where $\rho_{1,2\&3}$, $\rho_{2,1\&3}$, and $\rho_{3,1\&2}$ are the multiple correlation coefficients that are not affected by the monitoring efforts:

$$\rho_{1,2\&3} = \frac{p_{1,2,3} - p_1 p_{2,3}}{\sqrt{p_1 (1 - p_1) p_{2,3} (1 - p_{2,3})}} = \frac{p_{1,2,3}^m - (1 - m_1) p_1 p_{2,3}^m}{\sqrt{(1 - m_1) p_1 \{1 - (1 - m_1) p_1\} p_{2,3}^m [1 - p_{2,3}^m]}}. \quad (118)$$

$$\rho_{2,1\&3} = \frac{p_{1,2,3} - p_2 p_{1,3}}{\sqrt{p_2 (1 - p_2) p_{1,3} (1 - p_{1,3})}} = \frac{p_{1,2,3}^m - (1 - m_2) p_2 p_{1,3}^m}{\sqrt{(1 - m_2) p_2 \{1 - (1 - m_2) p_2\} p_{1,3}^m [1 - p_{1,3}^m]}}. \quad (119)$$

$$\rho_{3,1\&2} = \frac{p_{1,2,3} - p_3 p_{1,2}}{\sqrt{p_3 (1 - p_3) p_{1,2} (1 - p_{1,2})}} = \frac{p_{1,2,3}^m - (1 - m_3) p_3 p_{1,2}^m}{\sqrt{(1 - m_3) p_3 \{1 - (1 - m_3) p_3\} p_{1,2}^m [1 - p_{1,2}^m]}}. \quad (120)$$

Authors' Contribution Statements

Paper 1

Abdulali Hadizada and Peter Nippel are co-authors of the first paper. They contributed equally to the conceptualization, the writing, and the mathematical analysis.

Date 10.12.2025

Signature



Date 10.12.2025

Signature



Papers 2 and 3

Abdulali Hadizada is the sole author of the papers 2 and 3. He gratefully acknowledges comments by Peter Nippel to have been very helpful for improving the papers.

Erklärung zum selbständigen Verfassen der Arbeit

Ich erkläre hiermit, dass ich meine Doktorarbeit „Essays on Nonlinear Contracting under Uncertainty and Asymmetric Information: Trade-off between Risk Sharing and Moral Hazard Problem“ selbstständig und ohne fremde Hilfe angefertigt habe. Alle von anderen Autoren wörtlich übernommenen Stellen, wie auch die sich an die Gedanken anderer Autoren eng anlehenden Ausführungen der aufgeführten Beiträge wurden besonders gekennzeichnet und die Quellen nach den mir angegebenen Richtlinien zitiert.

Datum *10.12.2025*

Unterschrift 