

# On the Maker-Breaker $k$ -Cycle Game

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*"The game is not about striving for balance.  
It is about finding the seeds of order, even in the  
midst of profound chaos."*

— vulcan proverb

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## Summary

In this thesis we study the biased Maker-Breaker  $k$ -cycle game on the complete graph  $K_n$  for a fixed  $k \in \mathbb{N}$ . This is a special case of Maker-Breaker subgraph games that are played by two players called Maker and Breaker who alternately claim edges of  $K_n$  that were previously unclaimed by either player. Maker wins the game if he is able to claim a complete copy of a certain subgraph  $C$ , which is a  $k$ -cycle in the case of Maker-Breaker  $k$ -cycle games studied in this thesis. Otherwise Breaker wins. These games are typically played with a bias  $q \in \mathbb{N}$ , meaning that Breaker is allowed to claim up to  $q$  edges per turn while Maker only claims one. M. Bednarska and T. Łuczak [BL00] proved that for every Maker-Breaker subgraph game there are constants  $c_0, c_1 > 0$  such that for large  $n$  Maker has a winning strategy if  $q \leq c_0 n^{m(C)}$  and Breaker can win if  $q \geq c_1 n^{m(C)}$ , where  $m(C)$  is only depending on the fixed subgraph  $C$  Maker has to build. They conjectured these constants could be chosen arbitrarily close to each other but this conjecture remains open if  $C$  contains a cycle. Bednarska and Łuczak did not explicitly calculate the constants that would emerge from their proof but L. Wolos [Wol23] gave upper and lower bounds for these constants for some example graphs  $C$ : even for the smallest graphs the constants are magnitudes apart from each other. To find explicit constants that are close to each other means that the corresponding strategies are significant and therefore it is one of the most major tasks regarding Maker-Breaker subgraph games. V. Chvátal and P. Erdős [CE78] find such constants for one of the most famous combinatorial games, the triangle game, by giving strategies for Maker and Breaker and prove that Maker can win the game if  $q < (2n + 2)n^{1/2} - 5/2$  and Breaker can win if  $q \geq 2n^{1/2}$ . The constant  $c_0 = \sqrt{2}$  remains the best known value for the triangle game to date, whereas for  $c_1$  several further improvements have been achieved. The most recent one is by C. Glazik and A. Srivastav [GS22] who were able to lower the constant to  $c_1 = \sqrt{8/3}$  by using a newly invented concept of balance functions. Using similar techniques, a strategy was given in the author's master's thesis [Sow21] that leads to a win for Breaker in the Maker-Breaker  $C_4$ -game if  $q > \frac{3}{\sqrt[3]{4}} n^{2/3}$ . Besides that, no explicit upper or lower bound constants  $c_0$  and  $c_1$  were found for any non-trivial Maker-Breaker subgraph game yet. In particular, no constructive Maker strategy with large constant  $c_0$  was known for  $C = C_k$ ,  $k > 3$ , and also no Breaker strategy was known for  $C = C_k$ ,  $k > 4$ . In Chapter 2 we introduce two new Maker-Breaker games strongly connected to cycle games and give winning strategies for Maker under certain assumptions. In Chapter 3 we use these results to give first explicit Maker strategies for the  $C_4$ -game and the  $C_6$ -game respectively: we prove that Maker has a winning strategy in the Maker-Breaker  $C_4$ -game if  $q < 0.16 n^{2/3}$  and a winning strategy in the Maker-Breaker  $C_6$ -game if  $q < 0.007 n^{4/5}$ . In Chapter 4 we generalize the ideas used in [GS22] and [Sow21] to prove a theorem that allows to find Breaker strategies for Maker-Breaker subgraph games by using balance functions and thereby settle, in the affirmative, a conjecture proposed by Joel Spencer in 2019. In Chapter 5 we apply this theorem to find Breaker strategies for Maker-Breaker  $k$ -cycle games with arbitrary  $k$  and prove that Breaker has winning strategies in these games if  $q > \sqrt[k-1]{(k-1) \binom{2(k-1)}{k}^{k-2} n^{k-2}}$ .

# Zusammenfassung

In dieser Dissertation untersuchen wir das Maker-Breaker- $k$ -Kreis-Spiel mit Bias auf dem vollständigen Graphen  $K_n$  für ein festes  $n \in \mathbb{N}$ . Dies ist ein Spezialfall von Maker-Breaker-Subgraphen-Spielen, oder kurz Subgraphen-Spielen, die von zwei Spielern, genannt Maker und Breaker, gespielt werden, welche abwechselnd bisher ungefärbte Kanten des  $K_n$  in ihrer jeweiligen Farbe färben. Maker gewinnt das Spiel, falls er es schafft, eine Kopie eines gewissen Subgraphen  $C$  in seiner Farbe zu färben. Anderenfalls gewinnt Breaker. Im Falle der in dieser Arbeit untersuchten  $k$ -Kreis-Spielen, ist dieser Subgraph ein Kreis der Länge  $k$ . Typischerweise werden diese Spiele mit einem sogenannten Bias  $q \in \mathbb{N}$  gespielt, was bedeutet, dass Breaker in jedem Zug bis zu  $q$  Kanten, aber Maker nur eine Kante, färben darf. M. Bednarska and T. Łuczak [BL00] zeigen für jedes Subgraphen-Spiel Konstanten  $c_0, c_1 > 0$  existieren, derart, dass Maker für genügend große  $n$  eine Gewinnstrategie besitzt, falls für den Bias  $q \leq c_0 n^{m(C)}$  gilt, wobei  $m(C)$  eine Konstante ist, die nur von dem festen Subgraphen  $C$  abhängt und vermuten außerdem, dass beide Konstanten beliebig dicht beieinander gewählt werden können. Diese Vermutung bleibt unbewiesen, falls  $C$  einen Kreis enthält. Bednarska and Łuczak berechnen die Konstanten nicht explizit, aber L. Wolos [Wol23] zeigt, dass diese Konstanten selbst für kleine Beispielgraphen  $C$  um mehrere Größenordnungen auseinander liegen. Konstanten zu finden, die dicht beieinander liegen, bedeutet, dass die dazugehörigen Strategien hochwertig sind, was es zu einer der Hauptaufgaben im Bereich der Subgraphen-Spiele macht. V. Chvátal and P. Erdős [CE78] finden solche Konstanten für eines der berühmtesten kombinatorischen Spiele, das Dreieck-Spiel, an, indem sie Strategien für Maker und Breaker angeben und beweisen, dass Maker das Spiel gewinnen kann, falls  $q < (2n+2)n^{1/2} - 5/2$  und Breaker es gewinnen kann, falls  $q \geq 2n^{1/2}$  gilt. Die Konstante  $c_0 = \sqrt{2}$  bleibt bis heute die beste bekannte, während für  $c_1$  mehrere Fortschritte erzielt wurden. Im jüngsten dieser Ergebnisse beschränken C. Glazik and A. Srivastav [GS22] die Konstante  $c_1$  von oben durch  $\sqrt{8/3}$ , indem sie ein neues Konzept von Balance-Funktionen einführen. In seiner Masterarbeit [Sow21] nutzt der Autor ähnliche Techniken, um eine Strategie für das  $C_4$ -Spiel anzugeben, die zu einem Sieg für Breaker führt, falls  $q > \frac{3}{\sqrt[3]{4}} n^{2/3}$  gilt. Bis auf diese Ergebnisse wurden bisher weder untere noch obere Schranken  $c_0$  and  $c_1$  für nicht triviale Subgraphen-Spiele gefunden. Insbesondere waren bisher keine konstruktiven Maker-Strategien mit großen Konstanten  $c_0$  für den Fall  $C = C_k$ ,  $k > 3$  und keine Breaker-Strategien für den Fall  $C = C_k$ ,  $k > 4$  bekannt. In Kapitel 2 führen wir zwei neue Maker-Breaker-Spiele ein, die enge Verbindungen zu Kreis-Spielen aufweisen und geben unter gewissen Bedingungen Gewinnstrategien für Maker an. In Kapitel 3 nutzen wir diese Resultate um erste explizite Maker-Strategien für das  $C_4$ - beziehungsweise das  $C_6$ -Spiel zu geben: Wir beweisen das Maker eine Gewinnstrategie im  $C_4$ -Spiel, falls  $q < 0.16 n^{2/3}$  gilt, und im  $C_6$ -Spiel hat, falls  $q < 0.007 n^{4/5}$  gilt. In Kapitel 4 verallgemeinern wir die Ideen aus [GS22] and [Sow21], um Breaker-Strategien für allgemeine Subgraphen-Spiele zu finden, was eine Vermutung von Joel Spencer von 2019 positiv beantwortet. In Kapitel 5 wenden wir dieses Theorem auf allgemeine  $k$ -Kreis-Spiele an und beweisen, dass Breaker in diesen Spielen Gewinnstrategien besitzt, falls  $q > \sqrt[k-1]{(k-1) \binom{2(k-1)}{k}^{k-2} n^{k-2}}$  gilt.

# Chapter 1

## Introduction

In this thesis we study the biased Maker-Breaker  $k$ -cycle game on the complete graph  $K_n$ . Maker-Breaker games are finite games with perfect information that are played by two players which are called Maker and Breaker. For a fixed  $n \in \mathbb{N}$  let  $K_n = (V_n, E_n)$  be the complete graph with  $n$  vertices. In Maker-Breaker games both players alternately claim edges of  $K_n$  that were previously unclaimed by either player. Maker starts. We call the set  $M$  of edges that are claimed by Maker the Maker graph and the set  $B$  of edges that are claimed by Breaker the Breaker graph respectively. Both players take turns until no unclaimed edges are left and Maker wins the game if the Maker graph  $(V_n, M)$  fulfills certain conditions. Otherwise Breaker wins which means that there are no draws in these games. We call this game the Maker-Breaker subgraph game, or the Maker-Breaker  $H$ -game, if Maker wins whenever his graph contains a copy of  $H$ , for a given subgraph  $H \subseteq K_n$ . If  $H$  is fixed and  $n$  is sufficiently large it is obvious that Maker has a big advantage in the  $H$ -game if both players claim exactly one edge per turn, which is called the unbiased variant of the game. To make the game fair and interesting on  $K_n$ , it is usually played with a bias  $q \in \mathbb{N}$ , meaning that Breaker is allowed to claim up to  $q$  edges in each of his turns while Maker claims only one per turn. We call this game the Maker-Breaker subgraph game with bias  $q$ . For each fixed  $n \in \mathbb{N}$  we call the minimal bias  $q$  for which Breaker has a winning strategy the threshold bias of the Maker-Breaker  $C$ -game. One of the central questions regarding Maker-Breaker subgraph games is how the threshold bias  $q = q(n)$  behaves as a function on the number of vertices.

### 1.1 Previous Work

Asymptotically, this question regarding the threshold bias is answered in the following theorem by M. Bednarska and T. Łuczak [BL00] in 2000.

**Theorem 1.1** *Let  $C$  be a graph which contains at least 3 non-isolated vertices and let*

$$m(C) := \max \left\{ \frac{|E|-1}{|V|-2} : (E, V) \text{ is a subgraph of } C \text{ with } |V| \geq 3 \right\}.$$

*Then there exist positive constants  $c_0$  and  $c_1$  and  $N \in \mathbb{N}$  such that for every  $n > N$  the following holds.*

(i) If  $q \leq c_0 n^{1/m(C)}$  then Maker has a winning strategy in the Maker-Breaker  $C$ -game with bias  $q$ .

(ii) If  $q \geq c_1 n^{1/m(C)}$  then Breaker has a winning strategy in the Maker-Breaker  $C$ -game with bias  $q$ .

They further conjectured that the constants  $c_0$  and  $c_1$  in Theorem 1.1 could be chosen arbitrarily close to each other. This conjecture remains open for every graph  $C$  that contains a cycle. Theorem 1.1 shows that it is not hard for a player to win the Maker-Breaker subgraph with a bias with the right asymptotics. A more viable sign of quality for a strategy would be that a player can win the Maker-Breaker subgraph with a bias that also have a high constant  $c_0$  in Maker's case, or a low constant  $c_1$  in Breaker's case respectively, ideally  $c_0$  and  $c_1$  should coincide. The proof in [BL00] does not provide concrete values for the constants  $c_0$  and  $c_1$  but L. Wolos [Wol23] extracted them for certain graphs from that proof in her bachelor's thesis. For example she was able to bound the ratio between both constants by  $\frac{c_1}{c_0} \leq 1.94 \times 10^{10}$  in the case of the triangle game  $C = C_3$  and by  $\frac{c_1}{c_0} \leq 5.64 \times 10^{47}$  in the case of  $C = C_4$ . In most of the Maker-Breaker subgraph games no better bounds were known. The only exception, besides the case where  $C$  is a forest, is the by far most studied Maker-Breaker  $C$ -game, the so called triangle game in which  $C$  is a triangle. The triangle game was introduced by V. Chvátal and P. Erdős in their pioneering paper [CE78]. They give simple Maker and Breaker strategies that lead to the following theorem.

**Theorem 1.2** *There is an  $N \in \mathbb{N}$  such that for all  $n > N$  the following holds.*

(i) If  $q \leq \sqrt{2} n^{1/2}$  then Maker has a winning strategy in the Maker-Breaker  $C_3$ -game with bias  $q$ .

(ii) If  $q \geq 2 n^{1/2}$  then Breaker has a winning strategy in the Maker-Breaker  $C_3$ -game with bias  $q$ .

Maker's strategy that leads to the lower bound of  $\sqrt{2} n^{1/2}$  is the following. Maker chooses an arbitrary vertex  $v \in V_n$  and builds a star in  $v$  by claiming an arbitrary edge  $\{v, w\}$  incident with  $v$  in each of his turns. To stop Maker from winning, Breaker has to claim every edge that connects two Maker neighbors of  $v$ , because otherwise Maker would be able to complete a triangle by claiming this edge himself. It is shown in [CE78] that at some point of the game the number of neighbors of  $v$  is too large for Breaker to counter these threats, and therefore Maker wins. Even if this strategy and the corresponding proofs are elementary, no improvements have been made to the lower bound for the Maker-Breaker triangle game, for decades until to date. In contrast to this, there are several results that refine the upper bound given in Theorem 1.2 (ii). J. Balogh and W. Samotij [BS11] slightly lower the upper bound by giving a randomized Breaker strategy. The analysis of this strategy gives the following theorem.

**Theorem 1.3** *There is an  $N \in \mathbb{N}$  such that for all  $n > N$  the following holds. If  $q \geq (2 - 1/24) n^{1/2}$  then Breaker has a winning strategy in the Maker-Breaker  $C_3$ -game with bias  $q$ .*

They also conjectured that the lower bound given in Theorem 1.2 (i) is in fact sharp. One of the biggest breakthroughs regarding Breaker strategies for Maker-Breaker subgraph games is the following result by C. Glazik and A. Srivastav [GS22] for the triangle game.

**Theorem 1.4** *There is an  $N \in \mathbb{N}$  such that for all  $n > N$  the following holds. If  $q > \sqrt{8/3}n^{1/2}$  then Breaker has a winning strategy in the Maker-Breaker  $C_3$ -game with bias  $q$ .*

With this result Glazik and Srivastav nearly closed the gap between the upper and the lower bound by reducing the ratio to  $\frac{c_1}{c_0} \leq \sqrt{\frac{4}{3}}$ . Their work is also remarkable for introducing a new kind of potential function approach. Breaker's strategy is trying to keep a healthy balance between Maker and Breaker edges in each vertex and by this prevents Maker from building the big stars that are the key in every Maker strategy. Notably the potential function introduced by Glazik and Srivastav does not necessarily decrease over time like many previous potential functions used for the analysis of positional games, for example by J. Beck [Bec82]. Instead more delicate techniques are used to show that the potential does not exceed a certain value and that also that this implies that Breaker can win the game. In the author's master's thesis [Sow21] similar techniques are applied to the Maker-Breaker  $C_4$ -game which leads to the following result.

**Theorem 1.5** *There is an  $N \in \mathbb{N}$  such that for all  $n > N$  the following holds. If  $q > \frac{3}{\sqrt[3]{4}}n^{2/3}$  then Breaker has a winning strategy in the Maker-Breaker  $C_3$ -game with bias  $q$ .*

This result is an important step forward in understanding Maker-Breaker subgraph games, as it marks the first instance since the introduction of the triangle game in [CE78], more than 40 years earlier, where a concrete Breaker strategy is given that leads to a explicit upper bound constant  $c_1$  in a non-trivial Maker-Breaker subgraph game other than the triangle game. Furthermore, a conjecture was raised by Joel Spencer in November 2019 during a visit of Kiel University that similar techniques could be used to find the first concrete Breaker strategies for the Maker-Breaker  $C_k$ -game, where  $k > 4$  is an arbitrary natural number.

## 1.2 New Maker and Breaker Strategies for the $C_k$ -Game

In this thesis we give new strategies for Maker and Breaker respectively for the Maker-Breaker  $C_k$ -game, where  $k \geq 5$  is an arbitrary natural number in Breaker's case and  $k \in \{4, 6\}$  in Maker's case. We show that these strategies will lead to wins by the respective players if the bias of the game has the asymptotics that are given in Theorem 1.1 and certain explicitly determined constants.

In Chapter 2 we introduce two games closely related to the minimum degree game, studied by J. Beck [Bec08], in which Maker's goal is to achieve a certain Maker degree in each vertex. In our first generalization of this game, which we call the partial minimum degree game, Maker does not have to achieve a given degree for every vertex. Rather it is sufficient that he is able to achieve this for every vertex of a subset  $X \subset V_n$  with  $|X| = \beta n$  for a fixed  $\beta \in [0, 1]$ . The second game we introduce in Chapter 2, we call the  $(d, \beta, k)$ -chain-game, which is a further generalization of the partial minimum degree game in which Maker's goal is to construct nested sets  $V = X_0 \supseteq X_1 \supseteq \dots \supseteq X_{k-1}$  with  $|X_i| \geq \beta |X_{i-1}|$  and  $|N_M(x) \cap X_{i-1}| \leq d$  for all  $x \in X_i, i \in [k-1]$ . We give Maker strategies for both games and show that Maker not only can win the games but can do so in relatively few turns. Both these games are interesting in their own right, but they also are helpful to develop new Maker strategies for Maker-Breaker subgraph-games.

In Chapter 3 we introduce strategies for the Maker-Breaker  $C_4$  game and the Maker-Breaker  $C_6$  game and use the results of Chapter 2 to prove that Maker has a winning strategy in the Maker-Breaker  $C_4$ -game if  $q < 0.16 n^{2/3}$  and a winning strategy in the Maker-Breaker  $C_6$ -game if  $q < 0.007 n^{4/5}$ . These are the first concrete winning strategies for Maker with explicit and reasonably large constants in any non-trivial Maker-Breaker subgraph game besides the classic result for the triangle game in [CE78].

In Chapter 4, we introduce balance functions, define the corresponding potential functions and explain how the so called potential function strategy for Breaker works given the corresponding balance function. We then prove our main theorem on Breaker strategies. This potential strategy theorem states that if a balance function fulfills certain conditions Breaker automatically wins the Maker-Breaker subgraph game if he plays the potential function strategy corresponding to the balance function at hand. This is the first universal approach to Maker-Breaker subgraph games that automatically leads to concrete deterministic Breaker strategies in Maker-Breaker subgraph games and thereby settles, in the affirmative, a conjecture proposed by Joel Spencer in 2019.

In Chapter 5 we show an application of the potential function theorem proven in Chapter 4 by defining explicit balance functions for the Maker-Breaker  $C_k$ -games for arbitrary natural numbers  $k \geq 3$  and prove that these balance functions in fact fulfill the assumptions required by the theorem. By applying the theorem to these balance functions we prove that for every  $k > 2$  Breaker has a winning strategy in the Maker-Breaker  $C_k$ -game if  $q < \sqrt[k-1]{(k-1)\left(\frac{2(k-1)}{k}\right)^{k-2} n^{k-2}}$ . In the case  $k \in \{3, 4\}$ , we recover the same upper bounds that were previously established in [GS22] and [Sow21], respectively. For  $k > 4$ , however, we obtain the first known

winning strategies for Breaker for the Maker-Breaker  $C_k$ -game with reasonable upper bound constants altogether. It is worth noting that in contrast to the constants one could obtain by Theorem 1.1 (ii), which may get very large with growing  $k$ , the constants we obtain in Chapter 5 are bounded by 2.16 and even converge to 2 for  $k \rightarrow \infty$ .

Combining the results of Chapter 3 and 5 we are able to bound the ratio of the upper and lower bounds in Theorem 1.1 by  $\frac{c_1}{c_0} < 11.82$  for the Maker-Breaker  $C_4$ -game and by  $\frac{c_1}{c_0} < 287.42$  for the Maker-Breaker  $C_6$ -game. Both these ratios are magnitudes apart from the previous best known bounds.

## 1.3 Formal Definition of the Maker-Breaker Subgraph Game

In this section, we give a formal definition of Maker-Breaker games. This definition is less intuitive, but the rigorous is required in Chapter 5 and 6.

### Definition 1.6 (Maker-Breaker Subgraph-Game)

For a set  $X$  let  $\mathcal{P}(X) := \{A : A \subseteq X\}$  denote the power set of  $X$ . For  $n \in \mathbb{N}$  let  $V_n := [n]$  be the set of Vertices and  $E_n := \{e \in \mathcal{P}([n]) : |e| = 2\}$  be the set of edges of the complete graph  $K_n := (V_n, E_n)$ . We define  $\mathcal{E}_n := \mathcal{P}(E_n)$ .

- (i) For  $q \in \mathbb{N}$  we call a function  $s : \mathcal{E}_n \times \mathcal{E}_n \rightarrow \mathcal{E}_n$  a  $q$ -strategy if  $|s(M, B)| \leq q$  and  $s(M, B) \subseteq M \cup B \Rightarrow M \cup B = E_n$  for all  $M, B \in \mathcal{E}_n$ .

The set  $s(M, B)$  contains the edges that a player claims when he plays with the strategy  $s$  if  $M$  is the Maker graph and  $B$  is the Breaker graph before his turn. For technical reasons it is allowed that this set contains edges that are already claimed. In this case he claims only the unclaimed edges of  $s(M, B)$ .

- (ii) We call

$$S(q) := \{s : s \text{ is a } q\text{-strategy}\}$$

the set of  $q$ -strategies and

$$S := \bigcup_{q \in \mathbb{N}} S(q)$$

the set of strategies.

- (iii) For strategies  $m, b \in S$  let

$$M^0(m, b) := \emptyset, B^0(m, b) := \emptyset.$$

For all  $t \in \mathbb{N}$  with  $t \geq 1$  we recursively define

$$M^{t+1}(m, b) := M^t(m, b) \cup \left( m(M^t(m, b), B^t(m, b)) \setminus B^t(m, b) \right),$$

$$B^{t+1}(m, b) := B^t(m, b) \cup \left( b(M^{t+1}(m, b), B^t(m, b)) \setminus M^{t+1}(m, b) \right).$$

$M^t(m, b)$  is the Maker graph and  $B^t(m, b)$  is the Breaker graph after the  $t$ -th turn, if  $m$  is the Maker strategy and  $b$  is the Breaker strategy.

- (iv) Further, for all  $t \in \mathbb{N}, m, b \in S$  we define  $G^t(m, b) := (M^t(m, b), B^t(m, b))$ .

- (v) Since  $G^t(m, b) \neq G^{t+1}(m, b)$  or  $M^t(m, b) \cup B^t(m, b) = E_n$  for all  $t \in \mathbb{N}$ , we have

$$T := \{t' \in \mathbb{N} : \forall t \in \mathbb{N} \setminus [t'] G^{t'}(m, b) = G^t(m, b)\} \neq \emptyset.$$

Let  $\bar{t}(m, b) := \min(T)$ ,  $G(m, b) := (G^t(m, b))_{t \in [\bar{t}(m, b)]}$ .

$\bar{t}$  is the point of time at which the game ends because all edges are claimed by either player.

- (vi) If  $M^{\bar{t}(m,b)}(m, b)$  contains a copy of  $C$ , Maker wins the Maker-Breaker  $C$ -game, otherwise Breaker wins.
- (vii) If Maker is allowed to claim only a single edge, but Breaker may claim at most  $q \in \mathbb{N}$  edges in each turn, in other words Maker choose a strategy from  $S(1)$  and Breaker choose one from  $S(q)$ , we call the game the Maker-Breaker  $C$ -game with bias  $q$ .

Most of the time we have fixed strategies  $m$  respectively  $b$  for Maker respectively for Breaker. In this case we usually omit them and write  $M^t$  instead of  $M^t(m, b)$ ,  $\bar{t}$  instead of  $\bar{t}(m, b)$  and so on.

# Chapter 2

## Games adjacent to Cycle Games

In this chapter we study two new Maker-Breaker games and give winning strategies for Maker under certain assumptions for both of them.

### 2.1 The Partial Minimum Degree Game

In this section we introduce a generalization of the Maker-Breaker minimum degree game. In the minimum degree game Maker's goal is it to claim a subset  $M$  of edges of  $K_n = (V, E)$  such that  $\deg_M(v) \geq d(n)$  for every vertex of  $v \in V$ , where  $d$  is a fixed function. Our generalization, which we call the partial minimum degree game, is the following: Given  $\beta \in (0, 1]$  Maker's goal is it to claim a subgraph  $M$  such that there is a subset  $X \subseteq V$  with  $|X| \geq n\beta$  and  $\deg_M(x) \geq d(n)$  for every vertex  $x \in X$ . Let us define the game formally.

**Definition 2.1** Let  $\beta \in (0, 1]$ ,  $d : \mathbb{N} \rightarrow \mathbb{R}$ , and

$$\mathcal{F}_{\beta, d(n)} := \left\{ M \subseteq E(K_n) \mid \exists X \subseteq V(K_n) : |X| \geq \beta n \wedge \forall x \in X \deg_M(x) \geq d(n) \right\}.$$

We call the Maker-Breaker game with the winning sets  $\mathcal{F}_{\beta, d(n)}$  the  $(\beta, d(n))$ -partial minimum degree game.

For  $\beta = 1$  the  $(1, d(n))$ -partial minimum degree game is just the well-known minimum degree game. Beck [Bec08] showed that Maker has a winning strategy in the minimum degree game if  $d(n) = \frac{1-\varepsilon}{q(n)+1}n$ , where

$$\varepsilon = \left( 1 + O\left( q \sqrt{\frac{\log(n)}{(q+1)(n-1)}} \right) \right) 2q \sqrt{\frac{\log(n)}{(q+1)(n-1)}}.$$

Our result for the partial minimum degree game is similar to the result of Beck, but in our setting  $\beta < 1$  and this means that it is sufficient for Maker to achieve a high minimal degree only on a subset of vertices. Due to this we are able not only to give a much simpler proof but also to show that the number of rounds Maker needs to win is at most  $\frac{dn}{2}$ , where  $d$  is the target degree. In fact, this bound on the number of turns is the key to tackle the  $C_4$ -game.

For any  $d > 0$  we state the  $d$ -degree strategy for Maker: In each of his turns Maker claims an unclaimed edge  $\{v, w\}$  that fulfills  $\deg_M(v) < d$  and  $\deg_M(w) < d$ , where  $M$  is the graph of Maker's current edges. If he cannot do so, he stops. We will show that at this point Maker has already won the the partial minimum degree game.

**Theorem 2.2** *Let  $\delta > 0$ ,  $\alpha, \beta \in (0, 1)$ , and  $d : \mathbb{N} \rightarrow \mathbb{R} : n \mapsto \delta n^{1-\alpha}$ .*

*We consider the partial minimum degree game with regard to  $\mathcal{F}_{\beta, d(n)}$  played on the edges of  $K_n$ . Let  $d = d(n) = \delta n^{1-\alpha}$ . For sufficiently large  $n \in \mathbb{N}$  and a bias  $q < \frac{(1-\beta)^2}{\delta} n^\alpha$  Maker wins in at most  $\frac{\delta}{2} n^{2-\alpha}$  turns by playing according to the  $d$ -degree strategy.*

*Proof.* Let  $c < \frac{(1-\beta)^2}{\delta}$ ,  $q = cn^\alpha$  and let  $n \in \mathbb{N}$  be large enough so that the inequality  $\frac{c\delta}{2}n^2 < \frac{(1-\beta)^2}{2}n^2 - \frac{1}{2}\left((1-\beta)n + \delta(1-\beta)n^{2-\alpha}\right)$  holds. This is possible since  $c\delta < (1-\beta)^2$  and the term  $\frac{1}{2}\left((1-\beta)n + \delta(1-\beta)n^{2-\alpha}\right)$  is negligible for large  $n$ . Maker plays according to the  $(\delta n^{1-\alpha})$ -degree strategy which means that he only claims edges  $\{v, w\}$  with  $\deg_M(v) < \delta n^{1-\alpha}$  and  $\deg_M(w) < \delta n^{1-\alpha}$  as long as he is able to do so, where  $M$  is the graph of Maker's edges. Note that after Maker has claimed more than  $\frac{\delta}{2}n^{2-\alpha}$  edges we have  $\sum_{v \in V} \deg_M(v) > \delta n^{2-\alpha}$  and thus, by the pigeonhole principle, there has to be a vertex  $v$  with  $\deg_M(v) > \delta n^{1-\alpha}$ . Therefore we know that after at most  $\frac{\delta}{2}n^{2-\alpha}$  turns there is no unclaimed edge  $\{v, w\}$  left with  $\deg_M(v) < \delta n^{1-\alpha}$  and  $\deg_M(w) < \delta n^{1-\alpha}$ . Let be  $M$  and  $B$  the sets of Maker- and Breaker-edges respectively at this moment, and define  $X := \{v \in V : \deg_M(v) \geq \delta n^{1-\alpha}\}$  and  $Y := V \setminus X$ . We assume for a moment that the assertion of the theorem is not true. Thus by Definition 2.1 of the winning sets,  $|X| < \beta n$ , therefore  $|Y| \geq (1-\beta)n$ . For each  $y \in Y$ , we have  $V \setminus \{y\} \subseteq X \cup N_M(y) \cup N_B(y)$  because otherwise for  $v \in (V \setminus \{y\}) \setminus (X \cup N_M(y) \cup N_B(y))$  the edge  $\{v, y\}$  would be still unclaimed, which is a contradiction to the assumption that Maker cannot claim an edge containing two vertices with degree smaller than  $\delta n^{1-\alpha}$ . Therefore,  $n-1 \leq |X| + |N_M(y)| + |N_B(y)|$  which implies  $(1-\beta)n - 1 - \delta n^{1-\alpha} \leq |N_B(y)|$ . This means, Breaker has occupied at least  $(1-\beta)n - 1 - \delta n^{1-\alpha}$  edges incident in every vertex of  $Y$ . So in total the number of edges claimed by Breaker is at least

$$\begin{aligned} & \frac{|Y|}{2} \left( (1-\beta)n - 1 - \delta n^{1-\alpha} \right) \\ & \geq \frac{(1-\beta)n}{2} \left( (1-\beta)n - 1 - \delta n^{1-\alpha} \right) \\ & = \frac{(1-\beta)^2}{2} n^2 - \frac{1}{2} \left( (1-\beta)n + \delta(1-\beta)n^{2-\alpha} \right) \\ & > \frac{c\delta}{2} n^2 = \frac{\delta}{2} n^{2-\alpha} q \end{aligned}$$

which is a contradiction to the fact that at most  $\frac{\delta}{2}n^{2-\alpha}$  rounds were played.  $\square$

The last theorem gives a strategy for the partial minimum degree game played on the edges of the complete graph  $K_n$ . We now consider the partial minimum degree game played on the edges of a subgraph of  $K_n$  with at most  $\binom{n}{2} - An^2$  edges for

a constant  $A \in [0, 1)$  and give a strategy for Maker in this game as well. Because of technical reasons we will in fact look at the game played on the complete graph and interpret the missing edges as claimed by Breaker in an additional headstart phase of the game. We also need the following further generalization of the game. Maker has to build the set  $X_2$  of vertices with high degree inside of a given set  $X_1$  of vertices. An additional change is that Maker's goal is to create vertices  $v$  with  $|N_M(x) \cap X_1| \geq d$  instead of just  $\deg_M(v) \geq d$ . We define the game formally.

**Definition 2.3** (Partial Minimum Degree Game with Headstart)

Let  $\delta > 0$ ,  $\alpha, \beta_1, \beta_2, A \in [0, \frac{1}{2})$ ,  $d : n \mapsto \delta n^{1-\alpha}$  and  $q : n \mapsto cn^\alpha$  for a constant  $c > 0$ . Let  $X_1 \subseteq V$  with  $|X_1| = \beta_1 n$ . At the beginning of the game Breaker is allowed to claim at most  $An^2$  edges at once and after that Maker and Breaker are alternately claiming edges. Maker claims one edge and Breaker claims up to  $q$  edges per round. Maker's goal is that at the end of the game there is a set of vertices  $X_2 \subseteq X_1$  with  $|X_2| \geq \beta_1 \beta_2 n$  and  $|N_M(x) \cap X_1| \geq d$  for all  $x \in X_2$ . We call this the partial minimum degree game with bias  $q$  and headstart  $A$ .

Note that for  $A = 0$  and  $\beta_1 = 1$  this is the usual partial minimum degree game discussed above. For the partial minimum degree game with headstart we have the following analog of Theorem 2.2.

**Theorem 2.4** *Let  $\delta > 0$ ,  $\alpha, \beta_1, \beta_2, A \in [0, 1)$ ,  $d : n \mapsto \delta n^{1-\alpha}$  and  $0 < c < \frac{\beta_1(1-\beta_2)^2}{\delta} - \frac{2A}{\delta\beta_1}$ ,  $q : n \mapsto cn^\alpha$ . Then for large enough  $n$ , Maker has a winning strategy for the partial minimum degree game with bias  $q$  and headstart  $A$  in at most  $\frac{\beta_1\delta}{2}n^{2-\alpha}$  rounds.*

*Proof.* The proof uses ideas seen in the proof of Theorem 2.2. We have to take into account that Breaker has already claimed  $An^2$  edges before Maker claims his first one. This leads to a larger bias  $q$ . Furthermore the size of  $X_1$  influences the number of rounds Maker needs to achieve his goal. Since  $c < \frac{\beta_1(1-\beta_2)^2}{\delta} - \frac{2A}{\delta\beta_1}$  we have  $\frac{c\delta\beta_1}{2} + A < \frac{\beta_1^2(1-\beta_2)^2}{2}$ . Thus there is a  $n \in \mathbb{N}$  large enough, that the inequality

$$\left(\frac{c\delta\beta_1}{2} + A\right)n^2 + \frac{1}{2}\left(\beta_1(1-\beta_2)n + \delta\beta_1(1-\beta_2)n^{2-\alpha}\right) < \frac{\beta_1^2(1-\beta_2)^2}{2}n^2 \quad (2.1)$$

holds. Maker plays according to the  $(\delta n^{1-\alpha})$ -degree strategy restricted to edges within  $X_1$  which means that he only claims edges  $\{v, w\}$  with  $v, w \in X_1$  and  $|N_M(v) \cap X_1| < \delta n^{1-\alpha}$  and  $|N_M(w) \cap X_1| < \delta n^{1-\alpha}$  as long as he is able to do so, where  $M$  is the graph of Maker's edges. Note that after Maker has claimed more than  $\frac{\beta_1\delta}{2}n^{2-\alpha}$  edges, we have

$$\sum_{v \in X_1} |N_M(v) \cap X_1| = 2|M| > \beta_1\delta n^{2-\alpha}.$$

Thus since  $|X_1| = \beta_1 n$ , by the pigeonhole principle, there has to be a vertex  $v \in X_1$  with  $|N_M(v) \cap X_1| > \delta n^{1-\alpha}$ . Therefore we know that after at most  $\frac{\beta_1\delta}{2}n^{2-\alpha}$  turns there is no unclaimed edge  $\{v, w\}$  left with  $v, w \in X_1$  and  $|N_M(v) \cap X_1| < \delta n^{1-\alpha}$  and  $|N_M(w) \cap X_1| < \delta n^{1-\alpha}$ . Therefore, at this point of the game Breaker has claimed at most  $q\frac{\beta_1\delta}{2}n^{2-\alpha} + An^2 = \left(\frac{c\delta\beta_1}{2} + A\right)n^2$  edges including the  $An^2$  headstart edges at the beginning. Let be  $M$  and  $B$  the sets of Maker- and Breaker-edges respectively at this moment, and define  $X_2 := \{v \in X_1 : |N_M(v) \cap X_1| \geq \delta n^{1-\alpha}\}$

and  $Y := X_1 \setminus X_2$ . We assume for a moment that Maker has not already won. Thus  $|X_2| < \beta_1\beta_2n$ , therefore  $|Y| \geq |X_1| - |X_2| \geq (1 - \beta_2)\beta_1n$ . For each  $y \in Y$ , we have  $X_1 \setminus \{y\} \subseteq X_2 \cup N_M(y) \cup N_B(y)$ . Otherwise for  $v \in (X_1 \setminus \{y\}) \setminus (X_2 \cup N_M(y) \cup N_B(y))$  the edge  $\{v, y\}$  would be still unclaimed, because  $v$  is neither a Maker nor a Breaker neighbor. This is not possible since Maker already claimed all edges  $\{v, w\}$  with  $v, w \in X_1$  and  $|N_M(v) \cap X_1| < \delta n^{1-\alpha}$  and  $|N_M(w) \cap X_1| < \delta n^{1-\alpha}$ . Note that  $X_1 \setminus \{y\} \subseteq X_2 \cup N_M(y) \cup N_B(y)$  implies  $X_1 \setminus \{y\} \subseteq X_2 \cup (N_M(y) \cap X_1) \cup (N_B(y) \cap X_1)$ . Therefore,

$$|X_1 \setminus \{y\}| \leq |X_2| + |N_M(y) \cap X_1| + |N_B(y) \cap X_1|.$$

Thus

$$\begin{aligned} |N_B(y) \cap X_1| &\geq |X_1| - 1 - |X_2| - |N_M(y) \cap X_1| \\ &> \beta_1n - 1 - \beta_1\beta_2n - \delta n^{1-\alpha} \\ &= \beta_1(1 - \beta_2)n - 1 - \delta n^{1-\alpha} \end{aligned}$$

This means, Breaker has occupied at least  $\beta_1(1 - \beta_2)n - 1 - \delta n^{1-\alpha}$  edges incident in every vertex of  $Y$ . So in total the number of edges claimed by Breaker is at least

$$\begin{aligned} &\frac{|Y|}{2} (\beta_1(1 - \beta_2)n - 1 - \delta n^{1-\alpha}) \\ &\geq \frac{\beta_1(1 - \beta_2)n}{2} (\beta_1(1 - \beta_2)n - 1 - \delta n^{1-\alpha}) \\ &= \frac{\beta_1^2(1 - \beta_2)^2}{2} n^2 - \frac{1}{2} (\beta_1(1 - \beta_2)n + \delta\beta_1(1 - \beta_2)n^{2-\alpha}) \\ &> \left(\frac{c\delta\beta_1}{2} + A\right)n^2. \end{aligned} \tag{by 2.1}$$

This is a contradiction to the fact proved above that breaker has claimed as most  $\left(\frac{c\delta\beta_1}{2} + A\right)n^2$  edges.  $\square$

## 2.2 The $(d, \beta, k)$ -chain Game

In this section we consider the following generalization of the partial minimum degree game: Let  $\delta > 0$ ,  $\alpha, \beta \in (0, 1)$  and  $d : n \mapsto \delta n^{1-\alpha}$ . Maker's goal is to claim edges such that there is a  $k$ -tuple  $(X_0, \dots, X_{k-1}) \in \mathcal{P}(V)^k$  satisfying  $X_0 = V$ ,  $X_i \subseteq X_{i-1}$ ,  $|X_i| \geq \beta |X_{i-1}|$  and  $|N_M(x) \cap X_{i-1}| \geq d(n)$  for all  $i \in [k-1]$  and  $x \in X_i$ . We call this the Maker-Breaker  $(d, \beta, k)$ -chain game. In the following definition we define it formally.

**Definition 2.5** ( $(d, \beta, k)$ -chain) Let  $k, n \in \mathbb{N}$ ,  $d \in \mathbb{R}$ ,  $\beta \in (0, 1)$ . For a subgraph  $(V, M)$  of  $K_n = (V, E)$  we call a  $k$ -tuple  $(X_0, \dots, X_{k-1}) \in \mathcal{P}(V)^k$  a  $(d, \beta, k)$ -chain in  $(V, M)$  if the following is true for all  $i \in [k-1]$ :

- (i)  $X_0 = V$
- (ii)  $X_i \subseteq X_{i-1}$
- (iii)  $|X_i| \geq \beta |X_{i-1}|$
- (iv)  $|N_M(x) \cap X_{i-1}| \geq d$  for all  $x \in X_i$

**Definition 2.6** ( $(d, \beta, k)$ -chain Game) Let  $\beta \in (0, 1)$ ,  $k \in \mathbb{N}$  and  $d : \mathbb{N} \rightarrow \mathbb{R}$ . We call the Maker-Breaker game played on the complete graph  $K_n$  where the goal for Maker is to build a  $(d(n), \beta, k)$ -chain. In other words the Maker-Breaker game has the winning sets

$$\mathcal{F} = \{M \subseteq E(K_n) \mid \text{there is a } (d(n), \beta, k)\text{-chain in } (V, M)\},$$

the Maker-Breaker  $(d, \beta, k)$ -chain game.

**Theorem 2.7** Let  $k \in \mathbb{N}$ ,  $\delta > 0$ ,  $\alpha, \beta \in (0, 1)$  and  $d : n \mapsto \delta n^{1-\alpha}$ . Let  $c < \frac{\beta^{2(k-1)}(1-\beta)^3}{\delta(1+\beta^{k-1})(1-\beta)}$  and  $q : n \mapsto cn^\alpha$ . Then for large enough  $n$  Maker has a winning strategy for the Maker-Breaker  $(d, \beta, k)$ -chain game with bias  $q(n)$  in at most  $\frac{\delta}{2(1-\beta)} n^{2-\alpha}$  rounds.

*Proof.* We fix  $n \in \mathbb{N}$ ,  $q := q(n)$  and  $d := d(n)$  and assume that  $n$  is sufficiently large, where we need it. For all  $j \in [k-1]$  we set  $A_j := \frac{c\delta}{2} \sum_{i=0}^{j-1} \beta^i$ . Note that for all  $j \in [k-1]$  we have

$$A_j \leq A_{k-1} < \frac{c\delta}{2(1-\beta)}. \quad (2.2)$$

We show per induction over  $j$  that the following statement is true for every  $j \in [k-1]$ : Maker has a winning strategy for the  $(d, \beta, j+1)$ -chain game with bias  $q(n)$  and he can win the game in at most  $\frac{A_j}{c} n^{2-\alpha}$  rounds. For  $j = k-1$  this is the statement of the theorem.

For  $j = 1$  the  $(d, \beta, 2)$ -chain game is the partial minimum degree game and Theorem 2.2 proofs the above statement.

For the induction step let  $1 < j \in [k-1]$ . We assume that the above statement is true for  $j-1$ . This means Maker can build a  $(d, \beta, j)$ -chain  $(X_0, \dots, X_{j-1})$  in at most  $\frac{A_j}{c} n^{2-\alpha}$  turns. Since  $q = cn^\alpha$  Breaker has not claimed more than  $A_j n^2$  edges.

Therefore by Theorem 2.4 (with  $\beta_1 := \beta^j$  and  $\beta_2 := \beta$ ) Maker can win the partial minimum degree game with headstart  $A_j$  in at most  $\frac{\beta^j \delta}{2} n^{2-\alpha}$  rounds, and creates a set  $X_j \subseteq X_{j-1}$  such that  $(X_0, \dots, X_j)$  is a  $(d, \beta, j+1)$ -chain in the Maker graph  $M$  if

$$c < \frac{\beta^j(1-\beta)^2}{\delta} - \frac{2A_j}{\delta\beta^j}. \quad (2.3)$$

Using the induction hypothesis, the total number of rounds played is at most  $\frac{A_j}{c} n^{2-\alpha} + \frac{\beta^j \delta}{2} n^{2-\alpha} = \frac{A_{j+1}}{c} n^{2-\alpha}$ . We complete the proof by showing that (2.3) holds. We have by definition of  $c$ :

$$c < \frac{\beta^{2(k-1)}(1-\beta)^3}{\delta(1+\beta^{k-1}(1-\beta))} = \frac{\beta^{k-1}(1-\beta)^2}{\delta(\frac{1}{\beta^{k-1}(1-\beta)} + 1)}.$$

This implies  $c(1 + \frac{1}{\beta^{k-1}(1-\beta)}) < \frac{\beta^{k-1}(1-\beta)^2}{\delta}$ . Therefore,

$$\begin{aligned} c &< \frac{\beta^{k-1}(1-\beta)^2}{\delta} - \frac{c}{\beta^{k-1}(1-\beta)} \\ &\leq \frac{\beta^{k-1}(1-\beta)^2}{\delta} - \frac{2A_{k-1}}{\delta\beta^{k-1}} && \text{(by 2.2)} \\ &\leq \frac{\beta^j(1-\beta)^2}{\delta} - \frac{2A_j}{\delta\beta^j}, \end{aligned}$$

and 2.3 is proved. □

# Chapter 3

## Maker Strategies for the Cycle Game

### 3.1 A Winning Strategy for Maker for the $C_4$ -Game

In this section we will give a constructive, asymptotically optimal Maker strategy for the Maker-Breaker  $C_4$ -Game. We use Theorem 2.2 for the partial minimum degree game to establish a new lower bound for the threshold bias for the  $C_4$ -Game.

**Theorem 3.1** *Let  $c < 0.16$  and  $q = q(n) = cn^{2/3}$ . Then for sufficiently large  $n$  there is a winning strategy for Maker in the Maker-Breaker- $C_4$ -Game.*

For the proof of the Theorem, we fix  $\delta > 1$  with  $c\delta < 0.16$  and  $\beta \in (0.6, 1 - \sqrt{c\delta})$  and assume that  $n$  is sufficiently large. The parameters  $c$ ,  $\delta$ ,  $q$ , and  $\beta$  are fixed for the remainder of this section.

**Maker's strategy for the  $C_4$ -game:** *Maker plays according to the  $\delta n^{1/3}$ -degree strategy. Afterwards he connects two vertices with high degree which creates more than  $q$  immediate threats for Breaker, so Maker can build a copy of  $C_4$  the next turn.* Note that by Theorem 2.2 Maker wins the  $(\beta, \delta n^{1/3})$ -partial minimum degree. We now show that Maker needs at most two additional turns to win the  $C_4$ -game. First we need a few definitions to analyze the game state at the point of time, when Maker would win the partial minimum degree game. We assume that at this point of time Maker has not yet won the  $C_4$ -Game and show that he needs only two more turns to win it.

**Definition 3.2** Let  $M, B \subseteq E(K_n)$  be Maker's graph and Breaker's graph respectively, and let  $X := \{v \in V : \deg_M(v) \geq \delta n^{1/3}\}$ .

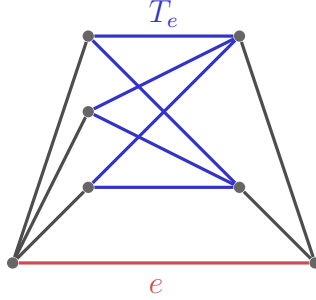
- (i) For each  $a = \{a_1, a_2\} \in E$  we call the edges in

$$T_a := \left\{ \{b_1, b_2\} \in E : b_1 \in N_M(a_1) \wedge b_2 \in N_M(a_2) \right\} \setminus \{a\}$$

the threats for Breaker induced by the edge  $a$ .

- (ii)  $D := \{e \in E : |T_e| \geq \delta^2 n^{2/3} - 1\}$  is the set of dangerous edges (for Breaker).

- (iii)  $D_d := D \cap B$  is the set of directly deactivated edges.
- (iv)  $D_i := \{e \in D : |T_e \setminus B| \leq q\}$  is the set of indirectly deactivated edges.
- (v)  $D_a := D \setminus (D_d \cup D_i)$  is the set of active dangerous edges.



Due to Maker's strategy, we have  $\deg_M(v) = \lceil \delta n^{1/3} \rceil$  for each vertex  $v \in X$ . Maker can create a  $C_4$  if he claims an edge  $e$  and also an edge from  $T_e$ . Because Breaker is allowed to claim  $q$  edges in each turn, Maker has to claim an edge that induces more than  $q$  threats in one of his turns in order to win the game. The edges of  $D$  could induce the highest number of threats and are therefore called dangerous. Breaker can stop Maker from claiming a dangerous edge by claiming the edge himself and thereby deactivating the danger of the edge. The set of edges that are dangerous but deactivated by Breaker in this direct way is  $D_d$ . The second way for Breaker to deactivate a dangerous edge  $e$  is claiming enough edges of  $T_e$  such that there are no more than  $q$  edges left in  $T_e$  unclaimed by Breaker. In this case Breaker would be able to claim all remaining threats if Maker claims  $e$ . The set of the edges that are deactivated in this indirect way is  $D_i$ . The set of edges that are dangerous and neither deactivated directly nor indirectly is  $D_a$ . These are the active dangerous edges and if at any point of time Maker is able to claim one of these he will present more threats than Breaker can deny and Maker will win in the next turn.

**Remark 3.3** *It is evident that for all  $e \in E$  the following two statements hold.*

- (i)  $e \in T_a \Leftrightarrow a \in T_e$  for all  $a \in E$ .
- (ii)  $T_e = \{a \in E : e \in T_a\}$ .

**Lemma 3.4** *Let  $e = \{v_1, v_2\} \in E$  be an edge with  $d := \deg_M(v_1) = \deg_M(v_2)$ . We assume that  $M$  does not contain a copy of  $C_4$ . Then  $|T_e| \in \{d^2 - 1, d^2\}$ .*

*Proof.* If  $N_M(v_1) \cap N_M(v_2) = \emptyset$ , we have  $|T_e| = \deg_M(v_1) \deg_M(v_2) = d^2$ . Otherwise let  $x, y \in N_M(v_1) \cap N_M(v_2)$ . Since  $M$  does not contain a copy of  $C_4$ , the closed walk  $(v_1, x, v_2, y, v_1)$  cannot be a  $C_4$ , therefore  $x = y$ . Thus  $|N_M(v_1) \cap N_M(v_2)| = 1$ . It follows

$$\begin{aligned} |T_e| &= |N_M(v_1) \setminus \{x\}| |N_M(v_2)| + |N_M(v_2) \setminus \{x\}| \\ &= (d - 1)d + (d - 1) = d^2 - 1 \end{aligned}$$

□

**Lemma 3.5** *If  $n$  is sufficiently large and  $M$  doesn't contain a copy of  $C_4$ , the following statements hold:*

- (i)  $|B| \leq \frac{c\delta}{2}n^2$
- (ii)  $|D| \geq \binom{\beta n}{2}$
- (iii)  $D \setminus D_d \subseteq D_i \cup D_a$
- (iv)  $|D \setminus D_d| \geq \binom{\beta n}{2} - \frac{c\delta}{2}n^2$
- (v)  $|B| \geq |D_i|(1 - \frac{c}{\delta^2} - n^{-2/3})$
- (vi)  $|D_a| > 0$

*Proof.* (i) By Theorem 2.2, Maker can win the  $(\beta, \delta n^{1/3})$ -partial minimum degree game in at most  $\frac{\delta}{2}n^{4/3}$  rounds, so Breaker has claimed at most  $q\frac{\delta}{2}n^{4/3} = \frac{c\delta}{2}n^2$  edges.

(ii) Since  $\deg_M(v) = \lceil \delta n^{1/3} \rceil$  for all  $v \in X$ , by Lemma 3.4 we have  $|T_e| \geq \delta^2 n^{2/3} - 1$  for each edge  $e \in E$  with  $e \subseteq X$ . Thus  $\binom{X}{2} \subseteq D$ . By Theorem 2.2, we have  $|X| \geq \beta n$  and therefore  $|D| \geq \binom{|X|}{2} \geq \binom{\beta n}{2}$ .

(iii) Since  $D_a = D \setminus (D_d \cup D_i)$ , the statement is evident.

(iv) We know from (i) that  $\frac{c\delta}{2}n^2 \geq |B| \geq |B \cap D| = |D_d|$ . The statement follows from (ii).

(v) Since Maker plays by the  $\delta n^{1/3}$ -degree strategy, each vertex has Maker degree of at most  $\delta n^{1/3}$ , which implies  $|T_e| \leq \delta^2 n^{2/3}$  for each  $e \in E$ .

It follows

$$\begin{aligned}
 & |D_i|((\delta^2 - c)n^{2/3} - 1) \\
 &= \sum_{e \in D_i} ((\delta^2 - c)n^{2/3} - 1) \\
 &= \sum_{e \in D_i} (\delta^2 n^{2/3} - 1 - q) \\
 &\leq \sum_{e \in D_i} (|T_e| - q) && \text{(as } D_i \subseteq D) \\
 &= \sum_{e \in D_i} (|T_e \cap B| + |T_e \setminus B| - q) \\
 &\leq \sum_{e \in D_i} |T_e \cap B| && \text{(by Definition 3.2(iv))} \\
 &= \sum_{e \in D_i} |\{b \in B : b \in T_e\}| \\
 &= \sum_{e \in D_i} |\{b \in B : e \in T_b\}| && \text{(by Remark 3.3)} \\
 &= \sum_{e \in D_i} \sum_{b \in B} \mathbb{1}_{T_b}(e) = \sum_{b \in B} \sum_{e \in D_i} \mathbb{1}_{T_b}(e)
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{b \in B} |T_b \cap D_i| \leq \sum_{b \in B} |T_b| \\
&\leq \sum_{b \in B} \delta^2 n^{2/3} = |B| \delta^2 n^{2/3}.
\end{aligned}$$

Hence  $|B| \geq |D_i| \frac{(\delta^2 - c)n^{2/3} - 1}{\delta^2 n^{2/3}} \geq |D_i|(1 - \frac{c}{\delta^2} - n^{-2/3})$ .

(vi) We assume  $D_a = \emptyset$ . Because  $\beta > 0.6$ , the following inequality is true:

$$(\beta^2 - (1 - \beta)^2)(1 - (1 - \beta)^2) > (1 - \beta)^2.$$

Therefore, for sufficiently large  $n$ ,

$$(\beta^2 - (1 - \beta)^2)(1 - (1 - \beta)^2 - n^{-2/3}) > (1 - \beta)^2. \quad (3.1)$$

We give upper and lower bounds for  $|B|$ :

$$\begin{aligned}
\frac{c\delta}{2}n^2 &\geq |B| && \text{(by (i))} \\
&\geq |D_i|(1 - \frac{c}{\delta^2} - n^{-2/3}) && \text{(by (v))} \\
&\geq |D \setminus D_d|(1 - \frac{c}{\delta^2} - n^{-2/3}) && \text{(by (iii))} \\
&\geq \left( \binom{\beta n}{2} - \frac{c\delta}{2}n^2 \right) (1 - \frac{c}{\delta^2} - n^{-2/3}) && \text{(by (iv))} \\
&= \left( \frac{1}{2}(\beta^2 - c\delta)(1 - \frac{c}{\delta^2} - n^{-2/3}) \right. \\
&\quad \left. - \frac{\beta}{2n}(1 - \frac{c}{\delta^2} - n^{-2/3}) \right) n^2 \\
&\geq \left( \frac{1}{2}(\beta^2 - c\delta)(1 - c\delta - n^{-2/3}) \right. \\
&\quad \left. - \frac{\beta}{2n}(1 - \frac{c}{\delta^2} - n^{-2/3}) \right) n^2 && \text{(as } \delta > 1) \\
&\geq \left( \frac{1}{2}(\beta^2 - (1 - \beta)^2)(1 - (1 - \beta)^2 - n^{-2/3}) \right. \\
&\quad \left. - \frac{\beta}{2n}(1 - \frac{c}{\delta^2} - n^{-2/3}) \right) n^2 && \text{(as } c\delta < (1 - \beta)^2) \\
&> \left( \frac{(1 - \beta)^2}{2} - \frac{\beta}{2n}(1 - \frac{c}{\delta^2} - n^{-2/3}) \right) n^2. && \text{(by (3.1))}
\end{aligned}$$

The above chain of inequalities gives

$$\begin{aligned}
c\delta &\geq (1 - \beta)^2 - \frac{\beta}{n}(1 - \frac{c}{\delta^2} - n^{-2/3}) \\
&= (1 - \beta)^2 - o(1).
\end{aligned} \quad (3.2)$$

Since  $c\delta < (1 - \beta)^2$ , (3.2) cannot hold for sufficiently large  $n$ . Thus the assumption  $D_a = \emptyset$  is false, and we have proved (vi).  $\square$

*Proof of Theorem 3.1.* We have shown in Lemma 3.5(vi) that  $D_a \neq \emptyset$ . To win the  $C_4$ -game Maker can now claim one of these active dangerous edges  $e \in D_a$ . Since  $e \notin D_i$ , we have  $|T_e| \setminus B > q$  and because Breaker could claim only  $q$  edges, in Maker's next turn at least one of the edges of  $T_e$  will still be not claimed and Maker completes a copy of  $C_4$  by claiming one of them.  $\square$

## 3.2 A Winning Strategy for Maker for the $C_6$ -Game

In this section we will give a constructive, asymptotically optimal Maker strategy for the Maker-Breaker  $C_6$ -Game. We use Theorem 2.7 for the  $(d, \beta, k)$ -chain game to establish a new lower bound for the threshold bias for the  $C_6$ -Game.

**Theorem 3.6** *Let  $c < 0.007$  and  $q = q(n) = cn^{4/5}$ . Then for sufficiently large  $n$  there is a winning strategy for Maker for the Maker-Breaker- $C_6$ -Game.*

The strategy for Maker consists of the following two steps. In the first step Maker plays the  $(d, \beta, 3)$ -chain game introduced in section 2.2 where  $\beta = 0.55$  and  $d$  is slightly larger than  $n^{1/5}$ . As a result he produces a  $(d, \beta, 3)$ -chain  $(X_0, X_1, X_2)$ . In the second step Maker connects two vertices of  $X_2$ . The key point in the proof is to show that this move creates more threats than Breaker can answer in a single round. For the analysis of step two of Maker's strategy we need the following two Lemmata.

**Lemma 3.7** *Let  $G = (V \uplus W, E)$  a connected bipartite graph with  $|V| > 2$ ,  $|W| > 1$  and  $\deg(w) > 1$  for all  $w \in W$ . Then there is a path  $(v_1, \dots, v_5)$  of length 4 in  $G$  with  $v_1, v_5 \in V$ .*

*Proof.* We assume that there is no path  $(v_1, \dots, v_5)$  of length 4 with  $v_1, v_5 \in V$ . Let  $v_1 \in V$ . Since  $G$  is connected there is a  $w_1 \in N(v_1)$  and since  $\deg(w_1) > 1$  there is a  $v_2 \in N(w_1) \setminus \{v_1\}$ . Because of  $|W| > 1$  there is a  $\bar{w} \in W \setminus \{w_1\}$ . Let  $i \neq j \in \{1, 2\}$  and  $w_2 \in W$  such that  $(w_1, v_i, w_2, \dots, \bar{w})$  is the shortest path from  $w_1$  to  $\bar{w}$  in  $G$ . If there is a  $v_3 \in N(w_2) \setminus \{v_1, v_2\}$  then  $(v_j, w_1, v_i, w_2, v_3)$  is a path of length 4 with endpoints in  $V$  which is a contradiction. Therefore we have  $N(w_2) \subseteq \{v_1, v_2\}$  and since  $\deg(w_2) > 1$  we have

$$N(w_2) = \{v_1, v_2\}. \quad (3.3)$$

Analogously we get

$$N(w_1) = \{v_1, v_2\}. \quad (3.4)$$

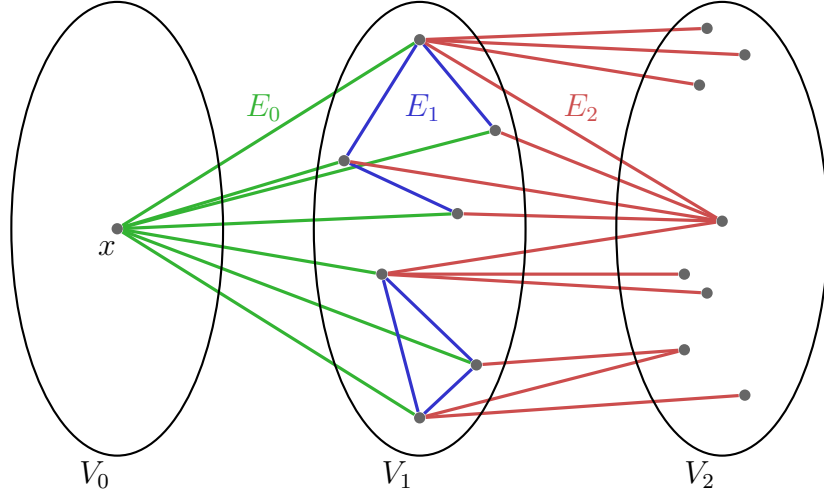
Since  $|V| > 2$  there is a  $v \in V \setminus \{v_1, v_2\}$ , a  $w \in W$  and a  $k \in \{1, 2\}$  with  $w \in N(v) \cup N(v_k)$ . From 3.3 and 3.4 it follows  $w \notin \{w_1, w_2\}$ . This implies that for  $l \in \{1, 2\} \setminus \{k\}$ ,  $(v, w, v_k, w_l, v_l)$  is a path of length 4 with endpoints in  $V$  which is a contradiction to the assumption at the beginning of the proof.  $\square$

**Lemma 3.8** *Let  $n \in \mathbb{N}$ ,  $G = (V, E)$  be a graph,  $x \in V$ ,  $V_0 := \{x\}$ ,  $V_1 := N(x)$ ,  $V_2 := N(V_1) \setminus (V_0 \cup V_1)$ ,  $\lambda < 1$ . The following conditions may hold:*

- (i)  $|V_1| = \lambda n^{1/5}$
- (ii)  $\deg(v) \geq \lambda n^{1/5}$  for all  $v \in V_1$
- (iii)  $G$  contains no copy of  $C_6$ .

*Then  $|V_2| \geq \frac{1}{2}\lambda^2 n^{2/5} - 2\lambda n^{1/5}$  if  $n$  is sufficiently large.*

*Proof.* For subsets  $A, B \subseteq V$  we write  $A \dot{\times} B := \{\{a, b\} \in E : a \in A, b \in B\}$ . Let  $E_0 := \{x\} \dot{\times} V_1$ ,  $E_1 := V_1 \dot{\times} V_1$ ,  $E_2 := V_1 \dot{\times} V_2$  and  $G_1 := (V_1, E_1)$ .



Note that  $G_1$  does not contain a path of length 4: Otherwise if  $P = (v_1, \dots, v_5)$  is a path of length 4 in  $G_1$ , then  $(x, v_1, \dots, v_5, x)$  is a cycle of length 6 in  $G$  which contradicts (iii). Let  $A = (V(A), E(A))$  be a connected component of  $G_1$ . We show

$$|E(A)| \leq \frac{3}{2}|V(A)|. \quad (3.5)$$

Since  $G_1$  contains no copy of  $P_4$ ,  $A$  does not contain any cycle  $C_k$  with  $k \geq 5$ . If  $A$  contains a copy of  $C_4$ ,  $V(A)$  cannot contain any further vertices because otherwise it would contain a copy of  $P_4$ . So in this case we have  $|E(A)| \leq 6 = \frac{3}{2}|V(A)|$ . If  $A$  contains a copy of  $C_3$  and no copy of  $C_4$ , at most one of the vertices  $v_1, v_2, v_3$  of the copy of  $C_3$  has further edges beyond the edges of the  $C_3$  and none of the additional vertices are connected to each other because otherwise  $A$  would contain a copy of  $C_4$  or  $P_4$ . So in this case we have  $|E(A)| = |V(A)|$ . If  $A$  does not contain any cycles, we have  $|E(A)| = |V(A)| - 1 < \frac{3}{2}|V(A)|$ , so in each case (3.5) holds.

We count the edges of  $E_2$  i.e. the edges that connect  $V_1$  and  $V_2$  and show

$$|E_2| \geq \lambda^2 n^{2/5} - 4\lambda n^{1/5}. \quad (3.6)$$

Since the neighbors of a vertex  $v \in V(A)$  are all contained in  $\{x\} \cup V(A) \cup V_2$  and there are exactly  $|V(A)|$  edges connecting  $x$  with  $V(A)$ , we have

$$\sum_{v \in A} \deg_G(v) = |V(A)| + 2|E(A)| + |V(A) \dot{\times} V_2|. \quad (3.7)$$

It follows

$$\begin{aligned} |V(A) \dot{\times} V_2| &= \sum_{v \in A} \deg_G(v) - |V(A)| - 2|E(A)| && \text{(by (3.7))} \\ &\geq |V(A)|\lambda n^{1/5} - |V(A)| - 2|E(A)| && \text{(by (ii))} \end{aligned}$$

$$\begin{aligned} &\geq |V(A)|(\lambda n^{1/5} - 1) - 2\left(\frac{3}{2}|V(A)|\right) && \text{(by (3.5))} \\ &= |V(A)|(\lambda n^{1/5} - 4). \end{aligned}$$

So for each connected component  $A$  of  $G_1$  we have

$$|V(A) \dot{\times} V_2| \geq |V(A)|(\lambda n^{1/5} - 4) \quad (3.8)$$

Counting the edges of  $E_2$  separated by the connected components of  $G_1$  we get

$$\begin{aligned} |E_2| &= |V_1 \dot{\times} V_2| = \sum_{\substack{A \text{ is comp.} \\ \text{of } G_1}} |V(A) \dot{\times} V_2| \\ &\geq \sum_{\substack{A \text{ is comp.} \\ \text{of } G_1}} |V(A)|(\lambda n^{1/5} - 4) && \text{(by (3.8))} \\ &= (\lambda n^{1/5} - 4) \sum_{\substack{A \text{ is comp.} \\ \text{of } G_1}} |V(A)| \\ &= (\lambda n^{1/5} - 4)|V_1| \\ &= \lambda^2 n^{2/5} - 4\lambda n^{1/5}, && \text{(by (i))} \end{aligned}$$

which proves 3.6. We show that there is at most one vertex  $v \in V_1$  with  $\deg_{E_1}(v) > \lambda n^{1/5} - 4$ . If there are two vertices  $v_1, v_2 \in V_1$  with  $\deg_{E_1}(v_i) > \lambda n^{1/5} - 4 = |V_1| - 4 > \frac{1}{2}|V_1|$  for  $i \in \{1, 2\}$ , by the pigeonhole principle there is a vertex  $v_3 \in N_{E_1}(v_1) \cap N_{E_1}(v_2)$ . So for all  $w_i \in N_{E_1}(v_i)$  we have the circle  $(x, w_1, v_1, v_3, v_2, w_2, x)$  in  $G$  which is a contradiction to (iii). Therefore for all vertices  $v$  except at most a single outlier we have  $\deg_{E_1}(v) \leq \lambda n^{1/5} - 4$ . This implies

$$\begin{aligned} \lambda n^{1/5} &\leq \deg_E(v) = \deg_{E_0}(v) + \deg_{E_1}(v) + \deg_{E_2}(v) \\ &\leq 1 + \lambda n^{1/5} - 4 + \deg_{E_2}(v). \end{aligned}$$

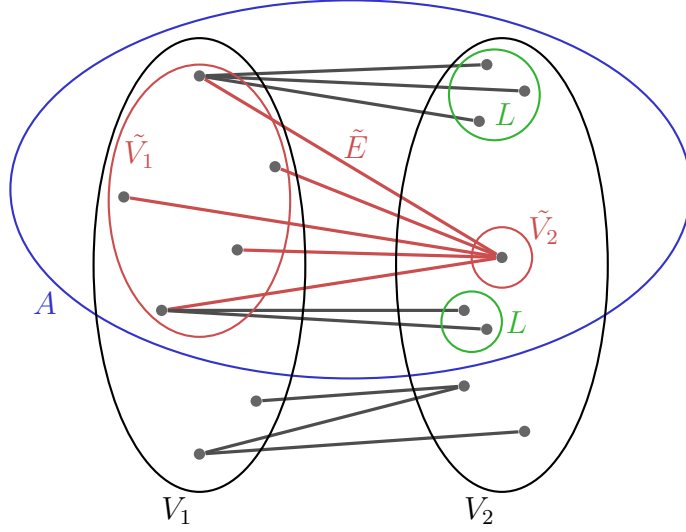
So for all vertices  $v \in V_1$  except at most a single one we have

$$3 \leq \deg_{E_2}(v). \quad (3.9)$$

In the following we consider the bipartite graph  $G_2 := (V_1 \cup V_2, E_2)$  and show that for each connected component  $A$  of  $G_2$  we have

$$|V_2 \cap V(A)| \geq \frac{1}{2}|E(A)|. \quad (3.10)$$

Let  $A = (V(A), E(A))$  be a connected component of  $G_2$ . We define  $\tilde{V}_1 := V_1 \cap V(A)$ ,  $\tilde{V}_2 := \{v \in V_2 \cap V(A) : \deg_A(v) > 1\}$ ,  $\tilde{E} := E(A) \cap (\tilde{V}_1 \dot{\times} \tilde{V}_2)$  and  $L := (V_2 \cap V(A)) \setminus \tilde{V}_2$ .  $\tilde{G}_2 := (\tilde{V}_1 \cup \tilde{V}_2, \tilde{E})$  is a connected bipartite graph in which  $\tilde{V}_2$  does not contain any vertices of degree one.



If  $|\tilde{V}_1| \geq 3$  and  $|\tilde{V}_2| \geq 2$ , by Lemma 3.7 there is a copy  $(v_1, \dots, v_5)$  of  $P_4$  in  $\tilde{G}_2$  with  $v_1, v_5 \in \tilde{V}_1$ . This would imply that  $(x, v_1, \dots, v_5, x)$  is a copy of  $C_6$  in  $G$  which is a contradiction to (iii). Therefore we have

$$|\tilde{V}_1| \leq 2 \text{ or } |\tilde{V}_2| \leq 1. \quad (3.11)$$

We show 3.10 by the cases

1.  $|\tilde{V}_1| \leq 2$
2.  $|\tilde{V}_1| > 2$  and  $|\tilde{V}_2| = 0$
3.  $|\tilde{V}_1| > 2$  and  $|\tilde{V}_2| = 1$ .

In the case of  $|\tilde{V}_1| \leq 2$  each vertex of  $V_2 \cap V(A)$  has at most 2 neighbors in  $V_1$ , which implies  $|V_2 \cap V(A)| \geq \frac{1}{2}|E(A)|$  directly.

Lets assume the case of  $|\tilde{V}_1| > 2$  and  $|\tilde{V}_2| = 0$ . In this case  $V_2 \cap V(A)$  consists of only vertices with degree one and we have  $|V_2 \cap V(A)| = E(A)$ .

For the third case lets assume  $|\tilde{V}_1| > 2$  and  $|\tilde{V}_2| = 1$ . We showed in 3.9 that for each vertex of  $v \in V_1$  except at most one we have  $deg_{E_2}(v) \geq 3$ . So we have at least  $|\tilde{V}_1| - 1$  vertices  $v \in \tilde{V}_1$  with  $deg_{E_2}(v) \geq 3$ . Since  $|\tilde{V}_2| = 1$  each of these vertices has to be connected to at least two vertices of  $L$  which implies

$$|E(A)| \geq |L| \geq 2(|V_1| - 1). \quad (3.12)$$

Note that since each vertex in  $L$  has a degree of one in  $G_2$  we have

$$|\tilde{V}_1 \times L| = |L| \quad (3.13)$$

and since  $|\tilde{V}_2| = 1$  we have

$$|\tilde{V}_1 \times \tilde{V}_2| = |\tilde{V}_1|. \quad (3.14)$$

It follows

$$\begin{aligned}
\frac{1}{2}|E(A)| &= |E(A)| - \frac{1}{2}|E(A)| \\
&\leq |E(A)| - |\tilde{V}_1| + 1 && \text{(by (3.12))} \\
&= |\tilde{V}_1 \dot{\times} (L \cup \tilde{V}_2)| - |\tilde{V}_1| + 1 \\
&= |(\tilde{V}_1 \dot{\times} L) \cup (\tilde{V}_1 \dot{\times} \tilde{V}_2)| - |\tilde{V}_1| + 1 \\
&= |L| + |\tilde{V}_1| - |\tilde{V}_1| + 1 && \text{(by (3.13) and (3.14))} \\
&= |L| + |\tilde{V}_2| = |V_2 \cap V(A)|
\end{aligned}$$

which completes the proof of 3.10. By applying 3.10 to each connected component  $A$  of  $G_2$  we get

$$\begin{aligned}
|V_2| &= \sum_{\substack{A \text{ is comp.} \\ \text{of } G_2}} |V_2 \cap V(A)| \\
&\geq \sum_{\substack{A \text{ is comp.} \\ \text{of } G_2}} \frac{1}{2}|E(A)| && \text{(by (3.10))} \\
&= \frac{1}{2} \sum_{\substack{A \text{ is comp.} \\ \text{of } G_2}} |E(A)| = \frac{1}{2}|E_2| \\
&\geq \frac{1}{2}\lambda^2 n^{2/5} - 2\lambda n^{1/5}. && \text{(by (3.6))}
\end{aligned}$$

□

For the proof of Theorem 3.6, we fix  $c < 0.007$ ,  $\beta = 0.55$ . Since  $c < \frac{\beta^4(1-\beta)^3}{1+\beta^2(1-\beta)}$  there is a  $\delta > 1$  with

$$c\delta < \frac{\beta^4(1-\beta)^3}{1+\beta^2(1-\beta)}. \quad (3.15)$$

We set  $d : n \mapsto \delta n^{1/5}$  and assume that  $n$  is sufficiently large whenever needed. The parameters  $n$ ,  $c$ ,  $\delta$ ,  $q = cn^{4/5}$ ,  $\beta$  and  $d$  are fixed for the remainder of this section.

***Maker's strategy for the  $C_6$ -game:***

*Maker plays with the strategy given in the proof of Theorem 2.7 for the  $(d, \beta, 3)$ -chain game with bias  $q$ . This means:*

*Phase 1:* *In the first phase Maker only connects vertices with degree less than  $\delta n^{1/5}$  until this isn't possible any more. Set  $X_1 := \{v \in V : \deg_M(v) \geq \delta n^{1/5}\}$  where  $M$  is Maker's graph at the end of this phase.*

*Phase 2:* *In the second phase Maker only connects vertices  $v, w$  of  $X_1$  with  $|N_M(v) \cap X_1| < \delta n^{1/5}$  and  $|N_M(w) \cap X_1| < \delta n^{1/5}$  until this isn't possible any more where  $M$  is the Maker degree before his turn. Set  $X_2 := \{v \in X_1 : |N_M(w) \cap X_1| \geq \delta n^{1/5}\}$  where  $M$  is Maker's graph at the end of this phase.*

*Phase 3:* *In the third phase of the game Maker connects two specific vertices of  $X_2$  that create more than  $q$  immediate threats for Breaker, so Maker can build a copy of  $C_6$  the next turn.*

Note that by Theorem 2.7 Maker wins the  $(\delta n^{1/5}, \beta, 3)$ -chain game in at most  $\frac{\delta}{2(1-\beta)} n^{6/5}$  turns. We show that Maker needs at most two additional turns to win the  $C_6$ -game. First we need a few definitions to analyze the game state at the point of

time, when Maker would win the chain game. We assume that at the point of time of creating the  $(\delta n^{1/5}, \beta, 3)$ -chain Maker has not yet won the  $C_6$ -Game and show that he needs only two more turns to win it.

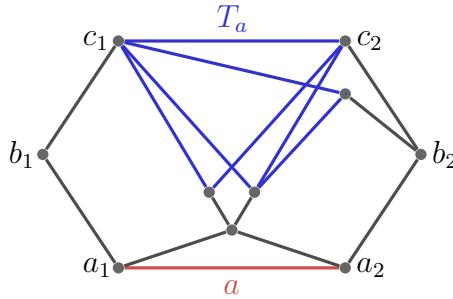
**Definition 3.9** Let  $M, B \subseteq E(K_n)$  be Maker's graph and Breaker's graph respectively.

- (i) For each  $a = \{a_1, a_2\} \in E$  we call the set of edges

$$T_a := \left\{ \{c_1, c_2\} \in \binom{V \setminus \{a_1, a_2\}}{2} : \begin{aligned} &\exists b_1 \neq b_2 \in V(K_n) \setminus \{c_1, c_2, a_1, a_2\} \\ &(a_1, b_1, c_1), (a_2, b_2, c_2) \text{ are paths in } M \end{aligned} \right\}$$

the threats for Breaker induced by the edge  $a$ .

- (ii)  $D := \left\{ e \in E : |T_e| \geq \left( \frac{1}{8}\delta^4 n^{4/5} - 2\delta^3 n^{3/5} + 5\delta^2 n^{2/5} + 6\delta n^{1/5} + \frac{3}{2} \right) \right\}$  is the set of dangerous edges (for Breaker).
- (iii)  $D_d := D \cap B$  is the set of directly deactivated edges.
- (iv)  $D_i := \left\{ e \in D : |T_e \setminus B| \leq q \right\}$  is the set of indirectly deactivated edges.
- (v)  $D_a := D \setminus (D_d \cup D_i)$  is the set of active dangerous edges.



**Remark 3.10** It is evident that for all  $e \in E$  the following two statements hold.

- (i)  $e \in T_a \Leftrightarrow a \in T_e$  for all  $a \in E$ .
- (ii)  $T_e = \{a \in E : e \in T_a\}$ .

**Lemma 3.11** Let  $\delta > 1$  and  $M \subseteq E(K_n)$  be a subgraph with maximum degree  $\Delta(M) \leq 2\delta n^{1/5}$ . Let  $a = \{v_0, w_0\} \in E \setminus M$  and  $V_0 := \{v_0\}$ ,  $W_0 := \{w_0\}$ ,  $V_1 := N_M(v_0)$ ,  $W_1 := N_M(w_0)$ ,  $V_2 := N_M(V_1) \setminus (V_0 \cup V_1)$ ,  $W_2 := N_M(W_1) \setminus (W_0 \cup W_1)$ . Furthermore we assume  $|V_1| \geq \delta n^{1/5} \leq |W_1|$  and  $|V_2| \geq \frac{1}{2}\delta^2 n^{2/5} - 2\delta n^{1/5} \leq |W_2|$ . Then for sufficiently large  $n$  we have

- (i)  $|T_a| \geq \left( \frac{1}{8}\delta^4 n^{4/5} - 2\delta^3 n^{3/5} + 5\delta^2 n^{2/5} + 6\delta n^{1/5} + \frac{3}{2} \right)$
- (ii)  $|T_a| \leq 16\delta^4 n^{4/5}$ .

*Proof.* For each  $v_2 \in V_2$  there is a  $v_1(v_2) \in V_1$  with  $v_2 \in N_M(v_1(v_2))$  and analogously for each  $w_2 \in W_2$  there is a  $w_1(w_2) \in W_1$  with  $w_2 \in N_M(w_1(w_2))$ . We define

$$\tilde{T}_a := \{(v_2, w_2) \in V_2 \times W_2 : v_2 \neq w_0, w_1(w_2) \notin \{v_2, v_1(v_2)\}, w_2 \notin \{v_0, v_1(v_2), v_2\}\}$$

and show for all  $v_2, w_2 \in V(K_n)$ :

$$(v_2, w_2) \in \tilde{T}_a \Rightarrow \{v_2, w_2\} \in T_a. \quad (3.16)$$

Let  $(v_2, w_2) \in \tilde{T}_a$ . Then by  $V_0 \cap V_2 = \emptyset = W_0 \cap W_2$  and the definition of  $\tilde{T}_a$  we have  $v_2, w_2 \notin \{v_0, w_0\}$ . Further by  $V_0 \cap V_1 = \emptyset = W_0 \cap W_1$ , the definition of  $\tilde{T}_a$  and  $a \notin M$  we have  $v_1(v_2), w_1(w_2) \notin \{v_0, w_0, v_2, w_2\}$  and  $v_1(v_2) \neq w_1(w_2)$ . Since  $(v_0, v_1(v_2), v_2)$  and  $(w_0, w_1(w_2), w_2)$  are paths in  $M$ , it follows  $\{v_2, w_2\} \in T_a$ . From 3.16 it follows directly

$$|\tilde{T}_a| \leq 2|T_a|. \quad (3.17)$$

Since  $\Delta(M) \leq 2\delta n^{1/5}$  we further have

$$\left| \{w_2 \in W_2 : w_1(w_2) \in \{v_2, v_1(v_2)\}\} \right| \leq |N_M(\{v_2, v_1(v_2)\})| \leq 4\delta n^{1/5}. \quad (3.18)$$

For  $W(v_2) := \{w_2 \in W_2 : w_1(w_2) \notin \{v_2, v_1(v_2)\}, w_2 \notin \{v_0, v_1(v_2), v_2\}\}$  it follows

$$\begin{aligned} |T_a| &\geq \frac{1}{2} \sum_{v_2 \in V_2 \setminus \{w_0\}} |W(v_2)| && \text{(by (3.17))} \\ &\geq \frac{1}{2} \sum_{v_2 \in V_2 \setminus \{w_0\}} (|W_2| - 4\delta n^{1/5} - 3) && \text{(by (3.18))} \\ &= \frac{1}{2} (|V_2| - 1) (|W_2| - 4\delta n^{1/5} - 3) \\ &\geq \frac{1}{2} (\frac{1}{2}\delta^2 n^{2/5} - 2\delta n^{1/5} - 1) (\frac{1}{2}\delta^2 n^{2/5} - 6\delta n^{1/5} - 3) \\ &= (\frac{1}{8}\delta^4 n^{4/5} - 2\delta^3 n^{3/5} + 5\delta^2 n^{2/5} + 6\delta n^{1/5} + \frac{3}{2}). \end{aligned}$$

The statement (ii) follows from

$$|N_M(N_M(v))| \leq \Delta(M) |N_M(v)| \leq \Delta(M)^2 \leq 4\delta n^{2/5}$$

for all  $v \in V(K_n)$  and the fact that by Definition 3.9(i) we have

$$T_a \subseteq N_M(N_M(a_1)) \dot{\times} N_M(N_M(a_2)).$$

□

In the following Lemma we assume that Maker already build the  $(\delta n^{1/5}, \beta, 3)$ -chain  $(X_0, X_1, X_2)$  by playing the strategy stated above. We show that there is at least one dangerous edge that is neither directly nor indirectly deactivated by Breaker.

**Lemma 3.12** *If  $n$  is sufficiently large and  $M$  doesn't contain a copy of  $C_6$ , the following statements hold:*

$$(i) \quad |B| \leq \frac{c\delta}{2(1-\beta)} n^2$$

$$(ii) |D| \geq \binom{\beta^2 n}{2}$$

$$(iii) D \setminus D_d \subseteq D_i \cup D_a$$

$$(iv) |D \setminus D_d| \geq \binom{\beta^2 n}{2} - \frac{c\delta}{2(1-\beta)} n^2$$

$$(v) |B| \geq |D_i| \left( \frac{1}{128} - \frac{c}{32\delta^4} - \frac{1}{8\delta n^{1/5}} + \frac{5}{16\delta^2 n^{2/5}} + \frac{3}{8\delta^3 n^{3/5}} + \frac{3}{32\delta^4 n^{4/5}} \right)$$

$$(vi) |D_a| > 0$$

*Proof.* (i) By Theorem 2.7, Maker can win the  $(\delta n^{1/5}, \beta, 3)$ -chain game in at most  $\frac{\delta}{2(1-\beta)} n^{6/5}$  rounds, so Breaker has claimed at most  $q \frac{\delta}{2(1-\beta)} n^{6/5} = \frac{c\delta}{2(1-\beta)} n^2$  edges.

(ii) Maker created the  $(\delta n^{1/5}, \beta, 3)$ -chain  $(X_0, X_1, X_2)$  with  $|X_2| \geq \beta |X_1| \geq \beta^2 |X_0| \geq \beta^2 n$  by playing the strategy given in the proof of Theorem 2.7. In the phase of the game in which Maker builds  $X_1$  he claims at most  $\delta n^{1/5}$  edges adjacent to a single vertex and in the phase of the game in which Maker builds  $X_2$  the same is true. Therefore at the time Maker has build the  $(\delta n^{1/5}, \beta, 3)$ -chain we have  $\Delta(M) \leq 2\delta n^{1/5}$ . By Lemma 3.8 we have  $|N_M(N_M(v)) \setminus (\{v\} \cup N_M(v))| \geq \frac{1}{2}\delta n^{2/5} - 2\delta n^{1/5}$  for each  $v \in X_2$ . So by Lemma 3.11 we have  $X_2 \times X_2 \subseteq D$ . It follows  $|D| \geq \binom{|X_2|}{2} \geq \binom{\beta^2 n}{2}$ .

(iii) Since  $D_a = D \setminus (D_d \cup D_i)$ , the statement is evident.

(iv) We know from (i) that  $\frac{c\delta}{1-\beta} n^2 \geq |B| \geq |B \cap D| = |D_d|$ . The statement follows from (ii).

(v) By Lemma 3.11(ii) we have

$$|T_e| \leq 16\delta^4 n^{4/5} \tag{3.19}$$

for each  $e \in E$ . For  $\varepsilon := -2\delta^3 n^{3/5} + 5\delta^2 n^{2/5} + 6\delta n^{1/5} + \frac{3}{2}$  it follows

$$\begin{aligned} & |D_i| \left( \left( \frac{1}{8}\delta^4 - c \right) n^{4/5} + \varepsilon \right) \\ &= \sum_{e \in D_i} \left( \left( \frac{1}{8}\delta^4 - c \right) n^{4/5} + \varepsilon \right) \\ &= \sum_{e \in D_i} \left( \frac{1}{8}\delta^4 n^{4/5} - q + \varepsilon \right) \tag{as } q = cn^{4/5} \\ &\leq \sum_{e \in D_i} (|T_e| - q) \tag{as } D_i \subseteq D \\ &= \sum_{e \in D_i} (|T_e \cap B| + |T_e \setminus B| - q) \\ &\leq \sum_{e \in D_i} |T_e \cap B| \tag{by Definition 3.9(iv)} \\ &= \sum_{e \in D_i} |\{b \in B : b \in T_e\}| \\ &= \sum_{e \in D_i} |\{b \in B : e \in T_b\}| \tag{by Remark 3.10} \end{aligned}$$

$$\begin{aligned}
&= \sum_{e \in D_i} \sum_{b \in B} \mathbb{1}_{T_b}(e) = \sum_{b \in B} \sum_{e \in D_i} \mathbb{1}_{T_b}(e) \\
&= \sum_{b \in B} |T_b \cap D_i| \leq \sum_{b \in B} |T_b| \\
&\leq \sum_{b \in B} 16\delta^4 n^{4/5} && \text{(by (3.19))} \\
&= |B| 16\delta^4 n^{4/5}.
\end{aligned}$$

Hence  $|B| \geq |D_i| \left( \frac{1}{128} - \frac{c}{32\delta^4} - \frac{1}{8\delta n^{1/5}} + \frac{5}{16\delta^2 n^{2/5}} + \frac{3}{8\delta^3 n^{3/5}} + \frac{3}{32\delta^4 n^{4/5}} \right)$ .

(vi) We assume  $D_a = \emptyset$ . Because  $\beta = 0.55$ , the following inequality is true:

$$\frac{1}{2} \left( \beta^2 - \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)} \right) \left( \frac{1}{128} - \frac{\beta^4(1-\beta)^3}{32(1+\beta^2(1-\beta))} \right) > \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)}.$$

Therefore, for sufficiently large  $n$ ,

$$\frac{1}{2} \left( \beta^2 - \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)} \right) \left( \frac{1}{128} - \frac{\beta^4(1-\beta)^3}{32(1+\beta^2(1-\beta))} + \varepsilon \right) > \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)}, \quad (3.20)$$

where  $\varepsilon := -\frac{1}{8\delta n^{1/5}} + \frac{5}{16\delta^2 n^{2/5}} + \frac{3}{8\delta^3 n^{3/5}} + \frac{3}{32\delta^4 n^{4/5}}$ . We give upper and lower bounds for  $|B|$ :

$$\begin{aligned}
\frac{c\delta}{1-\beta} n^2 &\geq |B| && \text{(by (i))} \\
&\geq |D_i| \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) && \text{(by (v))} \\
&\geq |D \setminus D_d| \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) && \text{(by (iii) and } D_a = \emptyset) \\
&\geq \left( \binom{\beta n}{2} - \frac{c\delta}{2(1-\beta)} n^2 \right) \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) && \text{(by (iv))} \\
&= \left( \frac{1}{2} (\beta^2 - \frac{c\delta}{1-\beta}) \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \right. \\
&\quad \left. - \frac{\beta}{2n} \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \right) n^2 \\
&\geq \left( \frac{1}{2} (\beta^2 - \frac{c\delta}{1-\beta}) \left( \frac{1}{128} - \frac{c\delta}{32} + \varepsilon \right) \right. \\
&\quad \left. - \frac{\beta}{2n} \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \right) n^2 && \text{(as } \delta > 1) \\
&\geq \left( \frac{1}{2} \left( \beta^2 - \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)} \right) \right. \\
&\quad \left( \frac{1}{128} - \frac{\beta^4(1-\beta)^3}{32(1+\beta^2(1-\beta))} + \varepsilon \right) \\
&\quad \left. - \frac{\beta}{2n} \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \right) n^2 && \text{(by (3.15))} \\
&> \left( \frac{\beta^4(1-\beta)^3}{(1+\beta^2(1-\beta))(1-\beta)} - \frac{\beta}{2n} \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \right) n^2. && \text{(by (3.20))}
\end{aligned}$$

The above chain of inequalities gives

$$\begin{aligned}
c\delta &\geq \frac{\beta^4(1-\beta)^3}{1+\beta^2(1-\beta)} - \frac{\beta(1-\beta)}{2n} \left( \frac{1}{128} - \frac{c}{32\delta^4} + \varepsilon \right) \\
&= \frac{\beta^4(1-\beta)^3}{1+\beta^2(1-\beta)} - o(1). && (3.21)
\end{aligned}$$

Since by (3.15) we have  $c\delta < \frac{\beta^4(1-\beta)^3}{1+\beta^2(1-\beta)}$  and therefore (3.21) cannot hold for sufficiently large  $n$ . Thus the assumption  $D_a = \emptyset$  is false, and we have proved (vi). □

*Proof of Theorem 3.6.* We have shown in Lemma 3.12(vi) that  $D_a \neq \emptyset$ . To win the  $C_6$ -game Maker can now claim one of the active dangerous edges  $e \in D_a$ . Since  $e \notin D_i$  we have  $|T_e \setminus B| > q$ , by Definition of  $D_i$ . Breaker can claim only  $q$  edges, so in Maker's next turn at least one of the edges of  $T_e$  will still be not claimed by Breaker, thus Maker completes a copy of  $C_6$  by claiming one of them.  $\square$

# Chapter 4

## The Potential Strategy Theorem

In this chapter we introduce balance functions and potential functions and prove our main theorem on Breaker strategies. This potential strategy theorem confirms the conjecture raised by Joel Spencer in 2019 during a visit of Kiel University, that the ideas used in [GS22] and [Sow21] could be generalized to find Breaker strategies for Maker-Breaker subgraph games.

### 4.1 Balance Functions and Potential Functions

#### Definition 4.1 (Obligatory and High-Potential Edges)

Let  $C$  be a subgraph of  $K_n$ . We write  $\mathcal{E} := \mathcal{E}_n$ ,  $V := V_n$  and consider the Maker-Breaker  $C$ -game.

- (i) We call  $pot : (\mathcal{E}, \mathcal{E}, V) \rightarrow \mathbb{R}, (M, B, v) \mapsto pot_{M,B}(v)$  a potential function and may think of  $M$  and  $B$  as the edge sets claimed by Maker and Breaker respectively.
- (ii) For all  $M, B \in \mathcal{E}$  and edges  $e = \{v, w\}$  let

$$pot_{M,B}(e) := pot_{M,B}(v) + pot_{M,B}(w).$$

- (iii) For all  $M, B \in \mathcal{E}$  let

$$obl(M, B) := \{e \in E_n \setminus (M \cup B) : M \cup \{e\} \text{ contains a copy of } C\}$$

be the set of obligatory edges.

- (iv) For  $X, Y \in \mathcal{E}$ ,  $Z \in \mathcal{E} \setminus \{\emptyset\}$  let

$$maximal_{X,Y}(Z) \in \{z \in Z : \forall e \in Z : pot_{X,Y}(e) \leq pot_{X,Y}(z)\}$$

be an edge with maximal potential in  $Z$ .

- (v) Let

$$\bar{f}(M, B) := \min\{q - |obl(M, B)|, n - 1 - |M| - |B| - |obl(M, B)|\}.$$

For all  $i \in [\bar{f}(M, B)]$  we recursively define  $X(M, B)^i$  and  $hp(M, B)^i$  as follows.  
For  $i = 1$  set

$$\begin{aligned} X(M, B)^1 &:= B \cup obl(M, B), \\ hp(M, B)^1 &:= maximal_{M, X(M, B)^1}(E_n \setminus (M \cup X(M, B)^1)) \end{aligned}$$

and if  $i \geq 2$  and  $X(M, B)^{i-1}$  and  $hp(M, B)^{i-1}$  were already defined, set

$$\begin{aligned} X(M, B)^i &:= X(M, B)^{i-1} \cup \{hp(M, B)^{i-1}\}, \\ hp(M, B)^i &:= maximal_{M, X(M, B)^i}(E_n \setminus (M \cup X(M, B)^i)). \end{aligned}$$

(vi) Finally

$$hp(M, B) := \{hp(M, B)^i : i \in [\bar{f}(M, B)]\}$$

is the set of high-potential edges.

**Remark 4.2** *The set  $obl(M, B)$  are the edges that Breaker has to claim immediately because otherwise Maker would win within his next turn. After he has claimed the obligatory edges, the unclaimed edge with the highest potential is  $hp(M, B)^1$ . If Breaker would also claim this edge and update the potentials, the new unclaimed edge with the highest potential would be  $hp(M, B)^2$  and so on.*

**Definition 4.3 (Potential Function Strategy)**

(i) Given an arbitrary potential function  $pot$  we call

$$b : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}, (M, B) \mapsto obl(M, B) \cup hp(M, B)$$

the potential function strategy for Breaker with respect to the potential function  $pot$ .

(ii) Breaker plays according to this strategy, if in each round he first claims every edge in  $obl(M, B)$  and thereafter he successively claims an edge with the highest potential among the unclaimed edges, and updates the potentials of all vertices and edges. He proceeds in this way until he has claimed  $q$  edges, completing his moves in this round.

The preceding definitions formally describe a Breaker strategy with respect to a general potential function. We proceed to specify a very special potential function for the Maker-Breaker  $C$ -game, which will lead to Breaker's win, if the potential function has certain properties.

**Definition 4.4 (Balance Function)** Let  $G = (V, E)$  be a graph,  $\theta, q \in \mathbb{R}$  and  $I, J \subseteq \mathbb{R}$  with  $0 \in I \cap J$ .

(i) We call  $bal : I \times J \rightarrow \mathbb{R}_{>0}$  a balance function if there exists a function  $b^* : \mathbb{R} \rightarrow \mathbb{R}$  such that for all  $m \in I$  and  $b \in J$  the following three conditions are true:

- (a)  $bal(m, \cdot) : J \rightarrow \mathbb{R}_{>0}$ ,  $z \mapsto bal(m, z)$  is strictly decreasing
- (b)  $bal(\cdot, b) : I \rightarrow \mathbb{R}_{>0}$ ,  $z \mapsto bal(z, b)$  is strictly increasing
- (c)  $bal_0 := bal(0, 0) = bal(m, b^*(m)) < 1$ .

(ii) We call

$$d_{bal} : \mathcal{P}(E) \times \mathcal{P}(E) \times V \rightarrow \mathbb{R} : (M, B, v) \mapsto b^*(deg_M(v)) - deg_B(v)$$

the deficit function of  $bal$ .

(iii) For all  $q \in \mathbb{R}$  we call the function

$$pot_{bal,q,\theta} : \mathcal{P}(E) \times \mathcal{P}(E) \times V \rightarrow \mathbb{R}, \mapsto pot_{bal,q,\theta}(M, B, v),$$

where

$$pot_{bal,q,\theta}(M, B, v) := \begin{cases} 0, & \text{if } deg_M(v) + deg_B(v) = |V| - 1 \\ (1 + \theta)^{d_{bal}(M, B, v)/q}, & \text{otherwise} \end{cases}$$

the potential function induced by  $(bal, q, \theta)$ .

**Definition 4.5 (Balance-Potential Function)** Let  $C$  be a fixed graph. Let  $\bar{n} \in \mathbb{N}$  and  $\theta : \mathbb{N} \rightarrow \mathbb{R}$  be a decreasing function with  $\lim_{n \rightarrow \infty} \theta(n) = 0$  and  $q : \mathbb{N} \rightarrow \mathbb{R}$  an arbitrary function. We consider the Maker-Breaker  $C$ -game with bias  $q(n)$  on the complete graph  $K_n = (V_n, E_n)$  for  $n \in \mathbb{N}$ . For all  $n \in \mathbb{N}$  let  $x_n \in \mathbb{R}$  and  $bal_n : [0, x_n] \times \mathbb{R} \rightarrow \mathbb{R}_{>0}$  be a balance function such that for all  $n > \bar{n}$ ,  $v \in V_n$ , any Maker graph  $M$ , and any Breaker graph  $B$  the following three properties hold:

- (i)  $(bal_n)_0 = (bal_{\bar{n}})_0$
- (ii)  $0 < pot_{bal_n, q(n), \theta(n)}(M, B, v) \leq 2n \Rightarrow deg_M(v) \neq \lceil x_n \rceil - 1$
- (iii) Suppose Breaker plays according to the potential function strategy with respect to the potential function  $pot_{bal_n, q(n), \theta(n)}$  and  $M$  is the graph of Maker,  $B$  is the graph of Breaker respectively and it holds  $\Delta(M) < x_n$ , and  $M'$  is the graph of Maker directly after his next move and  $obl(v) := \{o \in obl(M', B) \mid v \notin o\}$  are the obligatory Breaker edges not incident in  $v$ , then the following conditions are satisfied:

$$(a) \quad (bal_n)_0 \left( q(n) - |obl(M', B)| \right) \geq 1$$

$$(b) \quad (bal_n)_0 \left( q(n) - |obl(v)| \right) \geq d_{bal_n}(M', B, v) - d_{bal_n}(M, B, v)$$

Then we call the potential function  $pot_{bal_n, q(n), \theta(n)}$  a balance-potential induced by  $(bal_n)_{n \in \mathbb{N}}$ .

## 4.2 The Potential Strategy Theorem

We are now ready to state the central potential function strategy theorem.

**Theorem 4.6 (*Breaker's Win*)** *If Breaker plays the  $C$ -subgraph-game according to the potential function strategy with respect to a balance-potential, then there is a  $N \in \mathbb{N}$  such that he wins the game for any  $n > N$ .*

The proof of Theorem 4.6 requires considerable conceptual and technical effort. For the readers convenience we structure it in suitable subsections.

### 4.2.1 Properties of the potential during the $C$ -game

We assume that Breaker plays the Maker-Breaker  $C$ -game according to the potential strategy with respect to a balance-potential  $pot_{bal_n, q(n), \theta(n)}$  as defined in Definition 4.5. Let  $n_1 \in \mathbb{N}$  be a natural number with  $\bar{n} < n_1$  and

$$\theta(n_1) < \min\left\{\frac{1}{3}, \frac{1}{(bal_{n_1})_0} - 1\right\}. \quad (4.1)$$

Further let  $\gamma \in (0, 1)$ ,  $\eta \in (0, 1 - (bal_{n_1})_0(1 + \theta(n_1)))$  and  $\varepsilon \in (0, \frac{1}{2})$  with

$$\frac{1 - \eta}{(1 + \varepsilon)(1 + \theta(n_1))(bal_{n_1})_0} > 1. \quad (4.2)$$

For all  $n \in \mathbb{N}$  define

$$c(n) := \left\lceil \frac{1 - \ln(1 - \gamma)}{\ln(1 - \eta) - \ln(1 + \varepsilon) - \ln((1 + \theta(n))(bal_n)_0)} \right\rceil. \quad (4.3)$$

Since  $c(n)$  is bounded and  $\lim_{n \rightarrow \infty} \theta(n) = 0$ , we can find  $n \in \mathbb{N}$  with  $n_1 < n$  and

$$2c(n)\theta(n) < \min\left\{\eta\gamma, \eta(1 - \gamma)\left(1 - (1 + \varepsilon)^{-1/(bal_{n_1})_0}\right)\right\}. \quad (4.4)$$

Since  $n$  is fixed for the remainder of this proof we omit the dependency of  $\theta$  on  $n$  and write  $\theta$  instead of  $\theta(n)$ . Accordingly we write  $V, E, \mathcal{E}, x, bal, q, c$  instead of  $V_n, E_n, \mathcal{E}_n, x_n, bal_n, q_n, c(n)$ . Let  $\mu := 1 + \theta$  and  $b^*$  be as in Definition 1.5.(i). For all  $M, B \in \mathcal{P}(E)$  set

$$d_{M,B} : V \rightarrow \mathbb{R}, v \mapsto d_{bal}(M, B, v), \quad (4.5)$$

$$pot_{M,B} : V \rightarrow \mathbb{R}, v \mapsto pot_{bal, q, \theta}(M, B, v), \quad (4.6)$$

$$POT_{M,B} := \sum_{v \in V} pot_{M,B}(v). \quad (4.7)$$

Let  $b$  be Breaker's potential function strategy with respect to  $pot := pot_{bal, q, \theta}$  and let  $m$  be an arbitrary Maker strategy. Since these strategies are fixed for the remainder of the proof we omit the dependency on them and write  $\bar{t} := \bar{t}(m, b)$  and for each  $t \in [\bar{t}]$  we write  $M^t := M^t(m, b)$ ,  $B^t := B^t(m, b)$ . Assume  $e^t \in E$  is the edge with  $\{e^t\} = m(M^t, B^t)$ , i.e.  $e^t$  is the edge chosen by Maker in turn  $t$ . In the following definition we split up the change in the overall potential  $POT$  during a single turn into several parts.

**Definition 4.7** For all  $M, B \in \mathcal{E}$  let

$$M' := M \cup m(M, B), B^c := B \cup obl(M', B), B' := B^c \cup hp(M', B). \quad (4.8)$$

For  $u, v, w \in V$  with  $\{u, v\} = m(M, B)$  we define

$$(i) \quad \Delta_{M,B}^+(w) := \begin{cases} \mu^{\frac{d_{M',B}(w)}{q}} - \mu^{\frac{d_{M,B}(w)}{q}} & , \text{ if } w \in \{u, v\} \\ 0 & , \text{ otherwise} \end{cases}, \quad (4.9)$$

$\Delta_{M,B}^+(w)$  is the share of the change in the potential of  $w$  caused by Maker's move.

$$(ii) \quad \Delta_{M,B}^{heads}(w) := \begin{cases} \mu^{\frac{d_{M',B^c}(w)}{q}} - \mu^{\frac{d_{M',B}(w)}{q}} & , \text{ if } w \in \{u, v\} \\ 0 & , \text{ otherwise} \end{cases}, \quad (4.10)$$

$\Delta_{M,B}^{heads}(w)$  is the share of the change in the potential of  $w$  caused by obligatory edges of Breaker if  $w$  is one of the vertices in the edge claimed by Maker.

$$(iii) \quad \Delta_{M,B}^{tails}(w) := \begin{cases} 0 & , \text{ if } w \in \{u, v\} \\ \mu^{\frac{d_{M',B^c}(w)}{q}} - \mu^{\frac{d_{M',B}(w)}{q}} & , \text{ otherwise} \end{cases}, \quad (4.11)$$

$\Delta_{M,B}^{tails}(w)$  is the share of the change in the potential of  $w$  caused by obligatory edges of Breaker if  $w$  is contained in the edge claimed by Maker.

$$(iv) \quad \Delta_{M,B}^{hp}(w) := \mu^{\frac{d_{M',B'}(w)}{q}} - \mu^{\frac{d_{M',B^c}(w)}{q}}, \quad (4.12)$$

$\Delta_{M,B}^{hp}(w)$  is the share of the change in the potential of  $w$  caused by high-potential edges of Breaker.

$$(v) \quad \begin{aligned} \Delta_{M,B}^0(w) &:= pot_{M',B'}(w) - pot_{M,B}(w) - \Delta_{M,B}^+(w) \\ &\quad - \Delta_{M,B}^{heads}(w) - \Delta_{M,B}^{tails}(w) - \Delta_{M,B}^{hp}(w), \end{aligned} \quad (4.13)$$

$\Delta_{M,B}^0(w)$  is the share of the change in the potential of  $w$  caused by setting the potential to zero if all edges incident in  $w$  were claimed.

$$(vi) \quad \Delta_{M,B}^{h+}(w) := \begin{cases} \Delta_{M,B}^{heads}(w) & , \text{ if } \Delta_{M,B}^+(w) + \Delta_{M,B}^{heads}(w) > 0 \\ -\Delta_{M,B}^+(w) & , \text{ otherwise} \end{cases}, \quad (4.14)$$

$$\Delta_{M,B}^{h-}(w) := \begin{cases} 0 & , \text{ if } \Delta_{M,B}^+(w) + \Delta_{M,B}^{heads}(w) > 0 \\ \Delta_{M,B}^+(w) + \Delta_{M,B}^{heads}(w) & , \text{ otherwise} \end{cases}, \quad (4.15)$$

$\Delta_{M,B}^{h+}(w)$  and  $\Delta_{M,B}^{h-}(w)$  are defined to ensure  $\Delta_{M,B}^{h+}(w) + \Delta_{M,B}^{h-}(w) = \Delta_{M,B}^{heads}(w)$ ,  $\Delta_{M,B}^+(w) + \Delta_{M,B}^{h+}(w) \geq 0$ , and  $\Delta_{M,B}^+(w) + \Delta_{M,B}^{h-}(w) \leq 0$ .

(vii)

$$\Delta_{M,B}^+ := \sum_{w \in V} \Delta_{M,B}^+(w), \quad \Delta_{M,B}^{heads} := \sum_{w \in V} \Delta_{M,B}^{heads}(w), \quad (4.16)$$

$$\Delta_{M,B}^{tails} := \sum_{w \in V} \Delta_{M,B}^{tails}(w), \quad \Delta_{M,B}^{hp} := \sum_{w \in V} \Delta_{M,B}^{hp}(w), \quad (4.17)$$

$$\Delta_{M,B}^0 := \sum_{w \in V} \Delta_{M,B}^0(w), \quad \Delta_{M,B}^{h+} := \sum_{w \in V} \Delta_{M,B}^{h+}(w), \quad (4.18)$$

$$\Delta_{M,B}^{h-} := \sum_{w \in V} \Delta_{M,B}^{h-}(w), \quad \Delta_{M,B} := \Delta_{M,B}^+ + \Delta_{M,B}^{h+} + (1 - \eta)\Delta_{M,B}^{hp}, \quad (4.19)$$

$$r_{M,B} := \Delta_{M,B}^{h-} + \Delta_{M,B}^{tails} + \eta \Delta_{M,B}^{hp} + \Delta_{M,B}^0. \quad (4.20)$$

**Definition 4.8 (Critical Turn)** We call  $t \in [\bar{t} - 1]$  critical, if  $\Delta_{M^t, B^t} > 0$ .

**Proposition 4.9** Let  $M, B \in \mathcal{E}$  be Maker's graph and Breaker's graph respectively. Then the following holds:

(i)  $POT_{M', B'} - POT_{M, B} = \Delta_{M, B}^+ + \Delta_{M, B}^{heads} + \Delta_{M, B}^{tails} + \Delta_{M, B}^{hp} + \Delta_{M, B}^0$

(ii)  $\Delta_{M, B}^{heads} = \Delta_{M, B}^{h+} + \Delta_{M, B}^{h-}$

(iii)  $r_{M, B} \leq 0$

(iv) Every  $t \in [\bar{t} - 1]$  with  $POT_{M^{t+1}, B^{t+1}} - POT_{M^t, B^t} > 0$  is critical.

*Proof.* (i) follows from (4.13) by summation over all vertices.

(ii) For all  $w \in V$  by (4.14) and (4.15) we have

$$\Delta_{M, B}^{h+}(w) + \Delta_{M, B}^{h-}(w) = \Delta_{M, B}^{heads}(w)$$

if  $\Delta_{M, B}^+(w) + \Delta_{M, B}^{heads}(w) > 0$  and

$$\Delta_{M, B}^{h+}(w) + \Delta_{M, B}^{h-}(w) = -\Delta_{M, B}^+(w) + \Delta_{M, B}^+(w) + \Delta_{M, B}^{heads}(w) = \Delta_{M, B}^{heads}(w)$$

otherwise. (ii) follows by summation over all  $w \in V$ .

(iii) Since  $\Delta_{M, B}^{h-}$ ,  $\Delta_{M, B}^{tails}$ , and  $\Delta_{M, B}^{hp}$  are changes in the potential caused by Breaker's edges, all these values are non-positiv. With  $\Delta_{M, B}^0(v) \in \{0, -(1+\theta)^{d_{bal}(M', B', v)/q}\}$  for all  $v \in V$ , (iii) follows.

(iv) Let  $t \in [\bar{t} - 1]$  with  $POT_{M^{t+1}, B^{t+1}} - POT_{M^t, B^t} > 0$ . We have

$$\Delta_{M^t, B^t} = \Delta_{M, B}^+ + \Delta_{M, B}^{h+} + (1 - \eta)\Delta_{M, B}^{hp} \quad (\text{by (4.19)})$$

$$= \Delta_{M, B}^+ + \Delta_{M, B}^{heads} - \Delta_{M, B}^{h-} + (1 - \eta)\Delta_{M, B}^{hp} \quad (\text{by (ii)})$$

$$= POT_{M^{t+1}, B^{t+1}} - POT_{M^t, B^t} - r_{M, B} \quad (\text{by (4.13)})$$

$$> 0. \quad (r_{M, B} < 0)$$

Therefore  $t$  is critical. □

In the following we pass through a series of Lemmata, revealing useful information about the change of the potential in single turns of the game.

**Lemma 4.10** *Let  $M, B \in \mathcal{E}$  be Maker's graph and Breaker's graph respectively with  $\deg_M(v) < x$  for all  $v \in V$ . Let  $e \in E$  with  $\{e\} = m(M, B)$  and  $\{u, v\} \in E$  with  $M' := M \cup \{\{u, v\}\}$  and  $B' := B \cup \{\{u, v\}\} \neq B$ . Then*

$$(i) \text{ } pot_{M',B}(w) \leq \mu pot_{M,B}(w) \text{ for all } w \in V.$$

$$(ii) \text{ } \Delta_{M,B}^+ \leq (\mu - 1)pot_{M,B}(e)$$

$$(iii) \text{ } \sum_{w \in V} \left( \mu^{\frac{d_{M,B'}(w)}{q}} - \mu^{\frac{d_{M,B}(w)}{q}} \right) \leq (\mu^{-\frac{1}{q}} - 1)pot_{M,B}(\{u, v\})$$

*Proof.* (i) Let  $w \in V$  be an arbitrary vertex and  $obl(w) := \{o \in obl(M', B) \mid w \notin o\}$ . If  $pot_{M,B}(w) = 0$  or  $pot_{M',B}(w) = 0$ , then (i) is trivial. Otherwise, since  $bal_0 < 1$ , we obtain  $q - |obl(w)| \geq d_{M',B}(w) - d_{M,B}(w)$  and  $q \geq 1$  through Definition 4.5(iii). It follows

$$\begin{aligned} pot_{M',B}(w) - pot_{M,B}(w) &= pot_{M,B}(w) \left( \frac{pot_{M',B}(w)}{pot_{M,B}(w)} - 1 \right) \\ &= pot_{M,B}(w) \left( \mu^{\frac{d_{M',B}(w)}{q}} \left( \mu^{\frac{d_{M,B}(w)}{q}} \right)^{-1} - 1 \right) \\ &= pot_{M,B}(w) \left( \mu^{\frac{d_{M',B}(w) - d_{M,B}(w)}{q}} - 1 \right) \\ &\leq pot_{M,B}(w) \left( \mu^{\frac{q - |obl(w)|}{q}} - 1 \right) \\ &\leq pot_{M,B}(w) (\mu - 1). \end{aligned}$$

(ii) For  $v_1, v_2 \in V$  with  $e = \{v_1, v_2\}$  for the same reason as above we have

$$\begin{aligned} \Delta_{M,B}^+ &= \sum_{i \in [2]} \Delta_{M,B}^+(v_i) = \sum_{i \in [2]} \left( \mu^{\frac{d_{M',B}(v_i)}{q}} - \mu^{\frac{d_{M,B}(v_i)}{q}} \right) \\ &= \sum_{i \in [2]} \left( \mu^{\frac{d_{M,B}(v_i)}{q}} \left( \mu^{\frac{d_{M',B}(v_i) - d_{M,B}(v_i)}{q}} - 1 \right) \right) \\ &= \sum_{i \in [2]} \mu^{\frac{d_{M,B}(v_i)}{q}} (\mu - 1) = \sum_{i \in [2]} pot_{M,B}(v_i) (\mu - 1) \\ &= pot_{M,B}(e) (\mu - 1). \end{aligned}$$

(iii) For any  $w \in \{u, v\}$  we have

$$\begin{aligned} \mu^{\frac{d_{M,B'}(w)}{q}} - \mu^{\frac{d_{M,B}(w)}{q}} &= \mu^{\frac{d_{M,B}(w)}{q}} \left( \mu^{\frac{d_{M,B'}(w) - d_{M,B}(w)}{q}} - 1 \right) \\ &= \mu^{\frac{d_{M,B}(w)}{q}} \left( \mu^{-\frac{1}{q}} - 1 \right) \end{aligned}$$

$$= \text{pot}_{M,B}(w) \left( \mu^{-\frac{1}{q}} - 1 \right).$$

Since

$$\mu^{\frac{d_{M,B'}(w)}{q}} - \mu^{\frac{d_{M,B}(w)}{q}} = 0$$

for  $w \notin \{u, v\}$ , (iii) follows. □

**Lemma 4.11** *Let  $M, B \in \mathcal{E}$  be Maker's graph and Breaker's graph respectively with  $\Delta(M) < x$ . Let  $e \in E$  with  $\{e\} = m(M, B)$  and  $M' := M \cup m(M, B)$ . Then*

$$(i) \quad \Delta_{M,B}^+(w) + \Delta_{M,B}^{h^+}(w) \leq \left( \mu^{\frac{\text{bal}_0 \bar{f}(M', B)}{q}} - 1 \right) \text{pot}_{M,B}(w) \text{ for all } w \in V.$$

$$(ii) \quad \Delta_{M,B}^+ + \Delta_{M,B}^{h^+} \leq \left( \mu^{\frac{\text{bal}_0 \bar{f}(M', B)}{q}} - 1 \right) \text{pot}_{M,B}(e).$$

*Proof.* (i) Let  $w \in V$ . If  $\Delta_{M,B}^{h^+}(w) = -\Delta_{M,B}^+(w)$ , then (i) trivially holds. Otherwise we have  $\Delta_{M,B}^{h^+}(w) = \Delta_{M,B}^{\text{heads}}(w)$  and  $\Delta_{M,B}^+(w) + \Delta_{M,B}^{\text{heads}}(w) > 0$  by (4.14). Therefore  $w \in e$  and for  $B^c := B \cup \text{obl}(M', B)$  and  $\text{obl}(w) := \{o \in \text{obl}(M', B) \mid w \notin o\}$  we have

$$\begin{aligned} d_{M', B^c}(w) - d_{M,B}(w) &= d_{M', B^c}(w) - d_{M', B}(w) + d_{M', B}(w) - d_{M,B}(w) \\ &\leq \text{bal}_0 \left( q - |\text{obl}(w)| \right) + d_{M', B^c}(w) - d_{M', B}(w) \quad (\text{Def. 4.5(iii)}) \\ &\leq \text{bal}_0 \left( q - |\text{obl}(w)| + d_{M', B^c}(w) - d_{M', B}(w) \right) \\ &= \text{bal}_0 \left( q - |\text{obl}(w)| - |\{o \in \text{obl}(M', B) \mid w \in o\}| \right) \\ &= \text{bal}_0 \left( q - |\text{obl}(M', B)| \right) \\ &= \text{bal}_0 \bar{f}(M', B). \end{aligned} \quad (\text{Definition 4.1(v)})$$

Therefore

$$\begin{aligned} \Delta_{M,B}^+(w) + \Delta_{M,B}^{h^+}(w) &= \Delta_{M,B}^+(w) + \Delta_{M,B}^{\text{heads}}(w) \\ &= \mu^{\frac{d_{M', B^c}(w)}{q}} - \mu^{\frac{d_{M,B}(w)}{q}} \quad (\text{by (4.9), (4.10)}) \\ &= \left( \mu^{\frac{d_{M', B^c}(w) - d_{M,B}(w)}{q}} - 1 \right) \mu^{\frac{d_{M,B}(w)}{q}} \\ &= \left( \mu^{\frac{d_{M', B^c}(w) - d_{M,B}(w)}{q}} - 1 \right) \text{pot}_{M,B}(w) \\ &\leq \left( \mu^{\frac{\text{bal}_0 \bar{f}(M', B)}{q}} - 1 \right) \text{pot}_{M,B}(w). \end{aligned}$$

and (i) is proved.

(ii) From (i) and  $\forall v \in V \setminus e \quad \Delta_{M,B}^+(v) = \Delta_{M,B}^{h+}(v) = 0$  directly follows

$$\begin{aligned} \Delta_{M,B}^+ + \Delta_{M,B}^{h+} &= \sum_{v \in V} \left( \Delta_{M,B}^+(v) + \Delta_{M,B}^{h+}(v) \right) \\ &= \sum_{v \in e} \left( \Delta_{M,B}^+(v) + \Delta_{M,B}^{h+}(v) \right) \\ &\leq \left( \mu^{\frac{bal_0 \bar{f}(M',B)}{q}} - 1 \right) pot_{M,B}(e). \end{aligned}$$

□

**Lemma 4.12** *Let  $t \in [\bar{t}]$  be critical with  $\Delta(M^t) < x$  and  $e \in E_n \setminus (M^{t+1} \cup B^{t+1})$ . Then*

$$pot_{M^{t+1}, B^{t+1}}(e) < \frac{\mu bal_0}{(1-\eta)} pot_{M^t, B^t}(e^t).$$

*Proof.* For all  $i \in [\bar{f}(M^{t+1}, B^t)]$ , since  $X(M^{t+1}, B^t)^i \subseteq B^{t+1}$  and because in each step Breaker chooses an edge with maximal potential, we have

$$\begin{aligned} pot_{M^{t+1}, B^{t+1}}(e) &\leq pot_{M^{t+1}, X(M^{t+1}, B^t)^i}(e) \\ &\leq pot_{M^{t+1}, X(M^{t+1}, B^t)^i}(hp(M, B)^i). \end{aligned}$$

With Lemma 4.10.(iii) follows

$$\begin{aligned} \Delta_{M^t, B^t}^{hp} &= \sum_{w \in V} \left( \mu^{\frac{d_{M^{t+1}, B^{t+1}}(w)}{q}} - \mu^{\frac{d_{M^{t+1}, X(M^{t+1}, B^t)} \bar{f}(M^{t+1}, B^t)(w)}{q}} \right) \\ &\quad + \sum_{i \in [\bar{f}(M^{t+1}, B^t) - 1]} \left( \mu^{\frac{d_{M', X(M^{t+1}, B^t)^{(i+1)}(w)}{q}} - \mu^{\frac{d_{M', X(M^{t+1}, B^t)^i(w)}{q}} \right) \\ &\leq \sum_{i \in [\bar{f}(M^{t+1}, B^t)]} (\mu^{-\frac{1}{q}} - 1) pot_{M^{t+1}, X(M^{t+1}, B^t)^i}(hp(M, B)^i) \\ &\leq \bar{f}(M^{t+1}, B^t) (\mu^{-\frac{1}{q}} - 1) pot_{M^{t+1}, B^{t+1}}(e). \end{aligned}$$

Therefore we have

$$\bar{f}(M^{t+1}, B^t) (1 - \mu^{-\frac{1}{q}}) pot_{M^{t+1}, B^{t+1}}(e) \leq -\Delta_{M^t, B^t}^{hp} \quad (4.21)$$

Note that for all  $\alpha \geq 1$  we have

$$\alpha (1 - \mu^{-1/q}) \geq 1 - \mu^{-\alpha/q}. \quad (4.22)$$

Set  $\alpha := bal_0 \bar{f}(M^{t+1}, B^t)$ . Note that by Definition 4.1(v), Definition 4.4(i)(c), and Definition 4.5(iii)(a) we have

$$1 \leq \alpha \leq \bar{f}(M^{t+1}, B^t) \leq q. \quad (4.23)$$

Now

$$(1 - \eta) \bar{f}(M^{t+1}, B^t) (1 - \mu^{-\frac{1}{q}}) pot_{M^{t+1}, B^{t+1}}(e)$$

$$\begin{aligned}
 &\leq -(1 - \eta)\Delta_{M^t, B^t}^{hp} && \text{(by (4.21))} \\
 &< \Delta_{M^t, B^t}^+ + \Delta_{M^t, B^t}^{h+} && (t \text{ is critical}) \\
 &\leq \left(\mu^{\frac{\alpha}{q}} - 1\right) \text{pot}_{M^t, B^t}(e^t) && \text{(Lemma 4.11(ii))} \\
 &= \mu^{\frac{\alpha}{q}} \left(1 - \mu^{-\frac{\alpha}{q}}\right) \text{pot}_{M^t, B^t}(e^t) \\
 &\leq \mu \left(1 - \mu^{-\frac{\alpha}{q}}\right) \text{pot}_{M^t, B^t}(e^t) && \text{(by (4.22))} \\
 &\leq \alpha \mu \left(1 - \mu^{-\frac{1}{q}}\right) \text{pot}_{M^t, B^t}(e^t). && \text{(by (4.23))}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \text{pot}_{M^{t+1}, B^{t+1}}(e) &< \frac{\alpha \mu}{(1-\eta) f(M^{t+1}, B^t)} \text{pot}_{M^t, B^t}(e^t) \\
 &= \frac{\text{bal}_0 \mu}{(1-\eta)} \text{pot}_{M^t, B^t}(e^t). && \text{(Definition } \alpha)
 \end{aligned}$$

□

### 4.2.2 Bounding the overall Potential

In the following we will show that if a  $\hat{t} \in [\bar{t}]$  fulfills,  $POT_{M^t, B^t} < 2n$  for all  $t < \hat{t}$  it follows  $POT_{M^{\hat{t}}, B^{\hat{t}}} < 2n$  which implies that the overall Potential will never exceed the value  $2n$ , which is a crucial reason for the success of Breakers strategy. Because the proof of this Lemma is quite intricate, we split it into smaller Lemmata.

**Definition 4.13** (i) Let be  $\hat{t} \in [\bar{t}]$  with for all  $t \in [\hat{t} - 1]$   $POT_{M^t, B^t} < 2n$ . Since  $POT_{\emptyset, \emptyset} = n$ ,  $\{t \in [\hat{t}] : POT_{M^t, B^t} \leq n\} \neq \emptyset$ . Define

$$t_0 := \max\{t \in [\hat{t}] : POT_{M^t, B^t} \leq n\} + 1.$$

Note that  $t_0 - 1$  is a critical turn. Let  $u, v \in V$  such that  $\{u, v\} = e^{t_0-1}$  is the edge claimed by Maker in turn  $t_0 - 1$  and  $\text{pot}_{M^{t_0-1}, B^{t_0-1}}(v) \leq \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u)$ .

(ii) For all  $t \in [\bar{t}]$

$$K(t) := \{s \in [t - 1] : t_0 - 1 \leq s \text{ and } s \text{ is critical}\}$$

is the set of critical turns  $s$  with  $t_0 - 1 \leq s < t$ . Let  $k(t) := |K(t)|$ .

(iii) Let

$$T_1 := \left\{t \in \mathbb{N} : t_0 \leq t \leq \bar{t} \wedge \text{pot}_{M^t, B^t}(u) \leq (1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u)\right\}. \quad (4.24)$$

(iv) Let

$$\begin{aligned}
 T_2 := \left\{t \in \mathbb{N} : t_0 \leq t \leq \bar{t} \wedge \exists w \in V \exists s \in \mathbb{N} t_0 \leq s < t \right. \\
 \left. \wedge \text{pot}_{M^t, B^t}(w) \geq (1 + \varepsilon) \text{pot}_{M^s, B^s}(w)\right\}. && (4.25)
 \end{aligned}$$

(v) For all  $i \in [2]$  set

$$t_i := \begin{cases} \infty & , \text{ if } T_i = \emptyset \\ \min(T_i) & , \text{ otherwise } \end{cases} ,$$

$$t_* := \min\{t_1, t_2\} \text{ and } R := \sum_{s=t_0-1}^{t_*-1} r_{M^s, B^s}.$$

So  $t_0 - 1$  is the last turn in which the overall potential is not larger than  $n$ . In the following it will be shown that between turn  $t_0 - 1$  and turn  $t_*$  there are at most  $c$  critical turns and that in these  $c$  critical turns the overall potential can not rise above  $2n$ , where  $c$  is the constant defined in (4.3).

**Lemma 4.14** *Let  $t \in \mathbb{N}$  with  $t_0 \leq t < t_2$ . Then it holds  $POT_{M^t, B^t} < 2n$ .*

*Proof.* Since  $t < t_2$ ,  $t \notin T_2$ , so for all  $v \in V$  by (2.24) we have  $pot_{M^t, B^t}(v) < (1 + \varepsilon) pot_{M^{t_0}, B^{t_0}}(v)$ . It follows

$$\begin{aligned} POT_{M^t, B^t} &= \sum_{v \in V} pot_{M^t, B^t}(v) \\ &< \sum_{v \in V} (1 + \varepsilon) pot_{M^{t_0}, B^{t_0}}(v) \\ &= (1 + \varepsilon) POT_{M^{t_0}, B^{t_0}} \\ &\leq (1 + \varepsilon) \left( POT_{M^{t_0-1}, B^{t_0-1}} \right. \\ &\quad \left. + \Delta_{M^{t_0-1}, B^{t_0-1}}^+ + \Delta_{M^{t_0-1}, B^{t_0-1}}^{h^+} \right) \quad (\text{Proposition 4.9(i)}) \\ &\leq (1 + \varepsilon) \mu^{\frac{bal_0 \bar{f}(M^{t_0}, B^{t_0-1})}{q}} POT_{M^{t_0-1}, B^{t_0-1}} \quad (\text{Lemma 4.11(i)}) \\ &\leq (1 + \varepsilon) \mu POT_{M^{t_0-1}, B^{t_0-1}} \quad (bal_0 < 1, \bar{f} \leq q) \\ &\leq (1 + \varepsilon) \mu n \quad (\text{Definition 4.13(i)}) \\ &\leq \frac{3}{2} \frac{4}{3} n = 2n \quad (\text{Definition of } \mu, \varepsilon) \end{aligned}$$

□

The following Lemma is a refinement of the inequality in Lemma 4.12. It shows that the potential of an unclaimed edge is bounded by a term exponentially dependent on the number of critical turns.

**Lemma 4.15** *Let  $t \in [\bar{t}]$  with  $t_0 \leq t < t_2$  and  $e \in E_n \setminus (M^t \cup B^t)$ . Then we have*

$$pot_{M^t, B^t}(e) < \left( \frac{(1 + \varepsilon) \mu bal_0}{1 - \eta} \right)^{k(t)} 2 pot_{M^{t_0-1}, B^{t_0-1}}(u) .$$

*Proof.* The statement will be proved by induction over  $k(t)$ .  
 $k(t) = 1$ :

$$pot_{M^t, B^t}(e) < (1 + \varepsilon) pot_{M^{t_0}, B^{t_0}}(e) \quad (t < t_2, (4.25))$$

$$\begin{aligned}
 &< (1 + \varepsilon) \frac{\mu \text{bal}_0}{(1 - \eta)} \text{pot}_{M^{t_0-1}, B^{t_0-1}}(e^{t_0-1}) && \text{(Lemma 4.12)} \\
 &\leq \frac{(1 + \varepsilon) \mu \text{bal}_0}{(1 - \eta)} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u).
 \end{aligned}$$

$k(t) > 1$ : For all  $s \in [\bar{t}]$  with  $t_0 < s < t_2$ ,  $k(s) < k(t)$  and  $e' \in E_n \setminus (M^s \cup B^s)$  by the induction hypothesis we have

$$\text{pot}_{M^s, B^s}(e') < \left( \frac{(1 + \varepsilon) \mu \text{bal}_0}{1 - \eta} \right)^{k(s)} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u).$$

Put  $s := \max\{s' \in [t] : s' \text{ is critical}\} - 1$ . Then,  $k(s) = k(t) - 1$ , and with the induction hypothesis applied to  $e' = e^s$  we get

$$\text{pot}_{M^t, B^t}(e) < (1 + \varepsilon) \text{pot}_{M^{s+1}, B^{s+1}}(e) \quad (t < t_2, (4.25))$$

$$< (1 + \varepsilon) \frac{\mu \text{bal}_0}{(1 - \eta)} \text{pot}_{M^s, B^s}(e^s) \quad \text{(Lemma 4.12)}$$

$$< (1 + \varepsilon) \frac{\mu \text{bal}_0}{(1 - \eta)} \left( \frac{(1 + \varepsilon) \mu \text{bal}_0}{1 - \eta} \right)^{k(s)} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u)$$

$$= \left( \frac{(1 + \varepsilon) \mu \text{bal}_0}{1 - \eta} \right)^{k(s)+1} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u).$$

□

The following Lemma shows that between round  $t_0 - 1$  and  $t_*$  there are at most  $c$  critical turns, where  $c$  is the constant defined in (4.3).

**Lemma 4.16** *We have  $k(t_*) \leq c$ .*

*Proof.* Let  $t := t_* - 1$ . So  $t < t_1$  and by definition of  $T_1$  we have

$$\begin{aligned}
 0 &\leq (1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) \\
 &< \text{pot}_{M^t, B^t}(u). && (t < t_1, (4.24))
 \end{aligned}$$

Therefore, by Definition 4.4(iii), there is an  $e \in E_n \setminus (M^t \cup B^t)$  with  $u \in e$ , and we continue

$$\begin{aligned}
 &\left( \frac{(1 + \varepsilon) \mu \text{bal}_0}{(1 - \eta)} \right)^{k(t)} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) \\
 &> \text{pot}_{M^t, B^t}(e) && \text{(Lemma 4.15)} \\
 &> \text{pot}_{M^t, B^t}(u) \\
 &> (1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) && (t < t_1, (4.24))
 \end{aligned}$$

$$\geq \left( \frac{(1+\varepsilon)\mu \text{bal}_0}{1-\eta} \right)^c 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u). \quad (\text{by (4.3)})$$

Since  $\frac{(1+\varepsilon)\mu \text{bal}_0}{1-\eta} < 1$  by (4.2), we have  $k(t) < c$  thus  $k(t_*) \leq k(t) + 1 \leq c$ .  $\square$

**Lemma 4.17** *We have*

$$(i) \quad \sum_{s \in K(t_*)} \Delta_{M^s, B^s}^+ \leq 2c(\mu - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u)$$

$$(ii) \quad \text{POT}_{M^{t_*}, B^{t_*}} - \text{POT}_{M^{t_0-1}, B^{t_0-1}} \leq 2c(\mu - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R$$

*Proof.* (i): We have

$$\Delta_{M^{t_0-1}, B^{t_0-1}}^+ \leq (\mu - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(e^{t_0-1}) \quad (\text{Lemma 4.10(ii)})$$

$$\leq (\mu - 1) 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u). \quad (\text{Definition } u)$$

For all  $s \in K(t_*)$  with  $t_0 \leq s$  we have

$$\begin{aligned} \Delta_{M^s, B^s}^+ &\leq (\mu - 1) \text{pot}_{M^s, B^s}(e^s) && (\text{Lemma 4.10(ii)}) \\ &< (\mu - 1) \left( \frac{(1+\varepsilon)\mu \text{bal}_0}{1-\eta} \right)^{k(s)} 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) && (\text{Lemma 4.15}) \\ &\leq (\mu - 1) 2 \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u). && (\text{by (4.2)}) \end{aligned}$$

Lemma 4.16 concludes the proof of (i).

(ii): We have

$$\begin{aligned} &\text{POT}_{M^{t_*}, B^{t_*}} - \text{POT}_{M^{t_0-1}, B^{t_0-1}} \\ &= \sum_{s=t_0-1}^{t_*-1} \left( \text{POT}_{M^{s+1}, B^{s+1}} - \text{POT}_{M^s, B^s} \right) \\ &= \sum_{s=t_0-1}^{t_*-1} \left( \Delta_{M^s, B^s} + r_{M^s, B^s} \right) && (\text{Proposition 4.9(i)}) \\ &= R + \sum_{s \in K(t_*)} \Delta_{M^s, B^s} + \sum_{\substack{s \in \mathbb{N} \setminus K(t_*) \\ t_0-1 \leq s < t_*}} \Delta_{M^s, B^s} && (\text{Definition 4.13(v)}) \\ &\leq R + \sum_{s \in K(t_*)} \Delta_{M^s, B^s} && (\text{Definition 4.8}) \\ &\leq R + \sum_{s \in K(t_*)} \Delta_{M^s, B^s}^+ \\ &\leq 2c(\mu - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R. && (\text{by (i)}) \end{aligned}$$

$\square$

In the following two Lemmata it will be shown that if the game continues till turn  $t_*$ , then the overall potential at turn  $t_*$  is at most at the potential of turn  $t_0 - 1$ . This means the game will be over before turn  $t_*$ , because otherwise we have a contradiction to the definition of  $t_0$ .

**Lemma 4.18** *Let  $t_1 = t_* \leq \bar{t}$ . Then it holds*

$$POT_{M^{t_*}, B^{t_*}} \leq POT_{M^{t_0-1}, B^{t_0-1}}.$$

*Proof.* It holds

$$\begin{aligned}
 & POT_{M^{t_*}, B^{t_*}} - POT_{M^{t_0-1}, B^{t_0-1}} \\
 & \leq 2c(\mu - 1) pot_{M^{t_0-1}, B^{t_0-1}}(u) + R && \text{(Lemma 4.17(ii))} \\
 & \leq \eta \gamma pot_{M^{t_0-1}, B^{t_0-1}}(u) + R && \text{(by (4.4))} \\
 & \leq \eta \left( pot_{M^{t_0-1}, B^{t_0-1}}(u) - pot_{M^{t_*}, B^{t_*}}(u) \right) + R && (t_* = t_1, (4.24)) \\
 & = -\eta \left( \sum_{s \in K(t_*)} \Delta_{M^s, B^s}^+(u) + \Delta_{M^s, B^s}^{h^+}(u) \right. \\
 & \quad \left. + \Delta_{M^s, B^s}^{h^-}(u) + \Delta_{M^s, B^s}^{tails}(u) \right. \\
 & \quad \left. + \Delta_{M^s, B^s}^{free}(u) + \Delta_{M^s, B^s}^0(u) \right) + R && \text{(Prop. 4.9(i))} \\
 & \leq -\eta \left( \sum_{s \in K(t_*)} \Delta_{M^s, B^s}^{h^-}(u) + \Delta_{M^s, B^s}^{tails}(u) \right. \\
 & \quad \left. + \Delta_{M^s, B^s}^{free}(u) + \Delta_{M^s, B^s}^0(u) \right) + R && (\Delta^+ + h^+ \geq 0) \\
 & \leq - \left( \sum_{s \in K(t_*)} \Delta_{M^s, B^s}^{h^-}(u) + \Delta_{M^s, B^s}^{tails}(u) \right. \\
 & \quad \left. + \eta \Delta_{M^s, B^s}^{free}(u) + \Delta_{M^s, B^s}^0(u) \right) + R && (\gamma \leq 1 \text{ by (4.2)}) \\
 & \leq - \left( \sum_{s \in K(t_*)} r_{M^s, B^s} \right) + R && \text{(by (4.20))} \\
 & \leq 0. && \text{(Definition of } R)
 \end{aligned}$$

□

**Lemma 4.19** *Let  $t_1 \neq t_2 = t_* \leq \bar{t}$ . Then it holds*

$$POT_{M^{t_*}, B^{t_*}} \leq POT_{M^{t_0-1}, B^{t_0-1}}.$$

*Proof.* Since  $t_* = t_2$ , by (2.25), there exists  $w \in V$  and  $s_0 \in [t_* - 1]$  with  $t_0 \leq s_0$  and

$$(1 + \varepsilon) pot_{M^{s_0}, B^{s_0}}(w) \leq pot_{M^{t_*}, B^{t_*}}(w). \quad (4.26)$$

First, we show via induction over  $t$  that for all  $t \in [t_*]$  with  $s_0 \leq t$  the following inequality is true:

$$pot_{M^t, B^t}(w) \leq pot_{M^{s_0}, B^{s_0}}(w) \prod_{s=s_0}^t \mu^{\frac{bal_0 \bar{f}(M^s, B^{s-1})}{q}}. \quad (4.27)$$

For  $t = s_0$  the statement is obviously true. So let  $t \in [t_*]$  with  $s_0 < t$  and let the statement be true for  $t - 1$ . We have by Induction hypothesis

$$\begin{aligned}
 & pot_{M^t, B^t}(w) \\
 &= pot_{M^t, B^t}(w) - pot_{M^{t-1}, B^{t-1}}(w) + pot_{M^{t-1}, B^{t-1}}(w) \\
 &\leq \Delta_{M^{t-1}, B^{t-1}}^+(w) + \Delta_{M^{t-1}, B^{t-1}}^{h+}(w) \\
 &\quad + pot_{M^{t-1}, B^{t-1}}(w) \qquad \qquad \qquad \text{(by (4.14)-(4.19))} \\
 &\leq \left( \mu^{\frac{bal_0 \bar{f}(M^t, B^{t-1})}{q}} - 1 \right) pot_{M^{t-1}, B^{t-1}}(w) \\
 &\quad + pot_{M^{t-1}, B^{t-1}}(w) \qquad \qquad \qquad \text{(Lemma 4.11)} \\
 &= pot_{M^{t-1}, B^{t-1}}(w) \mu^{\frac{bal_0 \bar{f}(M^t, B^{t-1})}{q}} \\
 &\leq \left( pot_{M^{s_0}, B^{s_0}}(w) \prod_{s=s_0}^{t-1} \mu^{\frac{bal_0 \bar{f}(M^s, B^{s-1})}{q}} \right) \mu^{\frac{bal_0 \bar{f}(M^t, B^{t-1})}{q}} \\
 &= pot_{M^{s_0}, B^{s_0}}(w) \prod_{s=s_0}^t \mu^{\frac{bal_0 \bar{f}(M^s, B^{s-1})}{q}}.
 \end{aligned}$$

(4.26) and (4.27) give

$$\begin{aligned}
 (1 + \varepsilon) pot_{M^{s_0}, B^{s_0}}(w) &\leq pot_{M^{t_*}, B^{t_*}}(w) \qquad \qquad \qquad \text{(by (4.26))} \\
 &\leq pot_{M^{s_0}, B^{s_0}}(w) \prod_{s=s_0}^{t_*} \mu^{\frac{bal_0 \bar{f}(M^s, B^{s-1})}{q}}, \quad \text{(by (4.27))}
 \end{aligned}$$

therefore

$$\begin{aligned}
 (1 + \varepsilon) &\leq \prod_{s=s_0}^{t_*} \mu^{\frac{bal_0 \bar{f}(M^s, B^{s-1})}{q}} \\
 &= \mu^{\frac{bal_0 \sum_{s=s_0}^{t_*} \bar{f}(M^s, B^{s-1})}{q}}.
 \end{aligned}$$

Hence

$$\alpha := \frac{q \ln(1 + \varepsilon)}{bal_0 \ln(\mu)} \leq \sum_{s=s_0}^{t_*} \bar{f}(M^s, B^{s-1}). \quad (4.28)$$

Since  $t_* < t_1$  we have  $pot_{M^{t_*}, B^{t_*}}(u) > 0$  which means, there is still an unclaimed edge  $e$  with  $u \in e$ . By (4.23) and  $t_* < t_1$ , for all  $s \in [t_*]$  with  $t_0 \leq s$  and  $i \in [\bar{f}(M^s, B^{s-1})]$  we have

$$\begin{aligned}
 (1 - \gamma) pot_{M^{t_0-1}, B^{t_0-1}}(u) &< pot_{M^s, B^s}(u) \\
 &< pot_{M^s, B^s}(e)
 \end{aligned}$$

$$\begin{aligned} &\leq \text{pot}_{M^s, X(M^s, B^{s-1})}^i(e) \\ &\leq \text{pot}_{M^s, X(M^s, B^{s-1})}^i(\text{hp}(M^s, B^{s-1})^i). \end{aligned}$$

For all  $s \in [t_*]$  with  $t_0 \leq s$  and  $i \in [\bar{f}(M^s, B^{s-1})]$  it follows

$$\begin{aligned} &POT_{M^s, X(M^s, B^{s-1})}^{i+1} - POT_{M^s, X(M^s, B^{s-1})}^i \\ &\leq \sum_{v \in V} \mu^{\frac{d_{M^s, X(M^s, B^{s-1})}^{i+1}(v)}{q}} - \mu^{\frac{d_{M^s, X(M^s, B^{s-1})}^i(v)}{q}} \\ &\leq (\mu^{-1/q} - 1) \text{pot}_{M^s, X(M^s, B^{s-1})}^i(\text{hp}(M^s, B^{s-1})^i) \quad (\text{Lemma 4.10(iii)}) \\ &\leq (\mu^{-1/q} - 1)(1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u). \end{aligned}$$

So for all  $s \in [t_*]$  with  $t_0 \leq s$  we have

$$\begin{aligned} \Delta_{M^{s-1}, B^{s-1}}^{\text{hp}} &= \sum_{i=1}^{\bar{f}(M^s, B^{s-1})} (POT_{M^s, X(M^s, B^{s-1})}^{i+1} - POT_{M^s, X(M^s, B^{s-1})}^i) \\ &\leq \sum_{i=1}^{\bar{f}(M^s, B^{s-1})} (\mu^{-1/q} - 1)(1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) \\ &\leq \bar{f}(M^s, B^{s-1})(\mu^{-1/q} - 1)(1 - \gamma) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u), \end{aligned}$$

therefore by (4.28)

$$\alpha(1 - \gamma)(\mu^{-1/q} - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) \geq \sum_{s=s_0}^{t_*} \Delta_{M^{s-1}, B^{s-1}}^{\text{hp}} \quad (4.29)$$

It follows

$$\begin{aligned} &POT_{M^{t_*}, B^{t_*}} - POT_{M^{t_0-1}, B^{t_0-1}} \\ &\leq 2c(\mu - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R \quad (\text{Lemma 4.17(ii)}) \\ &\leq \eta(1 - \gamma)(1 - (1 + \varepsilon)^{-1/\text{bal}_0}) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R \quad (\text{by (4.4)}) \\ &\leq \eta(1 - \gamma)(1 - \mu^{-\alpha/q}) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R \quad (\text{by (4.28)}) \\ &\leq \alpha\eta(1 - \gamma)(1 - \mu^{-1/q}) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R \quad (\text{by (4.22)}) \\ &= -\eta\alpha(1 - \gamma)(\mu^{-1/q} - 1) \text{pot}_{M^{t_0-1}, B^{t_0-1}}(u) + R \\ &\leq -\eta \sum_{s=s_0-1}^{t_*-1} \Delta_{M^s, B^s}^{\text{hp}} + R \quad (\text{by (4.29)}) \\ &\leq - \sum_{s=s_0-1}^{t_*-1} r_{M^s, B^s} + R \quad (\text{by (4.20)}) \\ &\leq 0. \quad (\text{Definition of } R) \end{aligned}$$

□

We can now prove that the overall potential at turn  $\hat{t}$  is strictly smaller than  $2n$ .

**Lemma 4.20**  $POT_{M^{\hat{t}}, B^{\hat{t}}} < 2n$ .

*Proof.* By Lemma 4.14 it is sufficient to show  $\hat{t} < t_2$ . Assume for a moment  $t_2 \leq \hat{t}$ . Then we have  $t_* \leq t_2 \leq \hat{t} \leq \bar{t}$ . Since  $t_* \in \{t_1, t_2\}$  we get  $POT_{M^{t_*}, B^{t_*}} \leq POT_{M^{t_0-1}, B^{t_0-1}}$  by Lemma 4.18 in the case  $t_* = t_1$ , and by Lemma 4.19 in the case  $t_* = t_2$ . We know  $t_0 \leq t_*$  by (4.24) and (4.25) and therefore  $POT_{M^{t_0-1}, B^{t_0-1}} \leq n < POT_{M^{t_*}, B^{t_*}}$  by Definition 4.13. It follows  $n < POT_{M^{t_*}, B^{t_*}} \leq POT_{M^{t_0-1}, B^{t_0-1}} \leq n$ , which is a contradiction.  $\square$

With the help of the previous Lemma we can now prove that the Breaker wins the Maker-Breaker- $C$ -game with bias  $q(n)$  by playing the *pot*-strategy.

**Corollary 4.21** *Breaker wins the Maker-Breaker- $C$ -game with bias  $q = q(n)$  by playing according to the *pot*-strategy.*

*Proof.* We know  $POT_{M^t, B^t} < 2n$  for all  $t \in [\bar{t}]$ , because otherwise there would be a first point of time  $\hat{t}$  with  $POT_{M^{\hat{t}}, B^{\hat{t}}} \geq 2n$  which is not possible due to Lemma 4.20. Since  $POT_{M^t, B^t} = \sum_{v \in V} pot_{M^t, B^t}(v)$  we also have  $pot_{M^t, B^t}(v) < 2n$  for all  $t \in [\bar{t}]$ . By Definition 4.5(ii) we have

$$0 < pot_{M, B}(v) \leq 2n \Rightarrow deg_M(v) \neq \lceil x \rceil - 1,$$

which assures that  $\Delta(M^{\bar{t}})$  the maximum degree of Maker's graph at the end of the game is smaller than  $x$ . Therefore by Definition 4.5(iii)(a) we have

$$\begin{aligned} 1 &\leq bal_0 \left( q - |obl(M^{t+1}, B^t)| \right) \\ &\leq q - |obl(M^{t+1}, B^t)|. \end{aligned}$$

So, in every turn Breaker can claim all edges which may lead to Maker's win, and therefore Breaker wins the game.  $\square$

# Chapter 5

## A Breaker Strategy for the Cycle Game

In this section we apply Theorem 4.6 to the Maker-Breaker  $k$ -cycle game in which Maker tries to claim a cycle of length  $k \geq 3$ .

We prove that for  $q > \sqrt[k-1]{(k-1) \binom{2(k-1)}{k}^{k-2} n^{k-2}}$  Breaker has a winning strategy for sufficiently large  $n$ . For  $n \in \{3, 4\}$ , these are exactly the bounds shown in [GS22] and [Sow21], respectively. For  $n > 4$  these are the first explicitly known upper bound constants for the Maker-breaker  $k$ -cycle game.

### 5.1 A Balance Function for the $C_k$ -Game

**Theorem 5.1** *Let  $k \in \mathbb{N}_{\geq 3}$ ,  $\beta > \frac{2(k-1)}{k}$  and  $q(n) := \sqrt[k-1]{(k-1)\beta^{k-2}n^{k-2}}$ . There exists  $N \in \mathbb{N}$  so that for  $n > N$  Breaker has a winning strategy for the Maker-Breaker  $C_k$ -Game with bias  $q = q(n)$ .*

For the proof we define balance functions and show that these balance functions fulfill the assumptions of Theorem 4.6. Let  $k \in \mathbb{N}_{\geq 3}$ ,  $C := C_k$ ,  $n \in \mathbb{N}$ ,  $\bar{\beta} := \frac{2(k-1)}{k}$ ,  $\beta \in (\bar{\beta}, (k-1)\bar{\beta})$ ,  $\delta \in (0, 1 - \frac{\bar{\beta}}{\beta})$ ,  $q := \sqrt[k-1]{(k-1)\beta^{k-2}n^{k-2}}$ ,  $x := \sqrt[k-2]{\frac{q}{k-1}}$ ,  $\Theta := \frac{2\beta \ln(n)k}{\delta x}$ . Our balance function  $bal$  is defined as follows:

$$bal : [0, x] \times \mathbb{R} \rightarrow \mathbb{R}$$

$$(m, b) \mapsto \frac{n - b}{qx(1 - \delta)^{\frac{k+(k-2)\delta}{2(k-1)}} + m\left(\frac{k-2}{2} m x^{k-3} - q\right)}, \quad (5.1)$$

and with  $bal_0 := bal(0, 0)$ ,  $b^*$  is the function

$$b^* : \mathbb{R} \rightarrow \mathbb{R}$$

$$m \mapsto n - bal_0 \left( qx \left( 1 - \delta \right)^{\frac{k+(k-2)\delta}{2(k-1)}} + m \left( \frac{k-2}{2} m x^{k-3} - q \right) \right). \quad (5.2)$$

Further  $pot := pot_{bal, q, \Theta}$  is defined as in Definition 4.4(iii). We assume  $n$  is sufficiently large. Note that  $bal_0$  is independent on  $n$  since by definition of  $x$  and  $q$  we have

$$qx = \beta n. \quad (5.3)$$

We use the notions  $d_{M,B}(v)$  and  $pot_{M,B}(v)$  for  $d_{bal}(M, B, v)$  and  $pot_{bal,q,\theta}(M, B, v)$  as in (4.5) and (4.6).

## 5.2 A Winning Strategy for Breaker for the $C_k$ -Game

**Lemma 5.2** *We have*

$$(i) \quad bal_0 = \frac{1}{\beta(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}}}$$

$$(ii) \quad \frac{\bar{\beta}}{\beta} < bal_0 < \frac{\bar{\beta}}{\beta(1-\delta)} < 1 .$$

*Proof.* (i) By (5.3) we have

$$bal_0 = bal(0, 0) = \frac{n}{qx(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}}} = \frac{1}{\beta(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}}} .$$

(ii) Since  $(k-2)\delta > 0$ , we have

$$bal_0 < \frac{1}{\beta(1-\delta)^{\frac{k}{2(k-1)}}} = \frac{2(k-1)}{\beta(1-\delta)k} = \frac{\bar{\beta}}{\beta(1-\delta)} < 1$$

On the other hand, since  $(1-\delta)(k+(k-2)\delta) < k$ , we have

$$bal_0 = \frac{2(k-1)n}{qx(1-\delta)(k+(k-2)\delta)} > \frac{2(k-1)n}{kqx} = \frac{2(k-1)}{k\beta} = \frac{\bar{\beta}}{\beta}$$

□

**Lemma 5.3** *Let  $M, B \in \mathfrak{P}(E)$  and  $v \in V$ . For sufficiently large  $n$  we have*

$$0 < pot_{M,B}(v) \leq 2n \Rightarrow deg_M(v) \neq \lceil x \rceil - 1 .$$

*Proof.* We proof the contraposition of the statement. So, let  $0 < pot_{M,B}(v)$  and  $deg_M(v) = \lceil x \rceil - 1$ . We show  $pot_{M,B}(v) > 2n$  for sufficiently large  $n$ . By Definition 4.4(iii),  $pot_{M,B}(v) = (1 + \Theta)^{d_{M,B}(v)/q}$ . We proceed to show  $d_{M,B}(v) \geq \frac{2n\delta}{k}$ , and therefore a simple calculation will complete the proof. Note that by definition of  $x$ , we have  $x^{k-3\frac{k-2}{2}}x - q = -q\frac{k}{2(k-1)}$ . We bound  $b^*(deg_{M,B}(v))$  from below:

$$\begin{aligned} b^*(deg_{M,B}(v)) &= b^*(\lceil x \rceil - 1) \\ &= n - bal_0 \left( qx(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}} + (\lceil x \rceil - 1) \left( x^{k-3\frac{k-2}{2}} (\lceil x \rceil - 1) - q \right) \right) \\ &> n - bal_0 \left( qx(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}} + (x-1) \left( x^{k-3\frac{k-2}{2}}x - q \right) \right) \\ &= n - bal_0 \left( qx(1-\delta)^{\frac{k+(k-2)\delta}{2(k-1)}} + (x-1) \left( -q\frac{k}{2(k-1)} \right) \right) \end{aligned}$$

$$\begin{aligned}
 &= n + \frac{bal_0}{k-1} \left( qx\delta + q \left( \frac{x(k-2)\delta^2 - k}{2} \right) \right) \\
 &> n + \frac{bal_0 qx\delta}{k-1} && (x\delta^2 > \frac{k}{k-2} \text{ for large } n) \\
 &= n + \frac{bal_0 \beta n \delta}{k-1} && \text{(by (5.3))} \\
 &> n + \frac{2n\delta}{k}. && \text{(Lemma 5.2(ii))}
 \end{aligned}$$

With Definition 4.4(ii) of the deficit function it follows

$$d_{M,B}(v) = b^*(deg_M(v)) - deg_B(v) \geq n + \frac{2n\delta}{k} - n = \frac{2n\delta}{k}. \quad (5.4)$$

We also need

$$\left( \frac{1}{\Theta} + 1 \right)^{-1} \frac{2\delta n}{kq} = \frac{\Theta}{\Theta+1} \frac{2\delta n}{kq} > \frac{\Theta \delta n}{kq} = \frac{2\beta \ln(n)n}{qx} = 2 \ln(n). \quad (5.5)$$

From (5.2) and (5.3) we get

$$\begin{aligned}
 pot_{M,B}(v) &= (1 + \Theta)^{d_{M,B}(v)/q} \geq (1 + \Theta)^{2n\delta/kq} && \text{(by (5.2))} \\
 &= (1 + \Theta)^{\left(\frac{1}{\Theta} + 1\right)\left(\frac{1}{\Theta} + 1\right)^{-1} 2n\delta/kq} \\
 &> e^{2 \ln(n)} && \text{(by (5.3))} \\
 &= n^2 > 2n,
 \end{aligned}$$

for  $n$  large enough. □

**Lemma 5.4** *Let  $v \in V$ ,  $M, B \in \mathfrak{P}(E)$ ,  $e \in E \setminus (M \cup B)$ ,  $M' := M \cup \{e\}$  with  $\Delta(M) < x$  and let  $obl(v) := \{o \in obl(M', B) \mid v \notin o\}$ . Then*

$$(i) \quad bal_0 \left( q - |obl(M', B)| \right) \geq 1$$

$$(ii) \quad bal_0 \left( q - |obl(v)| \right) \geq d_{M',B}(v) - d_{M,B}(v)$$

*Proof.* Let  $u, w \in V$  with  $e = \{u, w\}$ . Note that the obligatory edges of Breaker  $obl(M', B)$  are those that connect the end points of a path in  $M'$  of length  $k-1$  that contain the last claimed Maker edge  $e$ . Therefore we can bound  $|obl(M', B)|$  from above:

$$|obl(M', B)| \leq (k-1)\Delta(M)^{k-2} \leq (k-1)x^{k-2} = q. \quad (5.6)$$

Also we can bound  $obl(w)$  from above:

$$|obl(v)| \leq (k-2)deg_{M'}(v)\Delta(M)^{k-3}. \quad (5.7)$$

(i) We have

$$bal_0 \left( q - |obl(M', B)| \right) \geq bal_0 \left( q - (k-1)\Delta(M)^{k-2} \right) \quad \text{(by (5.6))}$$

$$\begin{aligned}
&= \text{bal}_0 \left( (k-1)x^{k-2} - (k-1)\Delta(M)^{k-2} \right) && \text{(Definition of } x) \\
&\geq \text{bal}_0 \left( (k-1)x^{k-2} - (k-1)(x-1)^{k-2} \right) \\
&> \text{bal}_0(k-1) \\
&> \frac{\bar{\beta}}{\beta}(k-1) && \text{(Lemma 5.2(ii))} \\
&> 1.
\end{aligned}$$

(ii) Let  $v \in V$ . If  $v \notin e$ , then trivially

$$d_{M',B}(v) - d_{M,B}(v) \leq 0 \leq \text{bal}_0 \left( q - |\text{obl}(M', B)| \right).$$

Otherwise we have  $\text{deg}_{M'}(v) = \text{deg}_M(v) + 1$  and therefore by definition of  $d$  and  $b^*$

$$\begin{aligned}
d_{M',B}(v) - d_{M,B}(v) &= b^*(\text{deg}_{M'}(v)) - b^*(\text{deg}_M(v)) \\
&= \text{bal}_0 \left( q - (k-2)\text{deg}_M(v)x^{k-3} - \frac{k-2}{2}x^{k-3} \right) \\
&\leq \text{bal}_0 \left( q - (k-2)\text{deg}_M(v)\Delta(M)^{k-3} \right) \\
&\leq \text{bal}_0 \left( q - |\text{obl}(v)| \right). && \text{(by (5.5))}
\end{aligned}$$

□

*Proof of Theorem 5.1:* Lemma 5.3 and Lemma 5.4 ensure that our balance function fullfills the conditions in Definition 4.4 . Thus Theorem 4.6 can be applied and Breaker wins by playing according to the potential function strategy corresponding to this very balance function.

□

# Chapter 6

## Open Problems

In this chapter we highlight two of the most prominent open questions on Maker-Breaker  $C_k$ -games and discuss possible approaches to tackling them.

**Problem 1 (Reducing the gap):**

For the triangle game, the ratio between the upper bound and lower bound constant for the threshold bias  $\frac{c_1}{c_0} \leq \sqrt{\frac{4}{3}}$  is already relatively small. It is one of the most prominent open problems to prove the conjecture made in [BL00] that this ratio could be tightened to 1. For the  $C_4$ - and  $C_6$ -game we have decreased the respective ratios to reasonable size but there is potential for optimizing. For example our calculations of Maker's strategy don't take into account that Breaker has to do multiple tasks simultaneously. In the first phase of the game we assume he devotes all his effort in slowing down Maker's attempt to create high degree vertices. In the second phase we assume he does all he can to answer Maker's threats to complete a cycle. If one takes into account that Breaker has to compromise between these two tasks, it could be possible to get better constants for these two games without even altering the strategies given in Chapter 3.

**Problem 2 (Finding Maker strategies for  $C_k$  with  $k \notin \{3, 4, 6\}$ ):**

For the case of  $k \notin \{3, 4, 6\}$  unfortunately we do not have any good Maker strategies yet. Małgorzata Bednarska, in private communication with the author, suggested that the techniques used for the  $C_6$ -game could also be used for  $C_{2k}$ -games where  $k > 3$ . We too believe, that this is the case. The strategy for Maker could be the following. In the first step he plays the  $(d, \beta, k)$ -chain game for some  $d$  slightly larger than  $n^{1/(2k-1)}$  and some properly chosen  $\beta \in (0, 1)$ . By Theorem 2.7 he wins this chain-game in relatively few turns. To prove that this strategy leads to a win in the  $C_{2k}$ -game in a similar way as in the  $C_6$ -game one would need statements analogous to Lemma 3.8 and Lemma 3.7. A possible approach to this would be to use the results from [NS95] on bipartite graphs without even cycles of a fixed length. It seems that this approach would work, though the resulting constants appear to be rather weak. It may therefore be worthwhile to generalize the techniques from Lemmas 3.8 and 3.7 to this setting. Addressing  $C_k$ -games where  $k \geq 5$  is odd will likely require new ideas, yet we believe that the games introduced in Chapter 2 could be helpful in these cases as well.

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# Erklärung

Hiermit erkläre ich,

- dass diese Abhandlung - abgesehen von der Beratung durch den Betreuer - nach Inhalt und Form die eigene Arbeit ist,
- dass Teile der Kapitel 2 und 3 sowohl als Preprint unter <https://arxiv.org/abs/2405.04462> als auch in *The Electronic Journal of Combinatorics* (<https://doi.org/10.37236/13003>) veröffentlicht wurden,
- dass die Arbeit darüber hinaus weder ganz noch zum Teil schon einer anderen Stelle im Rahmen eines Prüfungsverfahrens vorgelegen hat, veröffentlicht worden ist oder zur Veröffentlichung eingereicht wurde,
- dass die Arbeit unter Einhaltung der Regeln guter wissenschaftlicher Praxis der Deutschen Forschungsgemeinschaft entstanden ist,
- dass mir kein akademischer Grad entzogen wurde.

Matthias Sowa

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Kiel, 2026